

MIRZO ULUG'BEK NOMIDAGI
O'ZBEKISTON MILLIY UNIVERSITETI



A.KARIMXODJAYEV

**NAZARIY MEXANIKA.
PRAKTIKUM**

O'ZBEKISTON RESPUBLIKASI
OLIY TA'LIM, FAN VA INNOVATSIYA VAZIRLIGI

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O'quv qo'llanma "Nazariy mexanika" faninig universitetlar uchun tuzilgan namunaviy dasturi asosida yozilgan bo'lib, zamonaviy standartlarga butkul to'g'ri keladi. Qo'llanma asosan fizika fakultetlarining kunduzgi va kechki bo'lim talabarlari uchun mo'ljallangan. Mazkur praktikum "osondan qiyinga", "soddadan murakkabga" tamoyili asosida saralab olingan masala va muammolarni hal qilish orqali talabalarda fan boblariga tegishli masalalarni yechish va tahlil qilish o'quv va ko'nikmalarini shakllantirishga qaratilgan. Unda mexanik sistemalarni tavsiflaydigan Lagranj va Gamilton usullari doirasida namunaviy masalalarni yechilishlari berilib, xuddi shunday turdagi masalalar mustaqil yechish uchun keltirilgan, hamda ularning javoblari yoki tegishli yo'riqlar ko'rsatilgan.

Mazkur qo'llanmadan amaliy mashg'ulotlarni olib boruvchi yosh o'qituvchilar, magistr va aspirantlar ham foydalanishi mumkin.

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Mas'ul muharrir:
Irgaziyev B.F. – professor

Taqrizchilar:

Imamov E.Z. –Toshkent Axborot texnologiyalari universiteti Fizika kafedrası professori, fizika-matematika fanlari doktori
Fayzullayev B.A. –O'zbekiston Milliy universiteti Nazariy fizika kafedrası dostenti, fizika-matematika fanlari nomzodi

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I. HARAKAT TENGLAMALARI.

Mexanik sistema holatini aniq belgilash uchun zarur bo'lgan mustaqil kattaliklar soni sistemaning *erkinlik darajasi* deyiladi. Masalan: N ta moddiy nuqtadan iborat sistemaning erkinlik darajasi $3N$ ga tengdir. Mustaqil kattaliklar albatta nuqtalarning dekart koordinatalari bo'lishi shart emas.

Erkinlik darajasi N bo'lgan sistema holatini to'la xarakterlovchi N ta istalgan q_1, q_2, \dots, q_N kattaliklar shu sistemaning *umumlashgan koordinatalari* deb,

$\dot{q}_i = \frac{dq_i}{dt}$ hosilalar esa uning *umumlashgan tezliklari* deb ataladi.

Tezlanishlar \ddot{q}_i ni koordinatalar q_i va tezliklar \dot{q}_i bilan bog'lovchi munosabatlar *harakat tenglamalari* deyiladi.

Mexanik sistemalar harakat qonunining eng umumiy ifodasi *eng kichik ta'sir prinsipi* (yoki *Gamilton prinsipi*) deb ataluvchi prinsip orqali beriladi.

Bu prinsipga ko'ra har bir mexanik sistema *Lagranj funksiyasi* deb ataluvchi

$$L(q_1, q_2, \dots, q_N, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_N, t) = (L, q, \dot{q}, t) \quad (1.1)$$

funksiya bilan xarakterlanadi.

Harakat tenglamasi esa *ta'sir* (ta'sir integrali, ta'sir funksiyasi) deb ataluvchi quyidagi ifodaning

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt \quad (1.2)$$

harakat davomida minimal (ekstremum) qiymatga erishishi shartidan topiladi ($\delta S = 0$) va *Lagranj tenglamalari* deb ataladi:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \quad (i=1, 2, \dots, N) \quad (1.3)$$

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Lagranj funksiyasi L ma'lum aniqlikda topiladi. Bir-biridan koordinata va vaqtning ixtiyoriy funksiyasidan olingan to'la hosilaga farqlanuvchi Lagranj funksiyalari bir xil harakat tenglamalariga olib keladi:

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t) \quad (1.4)$$

Lagranj funksiyasining fizik ma'nosi sistemaning kinetik va potentsial energiyalari ayirmasiga tengligidir:

$$L = T - U \quad (1.5)$$

T - kinetik energiya, U - potentsial energiya.

To'la energiya Lagranj funksiyasi orqali quyidagicha ifodalanadi:

$$E = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \quad (1.6)$$

Umumlashgan impulsning aniqlanishi esa quyidagicha:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \quad (1.7)$$

Umumlashgan kuch ham xuddi shuningdek aniqlanadi:

$$F_i = \frac{\partial L}{\partial q_i} \quad (1.8)$$

§1.1. Berilgan Lagranj funksiyasidan harakat tenglamalarini topish.

Har qanday fizik sistemani tavsiflashda, avvalom bor u sistemaning Lagranj funksiyasini tuzish lozim, so'ngra harakat tenglamasini topish kerak. Bu bo'limda avvaldan berilgan Lagranj funksiyalari orqali Lagranj tenglamalari, ya'ni harakat tenglamalarini tuzishga oid masalalar bilan tanishib chiqamiz.

1-masala.

$$L = \frac{\dot{q}^2}{2} - \frac{q^2}{2}$$

Yechilishi. Masalani yechishda Lagranj tenglamasining (1.3) ko'rinishidan foydalaniladi. Masalaning berilishiga asosan:

$$\frac{\partial L}{\partial q} = -q; \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{d}{dt} (\dot{q}) = \ddot{q}.$$

Demak, Lagranj tenglamasiga ko'ra $\ddot{q} = -q$.

2-masala.
$$L = \frac{t\dot{q}^2}{2}$$

Yechilishi. Lagranj funksiyasi vaqtga oshkora bog'liq holda berilgandir.

$$\frac{\partial L}{\partial q} = 0, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \Rightarrow \frac{d}{dt} (t\dot{q}) = \dot{q} + t\ddot{q}$$

Demak, $\dot{q} + t\ddot{q} = 0, \quad \ddot{q} = -\frac{\dot{q}}{t}$.

3-masala.
$$L = \frac{\dot{\theta}^2}{2} + \frac{\sin^2 \theta \cdot \dot{\phi}^2}{2} - \cos \theta.$$

Yechilishi. Lagranj funksiyasining berilishidan sistemaning erkinlik darajasi $N = 2$ ga tengdir. q_1 va q_2 larga esa θ va ϕ lar mos keladi. Lagranj tenglamasi (1.1) esa har ikkala koordinataga mos ravishda 2 ta tenglamadan iborat bo'ladi:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial L}{\partial \phi}.$$

Lagranj funksiyasining ko'rinishidan:

$$\frac{\partial L}{\partial \theta} = \sin \theta \cdot \cos \theta \cdot \dot{\phi}^2 + \sin \theta, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt} (\dot{\theta}) = \ddot{\theta}, \quad \frac{\partial L}{\partial \phi} = 0,$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\phi}} = \frac{d}{dt} (\dot{\phi} \sin^2 \theta) = \sin^2 \theta \cdot \ddot{\phi} + 2\dot{\phi} \cdot \sin \theta \cdot \cos \theta \cdot \dot{\theta}.$$

Demak, har bir erkinlik darajasiga mos ikkita tenglama ushbu ko'rinishga ega bo'ladi:

$$\ddot{\theta} = \frac{1}{2} \dot{\phi}^2 \sin 2\theta + \sin \theta, \quad \ddot{\phi} = -2\dot{\phi} \operatorname{ctg} \theta.$$

1.1.—Misollar.

Quyidagi Lagranj funksiyalari bilan xarakterlanuvchi fizik sistemalarning tezlanishlarini toping (harakat tenglamalarini tuzing):

1. $L = \frac{(1+q^2)\dot{q}^2}{2} - \frac{q^2}{2}$
2. $L = -\sqrt{1-\dot{q}^2} + q$
3. $L = \frac{\dot{r}^2}{2} + \frac{r^2}{2}\dot{\theta}^2 + \frac{1}{r}$
4. $L = \frac{1}{2}\dot{U}\dot{v} + \frac{1}{\sqrt{Uv}}$
5. $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + (xy - y\dot{x})$
6. $L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - (x^2 + y^2 + xy)$
7. $L = \frac{1}{2}(\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2) - \frac{1}{2}(x^2 - xy + y^2)$
8. $L = \frac{m}{2}(\dot{\xi}^2 + \rho^2\dot{\phi}^2) + \frac{1}{2}(\chi\xi^2 + mg\rho\phi^2)$
9. $L = \frac{m_1\xi_1^2}{2}\dot{\xi}_1^2 + \frac{m_2\xi_2^2}{2}\dot{\xi}_2^2 - \frac{k}{2}(\xi_2 - \xi_1)^2$
10. $L = \frac{ma^2}{2}\{\omega^2 + (\omega + \dot{\phi})^2 + 2\omega(\omega + \dot{\phi})\cos\phi\}$
11. $L = \frac{m}{2}(\dot{\xi}_1^2 + \dot{\xi}_2^2) - \chi(\xi_1^2 - \xi_1\xi_2 + \xi_2^2)$
12. $L = \frac{m_1\xi_1^2 + m_2\xi_2^2 + \chi(\xi_2 - \xi_1)^2}{2}$
13. $L = \frac{e^{at}(\dot{x}^2 - \omega^2 x^2)}{2}$

§1.2. Ekvivalent Lagranj funksiyalari.

Bir birlaridan doimiyga va koordinata hamda vaqtning ixtiyoriy funksiyasidan olingan to'la hosilaga farqlanadigan, shuningdek ixtiyoriy doimiyga ko'paytirish natijasida hosil bo'ladigan Lagranj funksiyalari bir xil harakat tenglamasiga olib keladi. Ushbu bo'limda aynan Lagranj funksiyasining zikr etilgan hossalardan foydalanilgan holda ekvivalent Lagranj funksiyalarni tuzish (topish) uquvi o'zlashtiriladi.

To'la hosilani tashlab yuborish yo'li orqali ekvivalent Lagranj funksiyasini tuzishga masalalar ko'raylik:

1-masala. $L' = \dot{x} \cdot \sin t$

Yechilishi. Berilgan ifodaga $x \cdot \cos t$ ni qo'shib va ayirib yozish natijasida quyidagi ifodani olamiz:

$$L' = \dot{x} \cdot \sin t + x \cos t - x \cos t = \frac{d}{dt}(x \cdot \sin t) - x \cos t = L + \frac{d}{dt}(x \cdot \sin t),$$

Demak, avvalgi Lagranj funksiyasiga mos yangi Lagranj funksiyasi quyidagi ko'rinishga ega ekan: $L = -x \cdot \cos t$.

Endi bu ikki Lagranj funksiyalari L va L' bir xil harakat tenglamasiga keltirishini ko'rsataylik:

$$\frac{\partial L'}{\partial x} = 0, \quad \frac{d}{dt} \frac{\partial L'}{\partial \dot{x}} = \frac{d}{dt}(\sin t) = (\cos t), \quad \cos t = 0.$$

$$\frac{\partial L}{\partial x} = -\cos t, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = 0 \quad \cos t = 0.$$

2-masala. Dekart koordinatalari bilan quyidagicha bog'lanishda bo'lgan koordinata sistemasida moddiy nuqtaning Lagranj funksiyasi yozilsin:

$$x = \frac{\varphi - v}{2}, \quad y = \sqrt{\varphi v}.$$

Yechilishi. Ma'lumki, moddiy nuqtaning Lagranj funksiyasi quyidagicha aniqlanadi:

$$L = T - U = \frac{m}{2} \left(\frac{dS}{dt} \right)^2 - U(q) = \frac{m}{2} \frac{dS^2}{dt^2} - U(q).$$

Bu yerda dS^2 - mazkur koordinata sistemasidagi uzunlik elementining kvadrati.

$U(q)$ - faqatgina koordinataga bog'liq potensial energiya. T - kinetik energiya.

Dekart koordinata sistemasida:

$$dS^2 = dx^2 + dy^2 + dz^2 \quad \text{va} \quad L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z).$$

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4. $L = \frac{1}{2} \dot{U} \dot{V} + \frac{1}{\sqrt{UV}}$
5. $L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - (xy - y\dot{x})$
6. $L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - (x^2 + y^2 + xy)$
7. $L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2} (x^2 - xy + y^2)$
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10. $L = \frac{ma^2}{2} \left\{ \omega^2 + (\omega + \dot{\phi})^2 + 2\omega(\omega + \dot{\phi}) \cos \phi \right\}$
11. $L = \frac{m}{2} (\dot{\xi}_1^2 + \dot{\xi}_2^2) - \chi (\xi_1^2 - \xi_1 \xi_2 + \xi_2^2)$
12. $L = \frac{m_1 \dot{\xi}_1^2 + m_2 \dot{\xi}_2^2 + \chi (\xi_2 - \xi_1)^2}{2}$
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Dekart koordinata sistemasida:

$$dS^2 = dx^2 + dy^2 + dz^2 \quad \text{va} \quad L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z).$$

Yuqoridagi bog'lanishdan: $\dot{x} = \frac{\dot{\phi} - \dot{v}}{2}$, $\dot{y} = \frac{\dot{\phi} v + \dot{v} \phi}{2(\phi v)^2}$ topiladi.

Yangi o'zgaruvchilardagi Lagranj funksiyasi esa ushbu ko'rinishni oladi:

$$L = \frac{m}{8} \left[\dot{\phi}^2 \left(1 + \frac{v}{\phi} \right) + v \dot{v}^2 \left(1 - \frac{\phi}{v} \right) \right] - U(\phi, v).$$

3-masala. Egri chiziq bo'ylab harakatlanayotgan zarraning Lagranj funksiyasi yozilsin.

Yechilishi. Egri chizikli ortogonal koordinata sistemalarning Dekart koordinata sistemalari bilan bog'lanish yaxshi ma'lum. Koordinata tekisliklari kesishgan yerda hosil bo'lgan chiziqlarning tenglamalari odatda parametrik ko'rinishida beriladi: $x_i = x_i(q)$, q - parametr.

$$L(q, \dot{q}) = \frac{1}{2} m h(q) \dot{q}^2 - V(q), \quad h(q) = \left(\frac{dx_i}{dq} \right)^2, \quad V(q) = U(\vec{r}(q)).$$

Bu yerda $h(q)$ - **Lame koeffitsiyentlari** (Ilovaga qarang).

Lagranj tenglamasi (1.3) dan:

$$m h(q) \ddot{q} + \frac{m}{2} \frac{dh}{dq} \dot{q}^2 - \frac{\partial V}{\partial q} = 0$$

Agarda yoyning uzunligini kiritsak:

$$S = \sqrt{h(q)} q, \quad \dot{S} = \sqrt{h(q)} \dot{q},$$

ekvivalent tenglamani olamiz.

$$m \dot{S} = - \frac{\partial U}{\partial \vec{r}} \frac{\partial \vec{r}}{\partial S} = - \frac{1}{\sqrt{h}} \frac{\partial V}{\partial q}.$$

Parametr sifatida yoy uzunligini olsak, $x_i = x_i(S)$, Lagranj funksiyasi

ushbu ko'rinishini oladi: $L = \frac{m}{2} \dot{S}^2 - U(\vec{r}(S))$

4-masala. Lagranj tenglamasini (1.4) almashtirishda

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{df}{dt}(q, t)$$

o'zgartirish qolishini ko'rsating.

Yechilishi. Haqiqatdan ham $\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum \frac{\partial f}{\partial q_i} \dot{q}_i$ ekanligini hisobga olsak (1.3)

tenglamada $L \rightarrow L'$ almashtirish bajarish natijasida uning birinchi hadida

$\frac{d}{dt} \frac{\partial f}{\partial q_i}$ ga teng "ortiqcha" qo'shiluvchi, ikkinchi hadida esa xuddi shunday

$\frac{\partial}{\partial q_i} \left(\frac{df}{dt} \right)$ "ortiqcha" qo'shiluvchilar paydo bo'lishadi va ular bir-birlari bilan

qisqarib ketishadi.

Xuddi shuningdek

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \varphi(t)$$

almashtirish ham Lagranj tenglamasini o'zgartirmay qolishini ko'rsatish qiyin emas.

1.2.- Misollar.

To'la hosilani tashlab yuborish yo'li bilan ekvivalent Lagranj funksiyasi tuzing:

- | | |
|--|--|
| 1. $L' = \frac{1}{2}(\dot{q} + t)^2$. | 2. $L' = \frac{1}{2}(\dot{q} + q)^2$. |
| 3. $L' = x\dot{y} - y\dot{x}$. | 4. $L' = t\dot{x}\dot{x}$. |
| 5. $L' = \dot{\phi} \cos t$. | 6. $L' = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2)$. |
| 7. $L' = \frac{m}{2}(2a\dot{x}t + a^2 t^2)$. | 8. $L' = \frac{m l^2}{2} \dot{\phi}^2 + m l \gamma \dot{\phi} \sin(\phi - \gamma t)$. |
| 9. $L' = \dot{\phi} \sin \gamma t \cos \phi + \frac{\sin^2 \gamma t}{2}$. | 10. $L' = \dot{\phi} \sin \phi \sin \gamma t + \cos^2 \gamma t$. |
| 11. $L' = \alpha \dot{\phi} e^{\alpha \phi} \sin \gamma t$. | 12. $L' = \frac{m}{2}(\dot{\phi}^2 + \dot{\phi} e^{\omega t} \cos \phi) + \frac{m e^{2\omega t}}{2}$. |

Moddiy nuqtaning Lagranj funksiyasini usbu hollarda yozing:

- Sferik koordinat sistemasida.
- Qutb koordinat sistemasida.
- Silindrik koordinat sistemasida.

16. Dekart koordinatalari bilan quyidagi bog'lanishda bo'lgan koordinatalarda bitta moddiy nuqtaning Lagranj funksiyasini yozing :

- a) $x = R(\phi + \sin \phi)$, $y = R(1 - \cos \phi)$ b) $x = \phi + \theta$, $y = \frac{\phi}{\theta}$.
- v) $x = Uv$, $y = v - U$, $z = z$. g) $x = \frac{U}{v}$, $y = vU$, $z = z$.
- d) $x = \frac{\phi}{\theta}$, $y = \phi - \theta$. e) $x = \sqrt{Uv}$, $y = \frac{U}{v}$, $z = z$.
- j) $x = x$, $y = \sqrt{\frac{U}{v}}$, $z = Uv$. z) $x = \sqrt{\xi\eta} \cos \phi$, $y = \sqrt{\xi\eta} \sin \phi$, $z = \frac{\xi - \eta}{2}$.
- i) $x = \sigma \sqrt{(\xi^2 - 1)(1 - \eta^2)} \cos \phi$, $y = \sqrt{(\xi^2 - 1)(1 - \eta^2)} \sin \phi$, $z = \sigma \xi \eta$.

17. $L = -\sqrt{1 - \dot{x}^2}$ Lagranj funksiyasida $x = qch\lambda + \tau sh\lambda$, $t = qsh\lambda + \tau ch\lambda$, almashtirish bajarib q va τ lar orqali Lagranj funksiyasining ifodasi topilsin.

§1.3. Mexanik sistemaning erkinlik darajasini topish.

Tavsiflanayotgan fizik sistemani, jumladan mexanik sistemani holatini tavsiflash uchun zaruriy kattaliklar sonini bilish muhim masaladir, chunki shunga mos ravishda harakat tenglamalari olinadi va ular hal qilinadi. Talabalarda ushbu uquvni hosil qilish uchun bu bobda chiziq bo'ylab yoki tekislikda harakatlanayotgan mexanik sistemalar uchun bog'lanishlarni inobatga olgan holda erkinlik darajasini topish masalalari ko'riladi. Bunday sistemalarning fazodagi harakati ham bundan mustasno emas. Ma'lumki, erkinlik darajasi deb fizik sistemani tavsiflash uchun zarur bo'lgan kattaliklar soniga aytiladi.

1-masala. Biror bir chiziq bo'yicha harakat qilayotgan moddiy nuqtaning erkinlik darajasi topilsin.

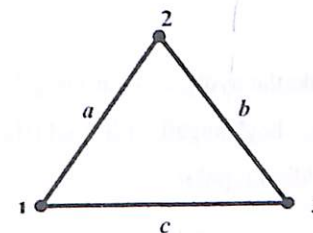
Yechilishi. Aniqlanishiga ko'ra, mexanik sistemaning erkinlik darajasi shu sistemani xarakterlaydigan, o'zaro bog'liq bo'lmagan kattaliklar sonidir. Chiziq bo'yicha harakatlanayotgan moddiy nuqtaning vaziyati bor yo'g'i bitta koordinata bilan aniqlanadi. Masalan, shu chiziqdagi muayyan nuqtagacha bo'lgan masofa bilan. Shuning uchun chiziq bo'ylab harakatlanayotgan moddiy nuqtaning erkinlik darajasi birga teng. Bu natijaga moddiy nuqtalar sistemasining dekart koordinat sistemasida erkinlik darajasi $3N$ ga teng degan ta'rifdan ham kelish mumkin. Bizning hol uchun $N=1$ ga teng, chunki bitta moddiy nuqta. Harakat esa bir o'lchamlidir, ya'ni chiziq bo'ylab. Demak, dekart koordinat sistemasini xarakterlovchi koeffitsiyent 3 o'rnida 1 bo'ladi.

2-masala. Fazoda harakatlanayotgan ikki moddiy nuqtadan iborat sistemaning erkinlik darajasi topilsin.

Yechilishi. Fazoda erkin harakatlanayotgan har bir moddiy nuqtaning erkinlik darajasi ta'rifga binoan $3N \Rightarrow 3$ ga teng. Ikki moddiy nuqtaning erkinlik darajasi $S = 3 \cdot 1 + 3 \cdot 1 = 3 \cdot 2 = 6$ ga teng.

3-masala. Fazoda bir-biridan o'zgarmas masofalarda joylashgan uchta moddiy nuqtadan iborat sistemaning erkinlik darajasi topilsin.

Yechilishi. 3 ta moddiy nuqtaning erkinlik darajasi $S = 3 \cdot 3 = 9$ ga teng.



1.1.-rasm.

Ammo bu sistemaning harakati 3 ta (masofalar) cheklanish (bog'lanish) bilan chegaralanganligi sababli $S = 9 - 3 = 6$ ga teng.

Demak, koordinatalararo har bir bog'lanish (tenglama) erkinlik darajasini bittaga kamaytiradi.

1.3.– Misollar.

1. Chiziq bo'ylab harakatlanayotgan quyidagi sistemalarning erkinlik darajasi topilsin:

- a) 2 ta bir-biri bilan bog'langan moddiy nuqtalarning,
- b) 9 ta moddiy nuqtalardan iborat zanjirning,
- v) N ta moddiy nuqtadan iborat zanjir.

2. Tekislikda harakatlanayotgan sistemalarning erkinlik darajasi topilsin:

- a) Bitta moddiy nuqta,
- b) 2 ta moddiy nuqta,
- v) 3 ta o'zaro bog'langan moddiy nuqtalar,
- g) 3 ta ketma-ket bog'langan moddiy nutalar,
- d) 2 ta o'zaro bog'langan moddiy nuqtalarni biri tekislikda, ikkinchisi esa chiziq ustida harakatlanadi.

e) Ikkita o'zaro bog'langan moddiy nuqtalarning har biri tekislikda harakatlanà oladi.

3. Fazoda harakatlanayotgan sistemalarning erkinlik darajasi topilsin:

- a) 2 ta bir-biri bilan bog'langan moddiy nuqtalar,
- b) 3 ta ketma-ket bog'langan moddiy nuqtalar,
- v) Ilgarilanma harakatlanayotgan bir-biri bilan o'zaro bog'langan moddiy nuqtalar,
- g) Aylanma harakatlanayotgan o'zaro bog'langan moddiy nuqtalar,
- d) 2 ta o'zaro bog'langan, biri tekislikda, ikkinchisi esa, fazoda harakatlanayotgan moddiy nuqtalar.

§1.4. Mexanik sistemaning Lagranj funksiyasini tuzish.

Lagranj metodi doirsida dinamik masalalarni hal qilishda yoki fizik sistemani tavsiflashda quyidagi amallar ketma-ketligini amalga oshirish nazarda tutiladi:

- fizik sistemasining erkinlik darajasini aniqlash;
- maqsadga muvofiq sanoq (koordinat) sistemasini va umumlashgan koordinatalarni tanlash;
- zarralar sistemasi koordinatalari va tezliklarini umumlashgan koordinatalar va tezliklar orqali ifodalash;
- zarralar sistemasining kinetik energiyasini umumlashgan koordinatalar va tezliklarning funksiyasi sifatida yozib olish;

Mazkur amallar natijasida ixtiyoriy fizik sistemaning Lagranj funksiyasini tuza bilish uquviga erishiladi. Yechimlari keltirilgan turli mexanik sistemalarning Lagranj funksiyasini tuzish masalalari esa, bu uquv va ko'nikmalarni shakllanishiga olib keladi.

1-masala. Bir tekis harakatlanayotgan sanoq sistemasidagi moddiy nuqtaning Lagranj funksiyasi yozilsin.

Yechilishi. Koordinatalari x, y, z bo'lgan sanoq sistemasiga nisbatan bir tekis harakatlanayotgan sanoq sistemasining koordinatalari x', y', z' bo'lsin. Harakat x yoqi bo'yicha v tezlik bilan sodir bo'layotgan bo'lsa, Galiley almashtirishlari quyidagicha yoziladi ($t = 0$ da koordinatalar boshi bir nuqtada bo'lgan deylik

$$x = x' + vt, \quad y = y', \quad z = z').$$

Bundan, $\dot{x} = \dot{x}' + v, \quad \dot{y} = \dot{y}', \quad \dot{z} = \dot{z}'$

Lagranj funksiyasi esa,

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U = \frac{m}{2}(\dot{x}'^2 + 2\dot{x}'v + v^2 + \dot{y}'^2 + \dot{z}'^2) - U;$$

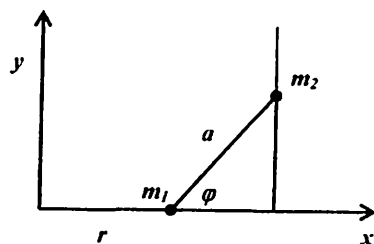
$(2\dot{x}'v + v^2)$ ni $(2v\dot{x}' + v^2)$ funksiyadan to'la hosila bo'lgani uchun Lagranj funksiyasi ifodasidan tushirib qoldirishimiz mumkin.

Demak, $L = \frac{m}{2}(\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2) - U,$

ya'ni Lagranj funksiyasi xuddi x, y, z koordinatasidagi kabi ko'rinishga ega. Bu Galileyning nisbiylik prinsipidan dalolatdir.

2-masala. Bir jinsli og'irlik maydonida joylashgan gorizontal chiziqda harakatlanuvchi m_1 va vertikal chiziqda harakatlanuvchi m_2 zarralarning Lagranj funksiyasi yozilsin. (1.2.-rasm). Ularning orasidagi masofa o'zgarmas va a ga teng.

Yechilishi. Ma'lumki, to'la Lagranj funksiyasi Lagranj funksiyalari yig'indisidan ibarat: $L = L_1 + L_2$.



1.2.-rasm.

Harakat tekislikda sodir bo'layotgani uchun har bir zarraning erkinlik darajasi 2 ga teng. Harakatni tasvirlash uchun masalan, x o'qini gorizontal bo'ylab tanlab olaylik (chizmaga qarang). Koordinata boshini esa m_1 nuqtadan r masofada joylaylik va a kesma bilan gorizontal chiziq orasidagi burchakni ϕ harfi bilan belgilaylik. U holda x_1, y_1, x_2, y_2 koordinatalarning o'zgarishi r va ϕ larning o'zgarishi bilan bog'lanadi. Chizmadan ko'rinib turibdiki

$$x_1 = r, y_1 = 0, \quad x_2 = r + a \cos \phi, y_2 = a \sin \phi.$$

Birinchi zarraning Lagranj funksiyasi

$$L_1 = \frac{m_1}{2} (\dot{x}_1^2 + \dot{y}_1^2) - U_1 \Rightarrow L_1 = \frac{m_1}{2} \dot{r}^2 - U_1.$$

Harakat davomida 1-zarraning potentsial energiyasi o'zgarmay qolgani sababli, uni keyinchalik tushirib qoldirsak ham bo'ladi (Lagranj funksiyasining xossalari asosan). 2-zarra uchun esa:

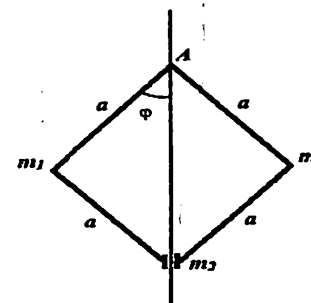
$$\frac{m_2}{2} (\dot{x}_2^2 + \dot{y}_2^2) - U_2 \Rightarrow \frac{m_2}{2} \left[(\dot{r} - a\dot{\phi} \sin \phi)^2 + (a\dot{\phi} \cos \phi)^2 \right] - U_2 =$$

$$= \frac{m_2}{2} (\dot{r}^2 - a^2 \dot{\phi}^2 - 2a\dot{r}\dot{\phi} \sin \phi) - U_2, \quad U_2 = mgy_2 = mgasin\phi.$$

Shunday qilib, to'la Lagranj funksiyasi:

$$L = \frac{m_1 + m_2}{2} \dot{r}^2 + \frac{m_2}{2} (a^2 \dot{\phi}^2 - 2a\dot{r}\dot{\phi} \sin \phi) - m_2 gasin \phi.$$

3-masala. 1.3.-rasmda ko'rsatilgan bir jinsli og'irlik maydonida joylashgan sistemaning Lagranj funksiyasi tuzilsin. Bu yerda m_2 zarra vertikal chiziq bo'yicha harakatlanadi. Sistema esa yalpi holda vertikal o'q atrofida doimiy burchak tezlik bilan aylanadi.



1.3.-rasm.

Yechilishi. Vertikal o'q bilan kesma orasidagi burchakni ϕ bilan belgilaylik. Vertikal o'q atrofida sistemaning burilish burchagini esa θ deylik. U holda $\dot{\theta}$ berilgan burchak tezlik ω ning o'zidir. Koordinata boshini A nuqtada tanlaylik va sistemaning Lagranj funksiyasini yozib olaylik:

$$L = L_1 + L_2, \quad L_1 = 2(T_1 - U_1), \quad L_2 = T_2 - U_2.$$

Har ikki xil zarraning kinetik energiyasi T_1 ni topish uchun har ikkala m_1 zarra uchun ko'chish elementi (uzunlik elementi) kvadratini yozib olaylik

$$dS^2 = (a d\varphi)^2 + (a \sin \varphi d\theta)^2.$$

Ikkinchi zarra esa, faqat vertikal o'q bo'yicha harakatlangani uchun. A nuqtadan m_2 gacha bo'lgan masofa $y_2 = 2a \cos \varphi$ ga teng va uning kinetik energiyasi

$$T_2 = \frac{m_2}{2} \dot{y}_2^2 = \frac{m_2}{2} (-2a\dot{\varphi} \sin \varphi)^2 = 2m_2 a^2 \dot{\varphi}^2 \sin^2 \varphi.$$

Har ikki tur zarraning potensial energiyasi esa, quyidagicha:

$$U_1 = -m_1 g a \cos \varphi, \quad U_2 = -2m_2 g a \cos \varphi \text{ ga teng.}$$

Shunday qilib

$$L = m_1 (a^2 \dot{\varphi}^2 + a^2 \sin^2 \varphi \cdot \omega^2) + 2m_2 a^2 \sin^2 \varphi \cdot \dot{\varphi}^2 + 2ga(m_1 + m_2) \cos \varphi.$$

4-masala. Massalari m_1 va m_2 bo'lgan qarama-qarshi zaryadli ikki zarra hosil qilgan dipol bir jinsli elektr maydon \vec{E} ta'sirida turibdi. Lagranj funksiyasi topilsin.

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$$\vec{r} = \vec{r}_1 - \vec{r}_2, \quad \vec{R} = \frac{\sum m_i \vec{r}_i}{\sum m_i} \Rightarrow \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad (1.9)$$

Har bir zarraning radius vektorini bu yangi o'zgaruvchilar orqali yozib olish mumkin:

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2} \vec{r}, \quad \vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2} \vec{r}. \quad (1.10)$$

Bu ikki zarradan iborat sistemaning Lagranj funksiyasi esa

$$L = T_1 + T_2 - U \Rightarrow \frac{m_1}{2} \dot{r}_1^2 + \frac{m_2}{2} \dot{r}_2^2 - U(r_1, r_2) \Rightarrow$$

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Quyidagi belgilashlarni kiritsak (m - keltirilgan massa):

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Lagranj funksiyasi ushbu ko'rinishni oladi:

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Demak, bu sistemaning Lagranj funksiyasi yalpi harakat \vec{R} va nisbiy harakat \vec{r} larga mos qismlarga ajraldi, potensial energiya esa faqatgina nisbiy joylashuvga bog'liq. Endi berilgan masalaga qaytsak, faqatgina potensial energiyani aniqlasak yetarli. Dipol momenti deganda quyidagi vektor tushuniladi:

$$\vec{d} = \sum_i \vec{r}_i e_i,$$

bu erda e_i - zarralar zaryadi. Bir jinsli elektr maydon \vec{E} ta'sirida turgan dipolning potentsial energiyasi $U = (\vec{E} \vec{d}) = Ed \cos \theta$ dan iborat.

Nisbiy harakat (aylanma harakat) ga mos Lagranj funksiyani sferik koordinat sistemasida yozadigan bo'lsak va dipolning uzunligi $l = |\vec{r}| = r$ o'zgarmas ekanligini hisobga olsak:

$$L_{ayl} = \frac{m}{2} \dot{r}^2 \Rightarrow \frac{ml^2}{2} (\dot{\varphi}^2 + \sin^2 \theta \cdot \dot{\theta}^2)$$

$$L = L_1 + L_2, \quad L_1 = 2(T_1 - U_1), \quad L_2 = T_2 - U_2.$$

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bu erda e_i - zarralar zaryadi. Bir jinsli elektr maydon \vec{E} ta'sirida turgan dipolning potensial energiyasi $U = (\vec{E}\vec{d}) = Ed \cos\theta$ dan iborat.

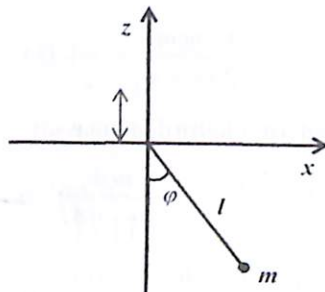
Nisbiy harakat (aylanma harakat) ga mos Lagranj funksiyani sferik koordinat sistemasida yozadigan bo'lsak va dipolning uzunligi $l = |\vec{r}| = r$ o'zgarmas ekanligini hisobga olsak:

$$L_{\text{ovl}} = \frac{m}{2} \dot{r}^2 \Rightarrow \frac{ml^2}{2} (\dot{\varphi}^2 + \sin^2\theta \cdot \dot{\theta}^2)$$

Demak,

$$L = \frac{M}{2} \dot{R}^2 + \frac{ml^2}{2} (\dot{\phi}^2 + \sin^2 \theta \cdot \dot{\theta}^2) - Ed \cos \theta.$$

5-masala. Mayatnikning osilish nuqtasi $S \approx S(t)$ qonuniyat bilan vertikal bo'yicha harakatlanadi. Lagranj funksiyasi tuzilib harakat tenglamasi topilsin (1.4.-rasm).



1.4.-rasm

Yechilishi. Koordinata boshini mayatnikning osilish nuqtasida tanlab, z o'qini vertikal bo'yicha yuqoriga yo'naltiraylik, x o'qi esa tebranma harakat tekisligida qolsin. Mayatnikning vertikaldan og'ish rolini umumlashgan koordinata ϕ o'ynaydi, mayatnik uzunligi l , massasi m bo'lsin.

U holda
$$\begin{cases} x = l \sin \phi \\ z = S - l \cos \phi \end{cases}$$

$$L = \frac{m}{2} (\dot{S}^2 + 2l\dot{S}\dot{\phi} \sin \phi + l^2 \dot{\phi}^2) - mg(S - l \cos \phi)$$

$$\dot{S}\dot{\phi} \sin \phi = -\frac{d}{dt} (\dot{S} \cos \phi) + \ddot{S} \cos \phi$$

ekanini hisobga olib, hamda Lagranj funksiyasi ifodasidan faqatgina vaqtga bog'liq funksiya va to'la hosilani tushirib qoldirsak:

$$L = \frac{m}{2} l^2 \dot{\phi}^2 + m(g + \ddot{S}(t)) l \cos \phi.$$

Bu yerdan harakat tenglamasini topish oson

$$\ddot{\phi} + l^{-1} (g + \ddot{S}(t)) \sin \phi = 0$$

2-usul. Agarda noinersial sanoq sistemasini osilish nuqtasida tanlab olsak:

$$x' = l \sin \phi, \quad z' = -l \cos \phi.$$

Umumlashgan potensial energiya

$$U_{yu} = -mg\vec{r}' + m\vec{W}(t)\vec{r}'$$

ko'rinishga ega bo'ladi. Bu yerda $\vec{W}(t)$ - noinersial sanoq sistemasining tezlanishidir:

$$\vec{W}(t) = (0, 0, \ddot{S}(t)).$$

Demak

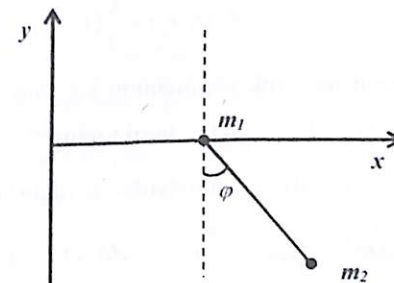
$$U_{yu} = -m(g + \ddot{S})l \cos \phi.$$

Ikkinchi tomondan $v'^2 = \dot{x}'^2 + \dot{z}'^2 = l^2 \dot{\phi}^2$ ekanligini eslasak, bu Lagranj funksiyasi ham yuqoridagi bilan bir xil ekanligini ko'ramiz.

$$L' = \frac{m}{2} v'^2 - U_{yu}.$$

Tabiiyki harakat tenglamasi ham yuqoridagidek bo'ladi.

6-masala. Bir jinsli og'irlik maydonida, gorizontol chiziqda turgan va vaziyati o'zgaradigan m_1 massali zarraga m_2 massali matematik mayatnik biriktirilgan. Lagranj funksiyasini tuzing, harakat tenglamasini, integrallarini yozing (1.5.-rasm).



1.5.-rasm

Yechilishi. Massasi m_1 ga teng zarraning gorizontaal chiziqdagi vaziyatini umumlashgan koordinata $S(t)$ aniqlasin, mayatnikning vertikal dan og'ishini umumlashgan koordinata ϕ aniqlasin. Koordinata o'qi y vertikal bo'yicha yuqoriga yo'nalgan bo'lsin, x esa gorizontaal chiziq bo'yicha. U holda

$$\begin{cases} x_1 = S & x_2 = l \sin \phi + S \\ y_1 = 0 & y_2 = -l \cos \phi \end{cases}$$

$$L = \frac{1}{2} m_1 \dot{S}^2 + \frac{1}{2} m_2 (\dot{S}^2 + 2l\dot{S}\dot{\phi} \cos \phi + l^2 \dot{\phi}^2) + m_2 g l \cos \phi$$

Harakat tenglamasi esa ushbu ko'rinishda

$$\frac{d}{dt} \{ (m_1 + m_2) \dot{S} + m_2 l \dot{\phi} \cos \phi \} = 0,$$

ya'ni umumlashgan impuls $P = \frac{\partial L}{\partial \dot{S}} = (m_1 + m_2) \dot{S} + m_2 l \dot{\phi} \cos \phi$ - saqlanuvchi kattalikdir, chunki S siklikdir. Ikkinchi harakat tenglamasi esa shunga o'xshash topiladi:

$$\frac{d}{dt} [m_2 (l \dot{S} \cos \phi + l^2 \dot{\phi})] = -m_2 g l \sin \phi.$$

Ya'ni
$$\ddot{\phi} + l^{-1} \{ \dot{S} \cos \phi + (g - \dot{S} \dot{\phi}) \sin \phi \} = 0.$$

7-masala. O'zaro ta'sir energiyasi

$$U(\vec{r}_1, \vec{r}_2) = \frac{k}{2} (\vec{r}_1 \cdot \vec{r}_2)^2$$

bo'lgan ikki zarradan iborat mexanik sistemaning Lagranj funksiyasini tuzing, harakat tenglamalari hamda $\vec{r}_1(t)$, $\vec{r}_2(t)$ larni toping.

Yechilishi: (1.9) va (1.10) formulalardan foydalansak:

$$L = \left(\vec{R}, \vec{r}, \dot{\vec{R}}, \dot{\vec{r}} \right) = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} m \dot{\vec{r}}^2 - \frac{k}{2} \vec{r}^2, \quad M \ddot{\vec{R}} = 0, \quad m \ddot{\vec{r}} = -k \vec{r}$$

harakat tenglamalarining yechimi esa:

$$\vec{R}(t) = \vec{R}(0) + \dot{\vec{R}}(0) \cdot t; \quad \vec{r}(t) = \vec{A} \cos \omega t + \vec{B} \sin \omega t; \quad \omega = \sqrt{\frac{k}{m}}$$

Bularni (1.10) ifodaga qo'yib, $\vec{r}_1(t)$ va $\vec{r}_2(t)$ larni topamiz.

Boshlang'ich shartlar (holat) ni shunday tanlaylikki, zarralar radiuslari $\frac{m_2}{M} a$

va $\frac{m_1}{M} a$ ga teng aylana bo'ylab harakatlanayotgan bo'lsin:

$$\vec{r}_1(0) = \frac{m_2}{M} \vec{a}, \quad \vec{r}_2(0) = -\frac{m_1}{M} \vec{a}, \quad \dot{\vec{r}}_1(0) = \frac{m_2}{M} \vec{v}_0, \quad \dot{\vec{r}}_2(0) = -\frac{m_1}{M} \vec{v}_0,$$

binobarin $\vec{v}_0 \cdot \vec{a} = 0$, $v_0 = \omega a$.

Bu holda $\vec{R}(0) = 0$, $\dot{\vec{R}}(0) = 0$, $\vec{A} = \vec{a}$, $\vec{B} = \frac{\vec{v}_0}{\omega}$ va

$$\vec{r}_1(t) = \frac{m_2}{M} \left(\vec{a} \cos \omega t + \frac{\vec{v}_0}{\omega} \sin \omega t \right), \quad \vec{r}_2(t) = -\frac{m_1}{M} \left(\vec{a} \cos \omega t + \frac{\vec{v}_0}{\omega} \sin \omega t \right)$$

Demak, zarralar $\frac{m_2}{M} a$ va $\frac{m_1}{M} a$ radiusli aylanalar bo'ylab $\frac{m_2}{M} v_0$

va $\frac{m_1}{M} v_0$ tezliklar bilan inersiya markazi atrofida bir-biridan a masofada

aylanar ekan.

8-masala. $N+1$ ta zarradan iborat sistemaning Lagranj funksiyasini koordinata boshi m_0 zarra ustida joylashgan sanoq sistemasida tuzing.

Yechilishi. Zarralarning radius-vektorlarini \vec{r}_a bilan belgilab ($a = 0, 1, \dots, N$) yangi vektor kiritaylik $\vec{r}_{ab} = \vec{r}_b - \vec{r}_a$. Sistemaning to'la massasi

$$M = m_0 + m_1 + \dots + m_N = \sum m_a$$

bo'lsin va $\vec{r}_0, \vec{r}_1, \dots, \vec{r}_N$ o'zgaruvchilardan $\vec{R}, \vec{r}_{01}, \vec{r}_{02}, \dots, \vec{r}_{0N}$ larga o'taylik, \vec{R} - inersiya markazining radius-vektori. Inersiya markazi bilan m_0 zarrani tutashdiradigan \vec{r}'_0 vektor kiritaylik.

U holda
$$\vec{r}_a = \vec{R} + \vec{r}'_a + \vec{r}_{0a}$$

chunki
$$\vec{R} = \frac{\sum m_a \vec{r}_a}{\sum m_a}, \quad \vec{r}'_0 = \vec{R} - \vec{r}_0, \quad \vec{r}_{0a} = \vec{r}_a - \vec{r}_0$$

Ushbuning $\sum m_a (\vec{r}'_0 + \vec{r}_{0a}) = 0$ o'rinligidan:

$$\vec{r}'_0 = -\frac{\sum m_a \vec{r}_{0a}}{\sum m_a} = -\frac{1}{M} \sum m_a \vec{r}_{0a}$$

Jumladan, bu sistemaning kinetik energiyasi quyidagicha:

$$\begin{aligned} T &= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \sum m_a (\dot{\vec{r}}'_0 + \dot{\vec{r}}_{0a})^2 = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \sum m_a (\dot{\vec{r}}_0^2 + 2\dot{\vec{r}}_{0a} \dot{\vec{r}}'_0 + \dot{\vec{r}}_{0a}^2) = \\ &= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} (\sum m_a \dot{\vec{r}}_{0a}^2 - M \dot{\vec{r}}_0^2) = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \sum m_a \dot{\vec{r}}_{0a}^2 - \frac{1}{2M} (\sum m_a \dot{\vec{r}}_{0a})^2 \end{aligned}$$

Bu zarralarning o'zaro ta'sir potensial energiyasi:

$$U = -\gamma \sum_{a>b} \frac{m_a m_b}{r_{ab}} = -\gamma \sum_{a>b} \frac{m_a m_b}{|\vec{r}_{0b} - \vec{r}_{0a}|}$$

Bu yerda γ - gravitasion doimiy. Yuqoridagi Lagranj funksiyasidan $N+1$ ta harakat tenglamasi olinadi:

$$\ddot{\vec{R}} = 0; \quad m_c \ddot{\vec{r}}_{0c} - \frac{m_c}{M} \sum_a m_a \ddot{\vec{r}}_{0a} = -\frac{\partial U}{\partial \vec{r}_{0c}}, \quad (c = 1, 2, \dots, N)$$

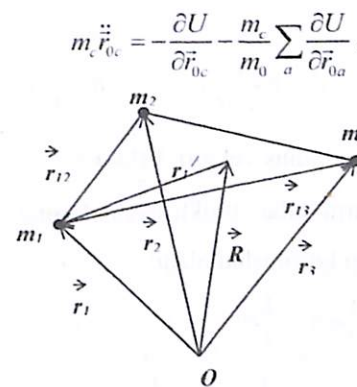
Ikkinchi tenglama tabiiyki, N ta va ularni bir-birlariga qo'shib chiqsak, hamda

$$\left(\sum_{c=1}^N m_c + m_0 \right) - m_0 = M - m_0$$

ekanligini hisobga olsak:

$$\frac{m_0}{M} \sum_a m_a \ddot{\vec{r}}_{0a} = -\sum \frac{\partial U}{\partial \vec{r}_{0a}}$$

Natijada, bularni qaytadan ushbu ko'rinishda yozib olishimiz mumkin:



1.6.-rasm

Agar $m_0 \gg m_i$ bo'lsa, oxirgi ifodadagi ikkinchi had birinchi hadga nisbatan g'alayon

rolini o'ynaydi. Yuqoridagi formulalarni uch jism masalasiga tadbiiq etaylik (1.6.-rasm). Koordinata boshini m_1 zarrada tanlab olaylik.

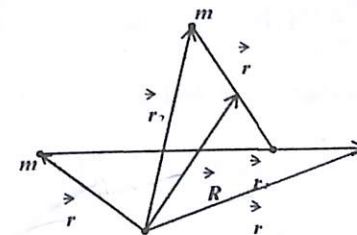
$$L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \frac{m_1 m_{13}}{M} \dot{\vec{r}}_{12}^2 + \frac{1}{2} \frac{m_1 m_{12}}{M} \dot{\vec{r}}_{13}^2 - \frac{m_2 m_3}{M} \dot{\vec{r}}_{12} \dot{\vec{r}}_{13} - U \quad (1.12)$$

$$U = -\gamma \frac{m_1 m_2}{r_{12}} - \gamma \frac{m_1 m_3}{r_{13}} - \gamma \frac{m_2 m_3}{|\vec{r}_{12} - \vec{r}_{13}|}$$

bu erda $m_{ab} = m_a + m_b$ va $M = m_1 + m_2 + m_3$ harakat tenglamalari:

$$\ddot{\vec{R}} = 0, \quad m_2 \ddot{\vec{r}}_{12} = -\frac{m_{12}}{m_1} \frac{\partial U}{\partial \vec{r}_{12}} - \frac{m_2}{m_1} \frac{\partial U}{\partial \vec{r}_{13}}, \quad m_3 \ddot{\vec{r}}_{13} = -\frac{m_{13}}{m_1} \frac{\partial U}{\partial \vec{r}_{13}} - \frac{m_3}{m_1} \frac{\partial U}{\partial \vec{r}_{12}}$$

Ko'pincha uch jism masalasida quyidagi Yakobi koordinatalari $\vec{R}, \vec{r}, \vec{r}_3$ keng qo'llaniladi (1.7.-rasm):



1.7.-rasm

$$\vec{r}_1 = \vec{R} - \frac{m_2}{M} \vec{r} - \frac{m_3}{m_{13}} \vec{r}_{13}, \quad \vec{r}_2 = \vec{R} + \frac{m_{13}}{M} \vec{r}, \quad \vec{r}_3 = \vec{R} - \frac{m_2}{M} \vec{r} - \frac{m_1}{m_{13}} \vec{r}_{13} \quad (1.13)$$

bu erda \vec{R} - inersiya markazi radius-vektori, vektor \vec{r} esa, m_1 va m_3 zarralarning inersiya markazini m_2 zarra bilan biriktiruvchi kesma. Bu o'zgaruvchilarda Lagranj funksiyasi quyidagi ko'rinishni oladi:

$$L = \frac{1}{2} m \dot{\vec{R}}^2 + \frac{1}{2} m \dot{\vec{r}}^2 + \frac{1}{2} m_{13} \dot{\vec{r}}_{13}^2 - U(\vec{r}, \vec{r}_{13}),$$

$$\frac{1}{m} = \frac{1}{m_2} + \frac{1}{m_{13}}, \quad \frac{1}{m_{13}} = \frac{1}{m_1} + \frac{1}{m_3} \quad (1.14)$$

$$U = -\gamma \frac{m_1 m_2}{\left| \vec{r} - \frac{m_3}{m_{13}} \vec{r}_{13} \right|} - \gamma \frac{m_1 m_3}{r_{13}} - \gamma \frac{m_2 m_3}{\left| \vec{r} - \frac{m_1}{m_{13}} \vec{r}_{13} \right|}$$

9-masala. Bir jinsli shar bilan zarra o'rtasidagi o'zaro ta'sirning potensial energiyasini toping. Bir jinsli og'irlik maydoniga o'tish chegarasini ko'ring.

Yechilishi. Massasi M radiusi R ga teng sharning markazidan r masofada turgan m massali zarra bilan o'zaro ta'sir potensial energiyasi (1.12) ga ko'ra:

$$U(\vec{r}) = -\gamma \int \frac{m dM}{|\vec{r} - \xi|}$$

Bu erda ξ , \vec{r} sharning elementar massasi $dM = \rho dV$ va m zarraning radius vektorlari. Sferik koordinatalar boshini shar markazida tanlab olsak:

$$U(\vec{r}) = -\gamma \rho \int_0^R \xi^2 d\xi \int_0^\pi \sin \theta d\theta \int_0^{2\pi} \frac{m d\varphi}{\sqrt{r^2 + \xi^2 - 2r\xi \cos \theta}} =$$

$$= -\frac{2\pi \gamma \rho m}{r} \int_0^R \xi (r + \xi - |r - \xi|) d\xi, \quad \rho = \frac{m}{V}, \quad V = \frac{4\pi}{3} R^3.$$

Bu yerdan shar $r > R$ bo'lsa, $r > \xi$ va $|r - \xi|$,

$$U(\vec{r}) = -\frac{2\pi \gamma \rho m}{r} \int_0^R 2\xi^2 d\xi = -\gamma \frac{mM}{r}$$

Aks holda $r < R$ va

$$U(\vec{r}) = -\frac{2\pi \gamma \rho m}{r} \left\{ \int_0^r 2\xi^2 d\xi + \int_r^R 2\xi r d\xi \right\} = -\gamma \frac{mM}{2R} \left(3 - \frac{r^2}{R^2} \right)$$

Shunday qilib, $U(x, y, z) = V(r)$, $r = \sqrt{x^2 + y^2 + z^2}$ uchun

$$V(r) = \begin{cases} -\gamma \frac{mM}{r}, & r \geq R \\ -\gamma \frac{mM}{2R} \left(3 - \frac{r^2}{R^2} \right), & 0 \leq r \leq R \end{cases} \quad (1.15)$$

Demak sharning zichligi sferik simmetrik bo'lganda shar tashqarisidagi maydon nuqtaviy zarraning maydonidan farqlanmaydi. Endi shar sirtiga yaqin masofadagi zarra uchun maydonning ko'rinishini qidiraylik (masalan, Yer sirtidagi, $\vec{r} = \vec{R} + \vec{r}'$, $\vec{r}' \ll \vec{R}$):

$$U(\vec{r}') \cong V(\vec{R}) + \vec{r}' \frac{\partial V}{\partial \vec{R}} = -\gamma \frac{mM}{R} - m \vec{g} \vec{r}', \quad \vec{g} = -\frac{\gamma M}{R^2} \vec{R}.$$

10-masala. Quyidagi erkin fizik sistemalar uchun Lagranj funksiyalari tuzilsin va Lagranj tenglamalari olinsin:

- erkin zarra;
- chiziqli garmonik ossillyator;
- simmetrik markaziy maydonda masofaning kvadratiga teskari proporsional bo'lgan tortishish kuchi ostida harakatlanayotgan zarra.

Yechilishi. a) Erkin zarraga hech qanday kuch ta'sir etmayotganligi uchun uning Lagranj funksiyasi faqatgina kinetic energiyasidan iborat bo'ladi. Umumlashgan koordinatalar sifatida oddiy dekart koordinatalarini olib x, y, z (hech qanday bog'lanishlar bo'lmaganligi tufayli), Lagranj funksiyasini ushbu ko'rinishda topamiz

$$L = \frac{m}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

Bu ifodadan Lagranj tenglamalarini tuzamiz:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{z}} \right) - \frac{\partial L}{\partial z} = 0,$$

Natijada ushbu tenglamalarni olamiz

$$m\ddot{x} = 0, \quad m\ddot{y} = 0, \quad m\ddot{z} = 0.$$

Ko'rinib turibdiki, Lagranj tenglamalari erkin zarraning dekart koordinatalarida yozilgan odatdagi harakat tenglamalaridan iborat ekan. Bulardan berilgan boshlang'ich shartlar asosidagi ($t = 0, \vec{r} = \vec{r}_0$ va $\vec{v} = \vec{v}_0$) erkin zarraning harakat qonunini topish mumkin:

$$\vec{r}(t) = \vec{v}_0 t + \vec{r}_0.$$

b) Chiziqli garmonik ossillyator deb, $F = -kx$ kuch ostida chiziq bo'ylab harakatlanayotgan m massali zarrachaga aytiladi, bu yerda k - doimiy musbat koeffitsient. Mazkur potrsial kuchga mos potensial energiya esa quyidagicha ifodalanadi:

$$\vec{F} = -\text{grad}U(x) = -\nabla U(x) = -(dU(x)/dx)\vec{e}_x = -kx\vec{e}_x.$$

Bu yerdan $\frac{dU(x)}{dx} = kx$, shuning uchun

$$U(x) = \frac{kx^2}{2},$$

agarda potensialni $x = 0$ da nolga teng desak. Ossillyatorning kinetik energiyasi

$$T = \frac{m\dot{x}^2}{2},$$

bo'lganligi uchun uning Lagranj funksiyasi quyidagicha bo'ladi:

$$L = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}.$$

Bundan Lagranj tenglamasini ushbu ko'rinishda topamiz:

$$m\ddot{x} + kx = 0.$$

Chastotani $\omega^2 = k/m$ ko'rinishda aniqlab, ossillyator uchun ma'lum bo'lgan harakat tenglamasini topamiz:

$$\ddot{x} + \omega^2 x = 0.$$

v) Har qanday markaziy - simmetrik maydondagi zarraning harakati impuls momenti salanganligi uchun, unga perpendikulyar bo'lgan tekislikda sodir bo'ladi.

Bu tekislikni xOy deb, unda umumlashgan kordinatalar sifatida qutb koordinatalarni ρ va φ kiritsak qulay bo'ladi. Tortishish kuchi koordinata boshida joylashgan markazgacha bo'lgan masofaning kvadratiga teskari proporsional bo'lganligi uchun, uni quyidagi ko'rinishda yozish mumkin:

$$\vec{F}(\rho) = -\frac{\alpha}{\rho^2}\vec{e}_\rho,$$

bu yerda α o'zaro ta'sir doimiysi. Kuch markazidan cheksiz uzoqdagi masofalarda potensialni nolga teng deb, potensial energiya uchun ushbu ifodani yozishimiz mumkin:

$$U(\rho) = -\frac{\alpha}{\rho}.$$

Qutb koordinatasida zarra tezligining kvadrati $v^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2$ bo'lganligi uchun Lagranj funksiyasi quyidagicha yoziladi:

$$L = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) + \frac{\alpha}{\rho}.$$

Lagranj tenglamalarini tuzib

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = 0, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0,$$

ikkinchi tartibli harakat tenglamalarini topamiz

$$m\ddot{\rho} - m\rho\dot{\varphi}^2 + \frac{\alpha}{\rho^2} = 0, \quad m\rho^2\ddot{\varphi} + 2m\rho\dot{\rho}\dot{\varphi} = 0,$$

va ular zarraning qutb koordina sistemasidagi harakat tenglamasining o'zidir, bu yerda w_ρ va w_φ lar radial va burchak tezlanishlardir:

$$mw_\rho = -\frac{\alpha}{\rho^2}, \quad mw_\varphi = 0,$$

Shuni eslatish lozimki, keyingi boblarda (II-bob) harakat tenglamalarini fizik sistemaning o'ziga tegishli bo'lgan harakat integrallari orqali yechish usuli qo'llaniladi. Jumladan, bu masalaning yuqoridagi Lagranj funksiyasi ifodasida φ oshkora qatnashmayotganligi tufayli (bunday koordinatalar siklik koordinatalar deyiladi), ularga mos *umumlashgan impulstar* saqlanadi, va u bizning hol uchun $p_\varphi = \partial L / \partial \dot{\varphi} = m\rho^2 \dot{\varphi}$, ko'rinishga ega bo'lib, qiymati impuls momentining M_z komponentasiga tengdir. Xuddi shuningdek, bu ifodalarda vaqt t ham "siklik" dir, va unga mos energiya (§1.5. bo'limga qarang) ham saqlanadi:

$$E = T + U = \frac{m}{2} (\dot{\rho}^2 + \rho^2 \dot{\varphi}^2) - \frac{\alpha}{\rho}.$$

Shunday qilib, markaziy maydonda siklik o'zgaruvchilarni hisobga olish **birinchi tartibli** tenglamalarga olib keladi:

$$\frac{m\dot{\rho}^2}{2} - \frac{\alpha}{\rho} + \frac{M_z^2}{2m\rho^2} = 0 \quad \text{va} \quad m\rho^2 \dot{\varphi} = M_z.$$

1.4.- Misollar.

1. Quyidagi hollar uchun zarralarning Lagranj funksiyasini yozing:

- a) tekis tezlanuvchan harakat qilayotgan sanoq sistemasi uchun,
- b) bir tekis aylanayotgan sanoq sistemasi uchun.

2. Zarralar sistemasining Lagranj funksiyasini yozing:

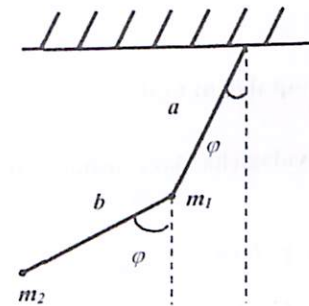
a) shar sirtida harakatlanayotgan bir jinsli og'irlik maydonidagi m massali zarra uchun (sferik mayatnik):

b) konus sirt ustida harakatlanayotgan m massali zarra uchun. Konus uchining burchagi -2α ga teng deyilsin, potensial energiya esa, konus uchidan bo'lgan masofaga teskari proporsional deyilsin;

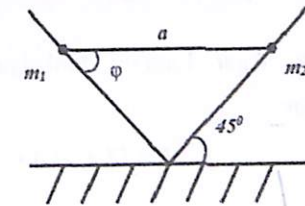
v) bir jinsli og'irlik maydonida joylashgan qo'sh yassi mayatnik uchun. (1.8.-rasm);

g) 1.9.-rasmda ko'rsatilgan bir jinsli og'irlik maydonida joylashgan m_1 va m_2 zarralar sistemasi uchun. Zarralar ko'rsatilgan chiziqlar bo'ylab harakatlanadi;

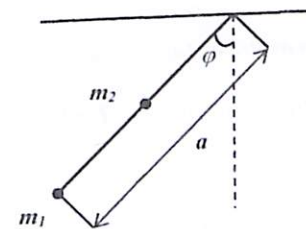
d) bir jinsli og'irlik maydonida joylashgan yassi mayatnik uchun (1.10.-rasm). Sterjen bo'ylab esa, m_2 zarra doimiy v tezlik bilan harakatlanadi.



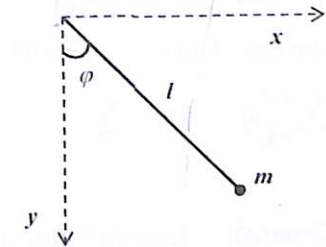
1.8.-rasm



1.9.-rasm



1.10.-rasm



1.11.-rasm

e) Osilish nuqtasi $y = a \sin \omega t$ qonun bo'yicha vertikal tebranadigan yassi mayatnik uchun (1.11.-rasm).

j) Shu sistema $x = a \sin \omega t$ qonun bo'yicha gorizontaal tebrangan hol uchun.

3. Massasi M bo'lgan yadro va n ta massasi m ga teng elektronlardan tashkil topgan atomning Lagranj funksiyasi yozilsin. Inersiya markazi harakati inobatga olinmasin va masalani n ta zarra harakati holiga keltirilsin.

4. Massalari m_1 va m_2 bo'lgan zarralar o'zaro va $r_3 = r_3(t)$ qonuniyat bilan harakatlanayotgan m_3 zarra bilan ta'sirlashayotgan bo'lsa Lagranj funksiyasini toping.

§1.5. Energiya va to'xtash nuqtalarini topish.

1-masala. Agar Lagranj funksiyasi quyidagicha ko'rinishda bo'lsa, energiya topilsin:

$$L = T(q_n, \dot{q}_n) + A(q_n, \dot{q}_n) - U(q_n),$$

bu erda A, T lar mos ravishda \dot{q}_n ning chiziqli va kvadratik funksiyalaridir.

Yechilishi. (1.6) formulaga asosan

$$E = \sum_n \frac{\partial L}{\partial \dot{q}_n} \dot{q}_n - L = \sum_n \frac{\partial T}{\partial \dot{q}_n} \dot{q}_n + \sum_n \frac{\partial A}{\partial \dot{q}_n} \dot{q}_n - T(q_n, \dot{q}_n) - A(q_n, \dot{q}_n) + U(q_n)$$

Lekin, bir jinsli funksiyalar uchun Eyler teoremasiga ko'ra

$$\sum_k \frac{\partial T}{\partial \dot{q}_k} \dot{q}_k = 2T, \quad \sum_n \frac{\partial A}{\partial \dot{q}_n} \dot{q}_n = A. \quad \text{Shuning uchun } E = T + U.$$

2-masala. Lagranj funksiyasi quyidagi boshlang'ich shartlarni qanoatlantiruvchi moddiy nuqta harakatining to'xtash nuqtasi tipilsin.

$$L = \frac{\dot{x}^2}{2} - \sin x, \quad x_0 = 0 \text{ da } \dot{x}_0 = 1$$

Yechilishi: Kinetik energiya nolga teng bo'ladigan nuqtalar ($E = U$ ni qanoatlantirgan, yoki $\dot{x}_0 = 0$ bo'lgan) to'xtash nuqtalari deyiladi.

Energiyaning saqlanish qonunidan foydalanamiz:

$$E = \frac{\partial L}{\partial \dot{x}} \cdot \dot{x} - L = \frac{\dot{x}^2}{2} + \sin x.$$

Boshlang'ich shartlarga asosan: $E_0 = \frac{\dot{x}_0^2}{2} + \sin x_0 = \frac{1}{2}$ ga teng.

Zarra to'xtaganda $\dot{x} = 0$ bo'ladi, ya'ni $E = U = E_0$. Demak, $\sin x = \frac{1}{2}$.

Bu erdan $x = \frac{\pi}{6}$ to'xtash nuqtasi ekanligini topamiz.

3-masala. Lagranj funksiyasi quyidagicha bo'lgan sistemaning energiyasi topilsin:

$$L = \frac{ml^2}{2} \left(\dot{\phi}^2 + \frac{1}{2} \dot{\theta}^2 + \dot{\phi} \dot{\theta} \right) + 3l \cos \phi.$$

Yechilishi:

$$\begin{aligned} E &= \sum \frac{\partial L}{\partial \dot{q}_i} \cdot \dot{q}_i - L \Rightarrow \frac{ml^2}{2} \left\{ (2\dot{\phi} + \dot{\theta})\dot{\phi} + (\dot{\theta} + \dot{\phi})\dot{\theta} \right\} - \\ &= \frac{ml^2}{2} \left(\dot{\phi}^2 + \frac{1}{2} \dot{\theta}^2 + \dot{\phi} \dot{\theta} \right) - 3l \cos \phi = \\ &= \frac{ml^2}{2} \left(\dot{\phi}^2 + \frac{1}{2} \dot{\theta}^2 + \dot{\phi} \dot{\theta} \right) - 3l \cos \phi. \end{aligned}$$

1.5.-Misollar.

1. Quyidagi Lagranj funksiyalariga mos energiya topilsin.

a) $L = -\sqrt{1-v^2} + Av - \phi$, A, ϕ lar faqatgina q_k larning funksiyasi.

b) $L = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}$, v) $L = \dot{x}^2 - (e^x - 1)^2$,

g) $L = \frac{m_1 + m_2}{2} \dot{r}^2 + \frac{m_2}{2} (\dot{\phi}^2 - 2\dot{r}\dot{\phi} \sin \phi) - m_2 \sin \phi$

2. 1.1.-misollarda berilgan Lagranj funksiyasi bilan xarakterlanuvchi sistemalarning energiyasini toping.

3. Quyidagi Lagranj funksiyasi va boshlang'ich shartlarga tegishli sistemalarning to'xtash nuqtalarini toping:

a) $L = -\sqrt{1-\dot{x}^2} + x, \quad x_0 = 1, \quad \dot{x}_0 = \frac{1}{2}.$

b) $L = \frac{\dot{x}^2 - x^2 + x}{x}, \quad x_0 = 1, \quad \dot{x}_0 = 1.$

v) $L = \frac{m\dot{x}^2}{2} + \cos x, \quad x_0 = 0, \quad \dot{x}_0 = \frac{1}{\sqrt{m}}.$

g) $L = \frac{\dot{x}^2 + x\dot{x} - 1}{x^2}, \quad x_0 = 1, \quad \dot{x}_0 = \sqrt{3}.$

d) $L = \dot{x}^2 - \frac{1}{x^2}, \quad x_0 = 1, \quad \dot{x}_0 = \sqrt{8}.$

e) $L = \dot{x}^2 - tg^2x, \quad x_0 = 0, \quad \dot{x}_0 = 2.$

j) $L = \dot{x}^2 - e^x, \quad x_0 = 0, \quad \dot{x}_0 = 2.$

z) $L = \frac{\dot{x}^2 + x^2 + 1}{x}, \quad x_0 = 3, \quad \dot{x}_0 = 1.$

i) $L = \dot{x}^2 - \ln x, \quad x_0 = 1, \quad \dot{x}_0 = \ln e.$

k) $L = \frac{m\dot{x}^2}{2} - \frac{1}{x}, \quad x_0 = 2, \quad \dot{x}_0 = \frac{2}{\sqrt{m}}.$

l) $L = \frac{1}{2}m\dot{x}^2 + U_0 e^{x/a}, \quad x_0 = 0,$

m) $L = \frac{1}{2}m\dot{x}^2 + U_0 ch^{-2}kx, \quad E = -E_0 < 0.$

$$\dot{x}_0 = v_0 \geq \sqrt{\frac{2U_0}{m}}.$$

n) $L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 - \frac{1}{3!}ax^3, \quad x(0) = \frac{3k}{a}, \quad \dot{x}(0) = 0.$

1) $x = 0, \dot{x} = \frac{3k}{a}$ 2) $x = 0, \dot{x} = \sqrt{\frac{2k}{\lambda}}$ 3) $x = l\pi.$

o) $L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 - \frac{1}{4}\lambda x^4,$

p) $L = \frac{1}{2}m\dot{x}^2 + U_0 \cos\left(\frac{x}{l}\right), \quad x(0) = 0,$

$$x(0) = \sqrt{\frac{2k}{\lambda}}, \quad \dot{x}(0) = 0.$$

$$\dot{x}(0) = \sqrt{\frac{4U_0}{m}}.$$

§1.6. Elektromagnit maydonlardagi harakat.

Zarraning elektromagnit maydondagi harakatining Lagranj funksiyasini quyidagicha yozib olish mumkin:

$$L(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2}m\dot{\vec{r}}^2 - U_{ym}(\vec{r}, \dot{\vec{r}}, t),$$

$$U_{ym}(\vec{r}, \dot{\vec{r}}, t) = e\phi(\vec{r}, t) - \frac{e}{c}\dot{\vec{r}}\vec{A}(\vec{r}, t) \quad (1.15)$$

bu yerda e -zarraning zaryadi, c -yorug'lik tezligi, ϕ va \vec{A} mos ravishda elektromagnit maydonning skalyar va vektor potentsiallari, U_{ym} -umumlashgan potentsial energiya (ahamiyat bering, potentsial energiya tezlikka bog'liq).

Odatda elektromagnit potentsiallari ϕ va \vec{A} lar elektr va magnit maydon kuchlanganliklari \vec{E} va \vec{H} lar bilan o'zaro bog'liqlarlar:

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \text{grad}\phi, \quad \vec{H} = \text{rot}\vec{A} \quad (1.16)$$

Umumlashgan impuls (1.7) dan odatdagidek topiladi:

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + \frac{e}{c}A_i(\vec{r}, t) \quad (1.17)$$

Bu yerdan Lagranj tenglamasi (harakat tenglamasi) (1.8) ni tuzish mumkin:

$$\frac{d}{dt}\left(m\dot{x}_i + \frac{e}{c}A_i\right) = -e\frac{\partial \phi}{\partial x_i} + \frac{e}{c}\dot{\vec{v}}\frac{\partial \vec{A}}{\partial x_i} \quad (1.18)$$

Lagranj tenglamasi (1.18) ni quyidagi holda yozib olish mumkin:

$$m\ddot{\vec{r}} = e\vec{E} + \frac{e}{c}[\dot{\vec{v}}\vec{H}] \quad (1.19)$$

Haqiqatdan ham (1.18) larni hisobga olib (1.16) dan:

$$m\ddot{x}_i = -e\frac{\partial \phi}{\partial x_i} + \frac{e}{c}\dot{x}_j\frac{\partial A_j}{\partial x_i} - \frac{e}{c}\left(\frac{\partial A_i}{\partial t} + \frac{\partial A_j}{\partial x_j}\dot{x}_j\right) = -e\left(\frac{\partial \phi}{\partial x_i} + \frac{1}{c}\frac{\partial A_i}{\partial t}\right) +$$

$$+\frac{e}{c}\left(\frac{\partial A_j}{\partial x_i} + \frac{\partial A_i}{\partial x_j}\right)\dot{x}_j = eE_i + \frac{e}{c}[\dot{\vec{r}}\vec{H}], \quad (1.20)$$

chunki $\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = \varepsilon_{ijk}H_k, \quad \varepsilon_{ijk}\dot{x}_jH_k = [\dot{\vec{r}}\vec{H}],$

Xuddi shuningdek, "umumlashgan" impuls momenti (1.17) dan aniqlanadi:

$$\vec{M} = \left[\vec{r} \left(m\dot{\vec{r}} + \frac{e}{c}\vec{A} \right) \right] \quad (1.21)$$

Zarraning bir jinsli o'zgarimas elektr va magnit

$$\phi = -\vec{E}\vec{r}, \quad \vec{A} = \frac{1}{2}[\vec{H}\vec{r}] \quad (1.22)$$

maydonlardagi harakatida yuqoridagi ifodalar "tanish" ko'rinishga ega bo'ladi:

$$L = \frac{1}{2}m\dot{r}^2 + e\vec{E}\vec{r} + \frac{e}{2c}\vec{v}[\vec{H}\vec{r}], \quad E = \frac{1}{2}m\dot{r}^2 - e\vec{E}\vec{r},$$

$$\vec{p} = m\dot{\vec{r}} + \frac{e}{2c}[\vec{H}\vec{r}], \quad \dot{\vec{p}} = e\vec{E} + \frac{e}{2c}[\vec{v}\vec{H}], \quad (1.23)$$

$$\vec{M} = [\vec{r}\vec{p}], \quad \dot{\vec{M}} = e[\vec{r}\vec{E}] - \frac{e}{2c}[\vec{H}[\vec{r}\vec{v}]].$$

Ma'lumki, maydon kuchlanganliklarining potentsiallar orqali aniqlanishi bir qiymatli emas. Haqiqatdan ham, (1.16) dagi ifodalarda

$$\phi \rightarrow \phi' = \phi + \frac{1}{c} \frac{df}{dt}, \quad \vec{A} \rightarrow \vec{A}' = \vec{A} - \vec{\nabla}f, \quad (1.24)$$

almashtirishlar bajarganimiz bilan \vec{E} va \vec{H} larning qiymati o'zgarmaydi. Bu yerda $f(x,t)$ -koordinata va vaqtning ixtiyoriy funksiyasi. Bunday simmetriyani kalibrovka invariantligi deyiladi.

II. HAKARAT TENGLAMALARINI INTEGRALLASH

§2.1. Bir o'Ichovli harakat tenglamalarini integrallash.

Bir o'Ichovli harakatni ifodalovchi Lagranj funksiyasi:

$$L = \frac{m\dot{x}^2}{2} - U(x), \quad (2.1)$$

Unga mos keluvchi Lagranj tenglamasini yozamiz:

$$\frac{d}{dt} \frac{dL}{dx} - \frac{dL}{dx} = 0, \quad (2.2)$$

(2.1) funksiyani (2.2) tenglamaga qo'ysak

$$m\ddot{x} + \frac{\partial U(x)}{\partial x} = 0 \quad (2.3)$$

tenglama hosil bo'ladi. Bu tenglama ikkinchi tartibli oddiy differensial tenglama bo'lgani uchun uning yechimida ikkita o'zgarimas kattalik bo'ladi. Harakat tenglamasi yechimini aniqlashning yuqoridagiga qaraganda soddaroq va ko'p hollarda juda qulay bo'lgan yo'li quyidagicha: avval moddiy nuqta energiyasi aniqlanadi va Lagranj funksiyasi vaqtga oshkor bog'liq bo'lmaganligi sababli energiya saqlanuvchi kattalik bo'lishini hisobga olgan holda, koordinataning vaqtga bog'liqligini topamiz:

$$E = \dot{x} \frac{\partial L}{\partial \dot{x}} - L. \quad (2.4)$$

Bu ifodaga (2.1) funksiyani qo'yib, energiya ifodasini topamiz:

$$E = \frac{m\dot{x}^2}{2} + U(x) = const. \quad (2.5)$$

Bundan,

$$\frac{dx}{dt} = \dot{x} = \sqrt{\frac{2}{m}[E - U(x)]}.$$

Oxirgi ifoda o'zgaruvchilari ajraluvchi birinchi tartibli oddiy differensial tenglama bo'lgani uchun,

$$\int dt = \int \frac{dx}{\sqrt{\frac{2}{m}[E - U(x)]}} + C, \quad t = \int \frac{dx}{\sqrt{\frac{2}{m}[E - U(x)]}} + C. \quad (2.6)$$

Natijada (2.6) ifodaning o'ng tomonidagi integral yechilib, vaqt va koordinata orasidagi bog'lanish aniqlanadi. Yechimdagi ikkita o'zgarmas kattalik E va C larni aniqlashda boshlang'ich shartlardan foydalaniladi.

1-masala. Harakat tenglamasi integrallansin:

$$L = \dot{x}^2 + tg^2x \quad t=0 \text{ da } x=0, \dot{x}=2.$$

Yechilishi. Ko'rinib turibdiki, Lagranj funksiyasi vaqtga oshkora bog'liq emas, demak, energiya saqlanuvchi kattalik.

Energiya ifodasini topamiz:

$$E = \dot{x} \frac{\partial L}{\partial \dot{x}} - L = 2\dot{x}\dot{x} - \dot{x}^2 - tg^2x = \dot{x}^2 - tg^2x.$$

Biror vaqt momenti uchun energiya ifodasini topsak, u boshqa vaqt momentlarida ham o'zgarmaydi: $t=0$ da $E=4-0=4$.

Demak, $4 = \dot{x}^2 - tg^2x, \quad \frac{dx}{dt} = \dot{x} = \sqrt{4 + tg^2x}.$

Bu tenglamani yechamiz:

$$t = \int \frac{dx}{\sqrt{4 + tg^2x}} + C = \int \frac{dx}{\sqrt{4 + \frac{\sin^2 x}{\cos^2 x}}} + C = \int \frac{\cos x dx}{\sqrt{4\cos^2 x + \sin^2 x}} + C =$$

$$\int \frac{d(\sin x)}{\sqrt{4(1 - \sin^2 x) + \sin^2 x}} + C = \int \frac{d(\sin x)}{\sqrt{4 - 3\sin^2 x}} + C = \frac{1}{\sqrt{3}} \arcsin\left(\frac{\sqrt{3}}{2} \sin x\right) + C,$$

$t=0$ da $x=0$ ekanligidan $C=0$ ekanligi kelib chiqadi. Demak,

$$t = \frac{1}{\sqrt{3}} \arcsin\left(\frac{\sqrt{3}}{2} \sin x\right), \quad x = \arcsin\left(\frac{2}{\sqrt{3}} \sin \sqrt{3}t\right).$$

2-masala. Harakat tenglamasi integrallansin:

$$L = \dot{x}^2 - \frac{1}{x}, \quad t=0 \text{ da } x=1, \dot{x}=0.$$

Yechilishi. Bu holda ham to'la energiyani topamiz:

$$E = \dot{x}^2 + \frac{1}{x}, t=0 \text{ da } E = 0 + \frac{1}{1} = 1, \quad 1 = \dot{x}^2 + \frac{1}{x}.$$

Oxirgi tenglamadan tezlikni topamiz:

$$\frac{dx}{dt} = \dot{x} = \sqrt{1 - \frac{1}{x}}.$$

Bundan, esa

$$t = \int \frac{dx}{\sqrt{1 - \frac{1}{x}}} + C = \sqrt{x(x-1)} + \ln[\sqrt{x} + \sqrt{x-1}] + C$$

Yechimdan ko'rinib turibdiki, $x > 0$ va $x-1 > 0$.

Demak, $x < 1$ bo'lishi mumkin emas ekan. $t=0$ da $x=1$ ligidan $C=0$ hosil bo'ladi. Shunday qilib, yechim quyidagicha bo'lar ekan:

$$t = \sqrt{x(x-1)} + \ln[\sqrt{x} + \sqrt{x-1}]$$

3-masala. Harakat tenglamasi integrallansin:

$$L = \frac{\dot{x}^2}{x} + x + \frac{1}{x}, \quad t=0 \text{ da } E = -3, x = 3.$$

Yechilishi. Bu masalada avvalgilaridan farqli ravishda $t=0$ da \dot{x}_0 berilmagan, lekin to'la energiyaning qiymati ma'lum bo'lganligi uchun masala yechilishi mumkin. Xuddi avvalgi masalalardagidek energiyaning ifodasini topamiz:

$$E = \dot{x} \frac{\partial L}{\partial \dot{x}} - L \text{ edi, } \frac{\partial L}{\partial \dot{x}} = \frac{2\dot{x}}{x} \text{ dan } E = \frac{\dot{x}^2}{x} \dot{x} - \frac{\dot{x}^2}{x} - x - \frac{1}{x},$$

Berigan shartlardan $E = \frac{\dot{x}^2}{x} - x - \frac{1}{x} = -3.$

Bundan \dot{x} ni topamiz: $\frac{dx}{dt} = \dot{x} = \sqrt{x^2 - 3x + 1}, \quad t = \int \frac{dx}{\sqrt{x^2 - 3x + 1}} + C.$

Bu integral oson yechiladi

$$t = \operatorname{arcch} \frac{2x-3}{\sqrt{5}} + C, \quad t=0 \text{ da } x=3 \text{ dan } C = -\operatorname{arcch} \frac{3}{\sqrt{5}}.$$

Demak, berilgan boshlang'ich shartlarni qanoatlantiruvchi yechim quyidagicha:

$$t = \operatorname{arcch} \frac{2x-3}{\sqrt{5}} - \operatorname{arcch} \frac{3}{\sqrt{5}}.$$

4-masala. Harakat tenglamasi integrallansin:

$$L = -\sqrt{1-\dot{x}^2} + x, \quad t = 0 \text{ da } x = 2, \dot{x} = 0.$$

Yechilishi. Bu holda energiya ifodasini hisoblashdan boshqa yo'l yo'q (ya'ni, avvalgi misollardagidek potensial energiya ifodasi ishorasini o'zgartirib qo'ya qolish imkoniyati yo'q).

$$E = \dot{x} \frac{\partial L}{\partial \dot{x}} - L = \dot{x} \frac{\dot{x}}{\sqrt{1-\dot{x}^2}} + \sqrt{1-\dot{x}^2} - x = \frac{1}{\sqrt{1-\dot{x}^2}} - x.$$

Demak, energiya ifodasi:

$$E = \frac{1}{\sqrt{1-\dot{x}^2}} - x = \frac{1}{\sqrt{1-0}} - 2 = -1.$$

Bundan tezlikni topsak:

$$\frac{dx}{dt} = \dot{x} = \sqrt{1 - \frac{1}{(x-1)^2}}, \quad t = \int \frac{dx}{\sqrt{1 - \frac{1}{(x-1)^2}}} + C.$$

Bu integral oson yechiladi:

$$t = \int \frac{(x-1)dx}{(x-1)^2 - 1} + C = \int \frac{(x-1)dx}{x^2 - 2x} + C = \frac{1}{2} \int \frac{d(x^2 - 2x)}{x^2 - 2x} + C = \ln|x^2 - 2x| + C.$$

Boshlang'ich shartlardan foydalanib C ni topamiz:

$$t = 0 \text{ da } x_0 = 2 \text{ edi } 0 = \sqrt{4-4} + C, \quad C = 0.$$

Demak, yechim ushbu ko'rinishni oladi

$$t = \sqrt{x^2 - 2x}, \quad \text{yoki} \quad x = 1 + \sqrt{1+t^2}.$$

5-masala. Harakat tenglamasi integrallansin:

$$L = \dot{x}^2 + 2x^3 + x^6 \quad t = 0, x = 0, \dot{x} = 1$$

Yechilishi. Energiyaning ifodasini topamiz va uning boshlang'ich qiymatini topib olamiz:

$$E = \dot{x}^2 - 2x^3 - x^6 \quad t = 0 \Rightarrow E = 1 - 0 - 0 = 1, \quad 1 = \dot{x}^2 - 2x^3 - x^6$$

Bu erdan harakat tenglamasini yuqoridagi masalalardagi kabi tuzamiz:

$$\dot{x} = \frac{dx}{dt} = \sqrt{1+2x^3+x^6} = \sqrt{(x^3+1)^2} = x^3+1$$

Echimi topish qiyinchilik tug'dirmaydi, ammo bu keng tarqalgan integrallash usulini batafsil ko'rsataylik:

$$t = \int \frac{dx}{x^3+1} \Rightarrow x^3+1 = (x+1)(x^2-x+1) \Rightarrow$$

$$t = \int \frac{dx}{x^3+1} = A \int \frac{dx}{x+1} + \int \frac{Bx+C}{x^2-x+1} dx$$

$$\begin{cases} x^2 | 0 = A+B \\ x | 0 = -A+B+C \\ x^0 | 1 = A+C \end{cases} \Rightarrow A = \frac{1}{3}, B = -\frac{1}{3}, C = \frac{2}{3}$$

$$t = \int \frac{dx}{x^3+1} = \frac{1}{3} \ln|x+1| - \frac{1}{3} \int \frac{x-2}{x^2-x+1} dx = \frac{1}{3} \ln|x+1| - \frac{1}{3} \int \frac{(x-\frac{1}{2})dx}{x^2-x+1} + \frac{1}{2} \int \frac{dx}{(x-\frac{1}{2})^2 + \frac{3}{4}} = \frac{1}{3} \ln|x+1| - \frac{1}{6} \ln(x^2-x+1) + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} + C = \frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} + C$$

Boshlang'ich shartlardan:

$$t = 0, x = 0 \Rightarrow C = \frac{5\sqrt{3}\pi}{18}$$

Va nihoyat

$$t = \frac{1}{6} \ln \frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} + \frac{5\sqrt{3}\pi}{18}$$

6-masala. $L = \dot{x}^2 + 2\sin x + \sin^2 x \quad t = 0 \text{ da } x = 0, \dot{x} = 1$

Yechilishi. Xuddi yuqoridagilardak energiyanning ifodasidan:

$$\dot{x} = \frac{dx}{dt} = \sqrt{1+2\sin x + \sin^2 x}$$

Integrallash esa qiyinchilik tug'dirmaydi:

$$t = \int \frac{dx}{1 + \sin x} = - \int \frac{d\left(\frac{\pi-x}{2}\right)}{1 + \cos\left(\frac{\pi-x}{2}\right)} = - \int \frac{d\left(\frac{\pi-x}{4} - \frac{x}{2}\right)}{\cos^2\left(\frac{\pi-x}{4} - \frac{x}{2}\right)} = - \operatorname{tg}\left(\frac{\pi-x}{4} - \frac{x}{2}\right) + C$$

$$t=0, x=0 \Rightarrow C=1 \quad t = - \operatorname{tg}\left(\frac{\pi-x}{4} - \frac{x}{2}\right) + 1 \quad \operatorname{tg}\left(\frac{\pi-x}{4} - \frac{x}{2}\right) = 1-t$$

$$x = \frac{\pi}{2} - 2 \operatorname{arctg}(1-t)$$

7-masala. $L = \dot{x}^2 + e^{-x}$

Yechilishi. Yana energiyani saqlanishidan foydalanamiz, chunki Lagranj funksiyasi vaqtga oshkora bog'liq emas:

$$E = \dot{x}^2 - e^{-x} \quad \dot{x} = \sqrt{E + e^{-x}}$$

$$t = \int \frac{dx}{\sqrt{E + e^{-x}}} \Rightarrow \left\| dx = \frac{de^x}{e^x} \right\| \Rightarrow \int \frac{de^x}{\sqrt{Ee^{2x} + e^x + \frac{1}{4E} - \frac{1}{4E}}} = \int \frac{de^x}{\sqrt{\left(\sqrt{E}e^x + \frac{1}{2\sqrt{E}}\right)^2 - \frac{1}{4E}}} =$$

$$= \frac{1}{2\sqrt{E}} \int \frac{de^x}{\sqrt{(2Ee^x + 1)^2 - 1}} = \frac{1}{4E\sqrt{E}} \int \frac{d(2Ee^x + 1)}{\sqrt{(2Ee^x + 1)^2 - 1}} = \frac{1}{4E\sqrt{E}} \ln \left| (2Ee^x + 1) + \sqrt{(2Ee^x + 1)^2 - 1} \right|$$

8-masala. Zarra $U(x) = -U_0 \cos \frac{x}{l}$ maydonda harakat qilmoqda.

Boshlang'ich shartlar $x(0) = 0$, $\dot{x}(0) = \sqrt{\frac{4U_0}{m}} = v_0$ bo'lganda $x(t)$ ni toping.

Yechilishi. To'la energiyani boshlang'ich shartlardagi qiymati noldan kattadir

$$E = \frac{1}{2} m \dot{x}^2 - U_0 \cos \frac{x}{l} \xrightarrow{t=0} U_0, \quad E = U_0 > 0$$

va saqlanuvchi kattalik (integral) dir:

$$E = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - U_0 \cos \frac{x}{l} = U_0,$$

$$\dot{x} = \sqrt{\frac{2}{m} U_0 \left(1 + \cos \frac{x}{l}\right)} = \sqrt{\frac{4U_0}{m} \cos^2 \frac{x}{2l}} = v_0 \cos \frac{x}{2l},$$

$z = \sin \frac{x}{2l}$ almashtirish bajarib,

$$\frac{dx}{\cos \frac{x}{2l}} = 2l \frac{dz}{1-z^2} = l d \left[\ln \frac{1+z}{1-z} \right]$$

ni hisobga olib $\dot{x}(t)$ ni integrallasak

$$\frac{v_0 t}{l} = \ln \frac{1 + \sin \frac{x}{2l}}{1 - \sin \frac{x}{2l}}$$

ni olamiz, chunki boshlang'ich shartlardan $C=0$.

Bu yerdan $\sin \frac{x}{2l} = \operatorname{th} \frac{v_0 t}{2l}$ degan xulosaga kelamiz ($t \rightarrow \infty, x \rightarrow \pi l$).

Masaladagi $U(x)$ potensial funksiya matematik mayatnikning potensial energiyasidir, x - yoyning uzunligi. Solitonlar nazariyasida yuqoridagi yechim odatda quyidagi ko'rinishda ko'proq ishlatiladi:

$$\operatorname{tg}\left(\frac{\pi}{4} - \frac{x}{4l}\right) = \exp - \frac{v_0 t}{2l}.$$

2.1-misolalar.

Harakat tenglamalari integrallansin

- $L = \dot{x}^2 - \frac{1}{x^2}, t=0$ da, $x=1, \dot{x}=0$.
- $L = \dot{x}^2 + e^x, t=0$ da $x=0, \dot{x}=2$.
- $L = \frac{\dot{x}^2}{x} - x, t=0$ da $x=1, \dot{x}=1$.
- $L = \frac{\dot{x}^2 - x\dot{x} - 1}{x^2}, t=0$ da $x=1, \dot{x}=0$.
- $L = \frac{m\dot{x}^2}{2} + \alpha x^4, t=0$ da $E=0, x=0$.
- $L = \dot{x}^2 - 8x^2 + x^4, t=0$ da $x=2, \dot{x}=0$.

$$t = \int \frac{dx}{1 + \sin x} = - \int \frac{d\left(\frac{\pi}{2} - x\right)}{1 + \cos\left(\frac{\pi}{2} - x\right)} = - \int \frac{d\left(\frac{\pi}{4} - \frac{x}{2}\right)}{\cos^2\left(\frac{\pi}{4} - \frac{x}{2}\right)} = - \operatorname{tg}\left(\frac{\pi}{4} - \frac{x}{2}\right) + C$$

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Yechilishi. To'la energiyaning boshlang'ich shartlardagi qiymati noldan kattadir

$$E = \frac{1}{2} m \dot{x}^2 - U_0 \cos \frac{x}{l} \xrightarrow{\text{b.sh}} U_0, \quad E = U_0 > 0$$

va saqlanuvchi kattalik (integral) dir:

$$E = \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 - U_0 \cos \frac{x}{l} = U_0,$$

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ni hisobga olib $\dot{x}(t)$ ni integrallasak

$$\frac{v_0 t}{l} = \ln \frac{1 + \sin \frac{x}{2l}}{1 - \sin \frac{x}{2l}}$$

ni olamiz, chunki boshlang'ich shartlardan $C=0$.

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Harakat tenglamalari integrallansin

1. $L = \dot{x}^2 - \frac{1}{x^2}, t=0$ da, $x=1, \dot{x}=0$.
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4. $L = \frac{\dot{x}^2 - x\dot{x} - 1}{x^2}, t=0$ da $x=1, \dot{x}=0$.
5. $L = \frac{m\dot{x}^2}{2} + ax^4, t=0$ da $E=0, x=0$.
6. $L = \dot{x}^2 - 8x^2 + x^4, t=0$ da $x=2, \dot{x}=0$

$$7. L = \frac{m\dot{x}^2}{2} + \frac{m\omega^2 x^2}{2} + \frac{\alpha x^3}{2} + \frac{\beta^2 x^4}{2},$$

$$m=1, \alpha=2\omega\beta, \beta=1$$

$$9. L = \dot{x}^2 + ctg^2 x, \quad t=0 \text{ da } x = \frac{\pi}{2}, \dot{x} = 4$$

$$10. L = \dot{x}^2 + \frac{e^{-2ax}}{\sin^2 bx},$$

$$t=0 \text{ da } E=0, x = \frac{\pi}{b}$$

Zarraning quyidagi maydonlardagi harakati integrallansin:

$$11. U(x) = -U_0 e^{-x}, \quad x(0) = 0, \quad \dot{x}(0) = v_0 \geq \sqrt{\frac{2U_0}{m}}$$

$$12. U(x) = -\frac{1}{2}kx^2 + \frac{1}{3!}ax^3, \quad x(0) = \frac{3k}{a}, \quad \dot{x}(0) = 0$$

Ko'rsatma: $\dot{x} = -\sqrt{\frac{a}{3m}}x\sqrt{x_0-x}$, $x_0 = x(0)$, $x_0 - x = z^2$ almashtirish bajaring va

$$-\frac{dx}{x\sqrt{x_0-x}} = \frac{2dz}{x_0-z^2} = \frac{dz}{\sqrt{x_0}} \left(\frac{1}{\sqrt{x_0+z}} + \frac{1}{\sqrt{x_0-z}} \right),$$

$$t = \sqrt{\frac{m}{k}} \ln \left[\frac{\sqrt{x_0} + \sqrt{x_0-x}}{\sqrt{x_0} - \sqrt{x_0-x}} \right], \quad x(t) = \frac{2k}{a} ch^{-2} \left(\frac{1}{2} \sqrt{\frac{k}{m}} t \right)$$

13. Zarra $U(x) = U_0 \exp\left(-\frac{|x|}{a}\right)$ tashqi maydonda $E > U_0$ energiya bilan harakatlanmoqda. $X_1 = -\infty$ dan $X_2 = \infty$ gacha borish uchun shunday potensial to'siqli maydonda va erkin harakat davomida ketgan vaqtlarining ayirmasini toping.

§2.2. Harakat finitlik shartini va tebranish davrini energiyaga bog'liq holda aniqlash.

Har ikki tomondan chegaralanuvchi harakat *finit harakat* deyiladi. To'la energiya $E = T + U$ bo'lganligidan $T = 0$ holda $E = U(x)$ tenglamadan harakat chegaralarini aniqlab, qaysi hollarda harakat finit bo'lishini, finitlik sharti va tebranish davrini energiyaga bog'lanishini ko'raylik.

1-masala. $L = \dot{x}^2 - x^2.$

Yechilishi. Harakat davomida $E > U$, yani $T > 0$. Lekin, to'xtash nuqtalarida $T = 0$ bo'lib, $E = U(x)$ ligidan finitlik shartini topamiz. Avval energiyani topaylik: $E = \dot{x}^2 + x^2 = T + U$ to'xtash nuqtalarida $T = 0$, demak, $\dot{x} = 0$. Ya'ni to'xtash nuqtalari tezlik nolga teng bo'lgan holatlardir:

$$x^2 = E, \quad x = \pm\sqrt{E}, \quad x_1 = -\sqrt{E}, \quad x_2 = \sqrt{E}.$$

Bundan ko'rinib turibdiki, $E < 0$ bo'lishi mumkin emas. Agar $E = 0$ bo'lsa, to'xtash nuqtasi bitta (harakat bir tomondangina cheklangan) va harakat finit bo'la olmaydi. Shuning uchun, $E > 0$.

Endi harakat tenglamasini integrallaylik va bunda tebranish davri bir to'xtash nuqtasidan ikkinchi to'xtash nuqtasigacha harakat uchun ketgan vaqtning ikkilangani ekanligini hisobga olaylik, ya'ni

$$\frac{dx}{dt} = \dot{x} = \sqrt{E-x^2}, \quad t = \int \frac{dx}{\sqrt{E-x^2}} + C.$$

$$T = 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{E-x^2}} = 2 \int_{-\sqrt{E}}^{\sqrt{E}} \frac{dx}{\sqrt{E-x^2}} = 2 \arcsin \left(\frac{x}{\sqrt{E}} \right) \Big|_{-\sqrt{E}}^{\sqrt{E}} = 2\pi$$

Demak, $E > 0$ da harakat finit va tebranish davri $T = 2\pi$ ga teng.

2-masala. $L = \dot{x}^2 + \frac{2}{x} - \frac{1}{x^2}.$

Yechilishi. Energiyaning ifodasini topaylik: $E = \dot{x}^2 - \frac{2}{x} + \frac{1}{x^2}.$

To'xtash nuqtalarida $\dot{x} = 0$ edi, $E = -\frac{2}{x} + \frac{1}{x^2}$.

Bu tenglamani yechamiz:

$$x_1 = \frac{-1 + \sqrt{1+E}}{E}, \quad x_2 = \frac{-1 - \sqrt{1+E}}{E}$$

Ko'rinib turibdiki, $-1 < E < 0$ da harakat finit bo'ladi, $E < -1$ da yechimlar yo'q, $E \geq 0$ da esa harakat infinitdir. Finitlik holida to'la energiyaning ishorasi manfiyligini hisobga olgan holda E ni $(-E)$ bilan belgilab olsak,

$$x_1 = \frac{1 - \sqrt{1-E}}{E}, \quad x_2 = \frac{1 + \sqrt{1-E}}{E}$$

Endi harakat tenglamasini integrallaymiz:

$$\frac{dx}{dt} = \dot{x} = \sqrt{E + \frac{2}{x} - \frac{1}{x^2}}, \quad t = \int \frac{dx}{\sqrt{E + \frac{2}{x} - \frac{1}{x^2}}} + C.$$

$$\begin{aligned} T &= 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{-E + \frac{2}{x} - \frac{1}{x^2}}} = 2 \int_{x_1}^{x_2} \frac{x dx}{\sqrt{-Ex^2 + 2x - 1}} = \\ &= 2 \left[\frac{-1}{2E} \int_{x_1}^{x_2} \frac{d(2x - Ex^2 - 1)}{\sqrt{2x - Ex^2 - 1}} + \frac{1}{E} \int_{x_1}^{x_2} \frac{dx}{\sqrt{-1 + \frac{1}{E} - (Ex^2 - 2x + \frac{1}{E})}} \right] = \\ &= -\frac{2\sqrt{2x - Ex^2 - 1}}{E} \Big|_{x_1}^{x_2} + \frac{2}{(E)^{\frac{3}{2}}} \arcsin \frac{xE - 1}{\sqrt{1-E}} \Big|_{x_1}^{x_2} = \frac{2\pi}{E^{\frac{3}{2}}}. \end{aligned}$$

Demak, harakat $0 < E < 1$ da finit va davri $T = \frac{2\pi}{E^{\frac{3}{2}}}$ ga teng. Bunda to'la

energiya manfiyligini hisobiga E ni $(-E)$ bilan almashtirilganini unutmang.

3-masala. $L = \dot{x}^2 - tg^2 x$

Yechilishi. $E = \dot{x}^2 + tg^2 x, \quad \Rightarrow \quad tg^2 x = E, \quad tg x_1 = -\sqrt{E}, \quad tg x_2 = \sqrt{E}$

Harakat finitligi uchun $E > 0$ sharti kifoya.

$$\dot{x} = \sqrt{E - tg^2 x}, \quad t = \int \frac{dx}{\sqrt{E - tg^2 x}} + C,$$

$$\begin{aligned} T &= 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - tg^2 x}} = 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - \frac{\sin^2 x}{\cos^2 x}}} = 2 \int_{x_1}^{x_2} \frac{\cos x dx}{\sqrt{E \cos^2 x - \sin^2 x}} = \\ &= \frac{2}{\sqrt{1+E}} \arcsin \left[\sqrt{\frac{1+E}{E}} \sin x \right] \Big|_{x_1}^{x_2}. \end{aligned}$$

Bu integralni yechish I-paragrafning 1-masalasidagiga o'xshash.

To'xtash nuqtalarining hisoblangan qiymatlari ko'rinishini biroz o'zgartirsak

$$\sin x_1 = -\sqrt{\frac{E}{1+E}}, \quad \sin x_2 = \sqrt{\frac{E}{1+E}}, \quad T = \frac{2\pi}{\sqrt{1+E}}$$

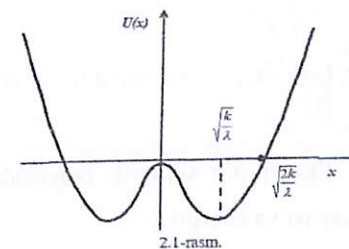
Demak, $E > 0, \quad T = \frac{2\pi}{\sqrt{1+E}}$.

4-masala. Zarra 2.1-rasmda ko'rsatilgan

$$U(x) = -\frac{1}{2} kx^2 + \frac{1}{4} \lambda x^4, \quad k, \lambda > 0$$

maydonda harakatlanyapti. Harakat tenglamasini integrallang, agarda

$$x_0 = x(0) = \sqrt{\frac{2k}{\lambda}}, \quad \dot{x}(0) = 0 \text{ bo'lsa.}$$



Yechilishi. To'la energiya boshlang'ich shartlarda

$$E = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 + \frac{1}{4} \lambda x^4 \equiv 0.$$

To'xtash nuqtalarida $\dot{x} = 0$ edi, $E = -\frac{2}{x} + \frac{1}{x^2}$.

Bu tenglamani yechamiz: $x_1 = \frac{-1 + \sqrt{1+E}}{E}$, $x_2 = \frac{-1 - \sqrt{1+E}}{E}$.

Ko'rinib turibdiki, $-1 < E < 0$ da harakat finit bo'ladi, $E < -1$ da yechimlar yo'q, $E \geq 0$ da esa harakat infinitdir. Finitlik holda to'la energiyaning ishorasi manfiyligini hisobga olgan holda E ni $(-E)$ bilan belgilab olsak,

$$x_1 = \frac{1 - \sqrt{1-E}}{E}, \quad x_2 = \frac{1 + \sqrt{1-E}}{E}.$$

Endi harakat tenglamasini integrallaymiz:

$$\frac{dx}{dt} = \dot{x} = \sqrt{E + \frac{2}{x} - \frac{1}{x^2}}, \quad t = \int \frac{dx}{\sqrt{E + \frac{2}{x} - \frac{1}{x^2}}} + C.$$

$$T = 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{-E + \frac{2}{x} - \frac{1}{x^2}}} = 2 \int_{x_1}^{x_2} \frac{x dx}{\sqrt{-Ex^2 + 2x - 1}} =$$

$$= 2 \left[\frac{-1}{2E} \int_{x_1}^{x_2} \frac{d(2x - Ex^2 - 1)}{\sqrt{2x - Ex^2 - 1}} + \frac{1}{E} \int_{x_1}^{x_2} \frac{dx}{\sqrt{-1 + \frac{1}{E} - \left(Ex^2 - 2x + \frac{1}{E}\right)}} \right] =$$

$$= -\frac{2\sqrt{2x - Ex^2 - 1}}{E} \Big|_{x_1}^{x_2} + \frac{2}{(E)^{\frac{3}{2}}} \arcsin \frac{xE - 1}{\sqrt{1-E}} \Big|_{x_1}^{x_2} = \frac{2\pi}{E^{\frac{3}{2}}}.$$

Demak, harakat $0 < E < 1$ da finit va davri $T = \frac{2\pi}{E^{\frac{3}{2}}}$ ga teng. Bunda to'la

energiya manfiyligini hisobiga E ni $(-E)$ bilan almashtirilganini unutmang.

3-masala. $L = \dot{x}^2 - tg^2 x$

Yechilishi. $E = \dot{x}^2 + tg^2 x$, $\Rightarrow tg^2 x = E$, $tg x_1 = -\sqrt{E}$, $tg x_2 = \sqrt{E}$

Harakat finitligi uchun $E > 0$ sharti kifoya.

$$\dot{x} = \sqrt{E - tg^2 x}, \quad t = \int \frac{dx}{\sqrt{E - tg^2 x}} + C,$$

$$T = 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - tg^2 x}} = 2 \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - \frac{\sin^2 x}{\cos^2 x}}} = 2 \int_{x_1}^{x_2} \frac{\cos x dx}{\sqrt{E \cos^2 x - \sin^2 x}} =$$

$$= \frac{2}{\sqrt{1+E}} \arcsin \left[\sqrt{\frac{1+E}{E}} \sin x \right] \Big|_{x_1}^{x_2}.$$

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$$\sin x_1 = -\sqrt{\frac{E}{1+E}}, \quad \sin x_2 = \sqrt{\frac{E}{1+E}}, \quad T = \frac{2\pi}{\sqrt{1+E}}$$

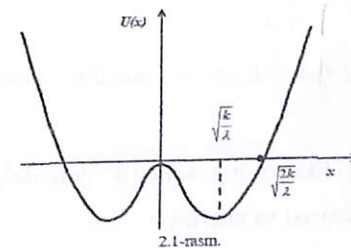
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$$U(x) = -\frac{1}{2} kx^2 + \frac{1}{4} \lambda x^4, \quad k, \lambda > 0$$

maydonda harakatlanyapti. Harakat tenglamasini integrallang, agarda

$$x_0 = x(0) = \sqrt{\frac{2k}{\lambda}}, \quad \dot{x}(0) = 0 \text{ bo'lsa.}$$



Yechilishi. To'la energiya boshlang'ich shartlarda

$$E = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2 + \frac{1}{4} \lambda x^4 \equiv 0.$$

U holda (2.6) formuladan x_0 dan x gacha borishi uchun ketgan vaqtni aniqlaylik (harakat x ning yo'nalishiga qarama-qarshi bo'lgani uchun ishora manfiy):

$$t = -\int_{x_0}^x \frac{dx}{\sqrt{\frac{2m}{m} \left[0 + \frac{1}{2} kx^2 - \frac{1}{4} \lambda x^4 \right]}} = -\sqrt{\frac{2m}{\lambda}} \int_{x_0}^x \frac{dx}{x\sqrt{x_0^2 - x^2}} =$$

$$= \frac{1}{2x_0} \sqrt{\frac{2m}{\lambda}} \ln \left[\frac{x_0 + x\sqrt{x_0^2 - x^2}}{x_0 - x\sqrt{x_0^2 - x^2}} \right]$$

Bu yerdan $x(t)$ ni topish mumkin:

$$x(t) = \sqrt{\frac{2k}{\lambda}} \operatorname{ch}^{-1} \left[\sqrt{\frac{k}{m}} t \right], \quad t \rightarrow \infty \text{ da } x(t) \rightarrow 0.$$

To'xtash nuqtasini energiyaga bog'liq holda aniqlaylik:

$$E + \frac{1}{2} kx^2 - \frac{1}{4} \lambda x^4 = 0, \quad x_{1,2} = \sqrt{\frac{k}{\lambda} \pm \sqrt{\left(\frac{k}{\lambda}\right)^2 + \frac{2E}{\lambda}}}$$

1) $E = 0$, unda $x_1 = 0$, $x_2 = \pm \sqrt{\frac{2k}{\lambda}}$.

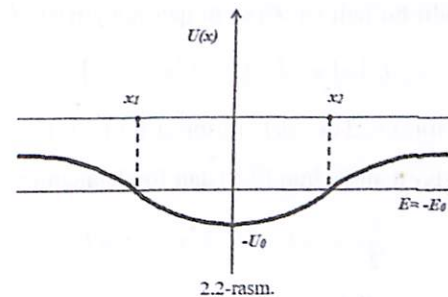
2) $E < 0$, lekin $E > E_{\min} = -\frac{k^2}{2\lambda}$ da $x_1 = 0$ va $x_{1,2} < \pm \sqrt{\frac{2k}{\lambda}}$ oraliqda finit harakat.

3) $E > 0$, $x_2 > \sqrt{\frac{2k}{\lambda}}$, $x_1 < -\sqrt{\frac{2k}{\lambda}}$.

5-masala. Zarra $U(x) = -U_0 \operatorname{ch}^{-2} kx$ maydonda harakatlanmoqda. Harakat tenglamasi integrallansin.

Yechilishi. Bu potensial Ekkartning simetrik potensali bo'lib, u 2.2-rasmda ko'rsatilgan. Lagranj funksiyasi va energiya:

$$L = \frac{1}{2} m\dot{x}^2 + U_0 \operatorname{ch}^{-2} kx, \quad E = \frac{1}{2} m\dot{x}^2 - U_0 \operatorname{ch}^{-2} kx.$$



Chizmadan ko'rinib turibdiki, $E > 0$ da harakat infinit, $E < 0$ da esa, harakat finit. Shuning uchun ikki xil holni ko'ramiz.

1. $E = -E_0 < 0$, $x(0) = 0$, $\dot{x}(0) = \dot{x}_0$.

U holda

$$t = \sqrt{\frac{m}{2}} \int_0^x \frac{dx}{\sqrt{E - U(x)}} = \sqrt{\frac{m}{2}} \int_0^x \frac{ch kx dx}{\sqrt{-E_0 \operatorname{ch}^2 kx + U_0}} = \frac{1}{k} \sqrt{\frac{m}{2}} \int_0^x \frac{d(shkx)}{\sqrt{U_0 - E_0 - E_0 \operatorname{sh}^2 kx}}$$

$$\xi = \sqrt{\frac{E_0}{U_0 - E_0}} shkx \quad \text{almashtirish bajarib}$$

$$shkx = \sqrt{\frac{U_0 - E_0}{E_0}} \sin(\omega t + \alpha), \quad \omega = k \sqrt{\frac{2E_0}{m}}$$

yechimini olamiz.

To'xtash nuqtalari $x_{1,2} = \frac{1}{k} \operatorname{arcch} \left(\pm \sqrt{\frac{U_0}{E_0}} \right)$.

Bu holda zarra $x_{1,2}$ nuqtalar orasida nohiziqli davriy tebranadi, harakat davri esa:

$$T = 2 \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - U(x)}} = \frac{2\pi}{k} \sqrt{\frac{m}{2E_0}} = \frac{2\pi}{\omega}, \quad E = U(x_{1,2})$$

Davrning to'la energiyaga bog'liqligi xarakatning noizoxronligidan dalolatdir. Potensial chuqur tubi atrofidagi harakatni ko'raylik, ya'ni

$$U = -U_0 + \varepsilon, \quad \varepsilon \ll U_0.$$

Bu hol $kx \ll 1$ da sodir bo'ladi va $U(x)$ ni qatorga yoyishimiz mumkin:

$$U(x) \approx -U_0(1 - k^2 x^2 + \dots).$$

Bu holda ham (2.6) formuladan $x(t)$ ni topsa bo'ladi, lekin $U(x)$ 2-darajali polinom ko'rinishda bo'lgani uchun (2.5) dan foydalanamiz:

$$\frac{1}{2}m\dot{x}^2 - U_0 + U_0 k^2 x^2 + \dots = E$$

Ikkala tomonini differensiallab

$$m\ddot{x} + 2U_0 k^2 x = 0$$

tenglamani olamiz. Bu tenglama kichik erkin tebranish ((3.2) tenglama) tenglamasi kabi bo'lib yechimi:

$$x = \frac{x_0}{\omega_0} \sin \omega_0 t, \quad \omega_0 = k \sqrt{\frac{2U_0}{m}}.$$

2. $E > 0$ bo'lsin. Boshlang'ich shartlar $x(0) = -L$, $\dot{x}(0) = v_0$

Zarra $x = L$ nuqtaga borguncha ketgan vaqtni topaylik. Biz uchun $kL \gg 1$ holi o'ta qiziqarli.

$$\xi = \sqrt{\frac{E_0}{E_0 - U_0}} shkx \left(\xi_0 = \sqrt{\frac{E_0}{E_0 - U_0}} shkx \right),$$

$$t = \frac{1}{k} \sqrt{\frac{m}{2E}} \int_{-\xi_0}^{\xi_0} \frac{d\xi}{\sqrt{1 + \xi^2}} = \frac{1}{k} \sqrt{\frac{m}{2E}} \ln \frac{\sqrt{\xi_0^2 + 1} + \xi_0}{\sqrt{\xi_0^2 + 1} - \xi_0} \cong \frac{1}{k} \sqrt{\frac{m}{2E}} \ln 4\xi_0^2 \cong \frac{1}{k} \sqrt{\frac{m}{2E}} \ln \frac{E}{E + U_0} e^{2kL} \cong 2L \sqrt{\frac{m}{2E}} + \frac{1}{k} \sqrt{\frac{m}{2E}} \ln \frac{E}{E + U_0}.$$

Agarda $E \approx \frac{1}{2}mv_0^2$ ekanini hisobga olsak 1-had zarraning erkin harakati vaqtini, 2-had esa shu zarraning tashqi maydon bilan o'zaro ta'siridan kelib chiqadigan o'zgarishni anglatadi. Albatta, harakat infinit va 2-had orqali tashqi potensial maydon ta'siri aniqlanadi.

2.2-misolalar.

a) $L = \frac{\dot{x}^2}{x} - x - \frac{1}{x}.$

b) $L = \dot{x} - (e^x - 1)^2.$

d) $L = \frac{\dot{x}^2 + 2x - 1}{x^2}$

e) $L = \dot{x}^2 - th^2 x.$

f) $L = \frac{1}{2}m\dot{x}^2 + U_0 ch^{-2} kx - \frac{1}{2}(U_2 + U_1) - (U_2 - U_1) thkx.$

$U_2 > U_1 > 0, U_0 > 0, E = -E < 0.$

g) $L = \frac{1}{2}m\dot{x}^2 - U_0 t g^2 \alpha x.$

§2.3. Siklik koordinata tushunchasidan foydalanib harakat tenglamalarini integrallash.

Lagranj funksiyasi oshkor bog'liq bo'lmagan koordinata siklik koordinata deyiladi va impulsning shu koordinataga mos tashkil etuvchisi saqlanuvchi kattalik bo'ladi. Chunki,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} = 0, \quad P_x = \frac{\partial L}{\partial \dot{x}} = const$$

Bu xildagi masalalarni ikki tipini ko'ramiz: boshlang'ich shartlar berilgan masalalar va boshlang'ich shartlar berilmagan masalalar.

1-masala. $L = \frac{t^2 \dot{x}^2}{2}, t=1$ da $x=0, \dot{x}=1.$

Yechilishi. Masalaning berilishidan ko'rinib turibdiki, boshlang'ich shartlar berilgan va

$$P_x = \frac{\partial L}{\partial \dot{x}} = t^2 \dot{x} = const.$$

Boshlang'ich shartlarga asosan $t=1$ da $P_x = 1 \cdot 1 = 1 \Rightarrow t^2 \dot{x} = 1,$

$$\frac{dx}{dt} = \dot{x} = \frac{1}{t^2}, \quad dx = \frac{dt}{t^2}, \quad x = \int \frac{dt}{t^2} + C = -\frac{1}{t} + C.$$

Yana boshlang'ich shartlarga binoan $C=1.$

ekanligini hisobga olsak,

$$y = \int x(P_y - 1) dt = \int \frac{x(P_y - 1) dx}{\sqrt{2Ex - x^2(P_y - 1)^2}} = -\frac{\sqrt{2Ex - x^2(P_y - 1)^2}}{P_y - 1} + \frac{E}{P_y - 1} \arcsin \frac{(P_y - 1)x - E}{E}$$

5-masala. §1.4ning 6-masalasidagi sistemaning harakat tenglamasi integrallansin.

Yechilishi. Umumlashgan koordinata S siklikdir:

$$p_s = (m_1 + m_2)\dot{S} + m_2 l \dot{\varphi} \cos \varphi = \text{const.}$$

Ko'pincha sistemani yaxlit holda tinch turibdi desak bo'ladi. U holda $\text{const} \equiv 0$ deb yuqoridagi tenglamani integrallasak:

$$(m_1 + m_2)S + m_2 l \sin \varphi = \text{const.}$$

Bu tenglik sistemani inersiya markazi gorizontol yo'nalishda tinch turganining matematik ifodasidir. To'la energiya bu ifodalar orqali quyidagi ko'rinishga keladi:

$$E = \frac{m_2 l^2 \dot{\varphi}^2}{2} \left(1 - \frac{m_2}{m_1 + m_2} \cos^2 \varphi \right) - m_2 g l \cos \varphi.$$

Bu yerdan
$$t = \sqrt{\frac{m_2}{2(m_1 + m_2)}} \int \sqrt{\frac{m_1 + m_2 \sin^2 \varphi}{E + m_2 g l \cos \varphi}} d\varphi.$$

Ikkinchi zarraning koordinatalarini $x_2 = l \sin \varphi + s$ $y_2 = -l \cos \varphi$

φ orqali ifodalab, m_2 zarraning harakat trayektoriyasi gorizontol va vertikal

yarim o'qlari $\frac{l m_1}{m_1 + m_2}$ hamda l ga teng ellipsning bir bo'lagi ekanligini ko'rish

qiyin emas. Agarda $m_1 \rightarrow \infty$ desak tebranish trayektoriyasi aylananing bo'lagidan iborat bo'lgan oddiy matematik mayatnik harakat holatlarini olamiz.

6-masala. 1.4-misollarning 2b-masalasidagi sistemaning harakat tenglamasini integrallang.

Yechilishi. qutb o'qi konus uchidan tik yuqoriga yo'nalgan sferik koordinata sistemasida (2.3-rasm):

$$L = \frac{m}{2} (\dot{r}^2 + r^2 \sin^2 \alpha \dot{\varphi}^2) - mgr \cos \varphi.$$

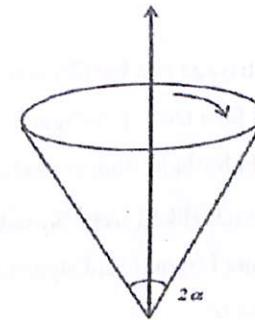
Ko'rinib turibdiki, φ -siklik koordinata va umumlashgan impuls - p_φ

saqlanadi: $p_\varphi = m r^2 \sin^2 \alpha \dot{\varphi} = M_z$. Energiya ham saqlanadi:

$$E = \frac{m \dot{r}^2}{2} + \frac{M_z^2}{2m r^2 \sin^2 \alpha} + mgr \cos \varphi.$$

Bu yerdan
$$t = \int \frac{dr}{\sqrt{\frac{2}{m} [E - U_{\varphi\varphi}(r)]}}$$
,

$$\varphi = \frac{M_z}{\sqrt{2m \sin^2 \alpha}} \int \frac{dr}{r^2 \sqrt{[E - U_{\varphi\varphi}(r)]}}$$



2.3-rasm.

$$U_{\varphi\varphi}(r) = \frac{M_z^2}{2m r^2 \sin^2 \alpha} + mgr \cos \varphi.$$

$E = U_{\psi\psi}(r)$ shart ($M_z = 0$ bo'lganda) r ga nisbatan uchinchi tartibli tenglamadir va uning ikkita ildizi musbatdir. Zarra trayektoriyasi esa, bu ildizlar bilan aniqlangan konus sirtidagi parallel ikki gorizontal aylana oralig'ida yotadi.

2.3-misolalar.

1. $L = \frac{\dot{x}^2}{2} + t^2 \dot{x}, t = 0, x = 0, \dot{x} = 1.$
2. $L = \frac{\dot{x}^2}{2} \cos t, t = 0, x = 0, \dot{x} = 1.$
3. $L = \frac{\dot{x}^2}{2} \cos^2 t, t = 0, x = 0, \dot{x} = 1.$
4. $L = \sqrt{t^2 + \dot{x}^2}, t = 3, x = 0, \dot{x} = 1.$
5. $L = \frac{\dot{x}^2 + x\dot{x} - 1}{x^2}, t = 0$ da
 $x = 1, \dot{x} = 0.$
6. $L = \frac{\dot{x}^2}{x} + x\dot{y}^2 + x.$
7. $L = \frac{\dot{x}^2 + 2\dot{x}\dot{y}x + \dot{y}^2}{2}.$
8. $L = -\sqrt{1 - \dot{x}^2 - \dot{y}^2} + x.$
9. $L = \frac{\dot{x}^2 + \dot{y}^2}{x}.$
10. 1.4-misolalarning 2a-masalasini integrallang.

§2.4 Markaziy maydondagi harakat tenglamalarini integrallash.

Maydon markaziy simmetriyaga ega bo'lsa, koordinata boshini maydon markaziga joylashtiriladi va fazo izotrop bo'lganligi uchun, impuls momenti vektori $\vec{M} = [\vec{r} \times \vec{p}]$ saqlanuvchi bo'ladi. Radius vektori $\vec{r} \perp \dot{\vec{r}}$ bo'lgani uchun markaziy maydondagi harakat tekislikda sodir bo'ladi va shuning uchun moddiy nuqta Lagranj funksiyasini quyidagicha ko'rinishda yozamiz:

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r) \quad (2.7)$$

Energiya esa,

$$E = \dot{r} \frac{\partial L}{\partial \dot{r}} + \dot{\varphi} \frac{\partial L}{\partial \dot{\varphi}} - L = \frac{m\dot{r}^2}{2} + \frac{mr^2\dot{\varphi}^2}{2} + U(r)$$

$$M = M_z = p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} \quad (2.8)$$

ekanligidan foydalansak:

$$E = \frac{m\dot{r}^2}{2} + \frac{M^2}{2mr^2} + U(r) \quad (2.9)$$

Energiya ifodasidagi ikkinchi had kinetik energiyaga tegishli edi, lekin impuls momenti M saqlanuvchi bo'lgani uchun uni endi potensial energiyaga taalluqli deyish kerak. Demak,

$$E = \frac{m\dot{r}^2}{2} + U_{\psi\psi}(r) = \frac{M^2}{2mr^2} + U(r) \quad (2.10)$$

va to'xtash nuqtalari koordinatalarini topishda $E = U_{\psi\psi}(r)$ tenglama yechiladi. Bu nuqtalarda $\dot{r} = 0$ bo'lsa ham, $\dot{\varphi} = 0$ bo'lmazligi mumkin. Shuning uchun, ba'zan, burilish nuqtalari ham deyiladi. Harakat tenglamalarini (bu holda koordinata ikkita ekanligini nazarda tutishimiz kerak) integrallash avvalgi paragrafdagidek amalga oshiriladi. (2.8) ifodadan

$$\frac{dr}{dt} = \dot{r} = \sqrt{\frac{m}{2}[E - U(r)] - \frac{M^2}{m^2 r^2}}; \quad dt = \frac{dr}{\sqrt{\frac{m}{2}[E - U(r)] - \frac{M^2}{m^2 r^2}}} \quad (2.11)$$

$$t = \int \frac{dr}{\sqrt{\frac{m}{2}[E - U(r)] - \frac{M^2}{m^2 r^2}}} + C_1 \quad (2.12)$$

(2.8) dan $\frac{d\varphi}{dt} = \dot{\varphi} = \frac{M}{2mr^2}, \quad d\varphi = \frac{M}{mr^2} dt$

ekanligini va (2.10) ni hisobga olsak,

$$\varphi = \int \frac{\frac{M}{r^2} dr}{\sqrt{\frac{m}{2}[E - U(r)] - \frac{M^2}{r^2}}} + C_2 \quad (2.13)$$

hosil bo'ladi. (2.12) va (2.13) lar qidirilgan yechimlardir. Lagranj tenglamasi har bir koordinata uchun ikkinchi tartibli oddiy differensial tenglama bo'lgani va masala ikki o'lovli bo'lgani uchun harakat tenglamalari yechimlarida to'rtta

o'zgaras kattalik bo'lishi kerak edi. Haqiqatan ham (2.12) va (2.13) da E, M, C_1, C_2 lar o'zgaras kattaliklardir.

Markazga eng yaqin va eng uzoq nuqtalar koordinatalarini energiya va impuls momentiga bog'liq holda aniqlash.

1-masala.
$$U = \frac{a}{r^2}, \quad a > 0.$$

Yechilishi. Yuqoridagi (2.10) ifodadan foydalanib, markazga eng yaqin va markazdan eng uzoq nuqtalar koordinatalarini aniqlash uchun quyidagi tenglamani yozamiz:

$$E = U \Rightarrow \frac{M^2}{2mr^2} + \frac{a}{r^2} = E,$$

Ko'rinib turibdiki,

$$r = \pm \sqrt{\frac{M^2 + 2ma}{2mE}}.$$

r manfiy bo'lishi mumkin bo'lmaganligi uchun harakat bir tomonlama cheklangan:

$$r = \sqrt{\frac{M^2 + 2ma}{2mE}}.$$

2-masala.
$$U = \frac{1}{2} \left(\frac{1}{r^2} - \frac{1}{r} \right), \quad m=1.$$

Yechilishi. Yana (2.10) ifodadan foydalansak:

$$\frac{M^2}{2r^2} + \frac{1}{2} \left(\frac{1}{r^2} - \frac{1}{r} \right) = E, \quad 2Er^2 + r - M^2 + 1 = 0, \quad r = \frac{-1 \pm \sqrt{1 + 8E(M^2 + 1)}}{4E}.$$

Agar $E > 0$ bo'lsa, kasr suratidagi ildiz birdan katta bo'lgani uchun tenglamaning yechimi bitta va harakat infinitdir.

$E < 0$ bo'lsa, $1 + 8E(M^2 + 1) > 0$ bo'lishi kerak.

Demak, $-\frac{1}{8(M^2 + 1)} < E < 0$ da harakat finit bo'ladi.

3-masala. Quyidagi potensial energiya bilan xarakterlanuvchi markaziy maydondagi harakat tenglamalari yechimlarini toping:

$$U = \frac{1}{2r^2}, \quad m=1.$$

Yechilishi. Buni (2.12) ga qo'ysak:

$$t = \int \frac{dr}{\sqrt{2E - \frac{1}{r^2} - \frac{M^2}{r^2}}} + t_0 = \int \frac{rdr}{\sqrt{2Er^2 - (M^2 + 1)}} + t_0 = \frac{1}{2E} \sqrt{2Er^2 - (M^2 + 1)} + t_0$$

$$t - t_0 = \frac{1}{2E} \sqrt{2Er^2 - (M^2 + 1)}.$$

Potensial energiya ifodasini (2.13) ga qo'ysak,

$$\varphi = \int \frac{\frac{M}{r^2} dr}{\sqrt{2E - \frac{1}{r^2} - \frac{M^2}{r^2}}} + \varphi_0 = - \int \frac{Md\left(\frac{1}{r}\right)}{\sqrt{2E - \frac{M^2 + 1}{r^2}}} + \varphi_0 = \frac{M}{\sqrt{1 + M^2}} \arccos \frac{\sqrt{M^2 + 1}}{\sqrt{2Er}} + \varphi_0.$$

$$\varphi - \varphi_0 = \frac{M}{\sqrt{1 + M^2}} \arccos \frac{\sqrt{M^2 + 1}}{\sqrt{2Er}}.$$

4-masala.
$$U = \frac{1}{2} \left(r^2 + \frac{1}{r^2} \right), \quad m=1.$$

Yechilishi. Bu ifodani (2.12) ga qo'yamiz.

$$t = \int \frac{dr}{\sqrt{2E - r^2 - \frac{1}{r^2} - \frac{M^2}{r^2}}} + t_0 = \int \frac{rdr}{\sqrt{2Er^2 - r^4 - (M^2 + 1)}} + t_0 = \int \frac{\frac{1}{2} d(r^2 - E)}{\sqrt{E^2 - (M^2 + 1) - (r^2 - E)^2}} + t_0 = \frac{1}{2} \arcsin \frac{r^2 - E}{\sqrt{E^2 - (M^2 + 1)}} + t_0.$$

$$r = \sqrt{E + \sqrt{E^2 - (M^2 + 1)} \cdot \sin 2(t - t_0)}.$$

Potensial energiyani (2.13) ga qo'ysak, quyidagicha bo'ladi:

f

$$\varphi = \int \frac{\frac{M}{r^2} dr}{\sqrt{2E - r^2 - \frac{1}{r^2} - \frac{M^2}{r^2}}} + \varphi_0 = \int \frac{\frac{M}{r^2} dr}{\sqrt{\frac{2E}{r^2} - 1 - \frac{M^2 + 1}{r^2}}} + \varphi_0 =$$

$$= -\frac{M}{2} \int \frac{d\left(\frac{1}{r^2}\right)}{\sqrt{\frac{2E}{r^2} - 1 - \frac{M^2 + 1}{r^2}}} + \varphi_0 = \frac{M}{2\sqrt{M^2 + 1}} \arccos \frac{\frac{M^2 + 1}{r^2} - E}{\sqrt{E^2 - M^2 - 1}} + \varphi_0.$$

$$r = \sqrt{\frac{M^2 + 1}{E + \sqrt{E^2 - M^2 - 1} \cos \left[2\sqrt{M^2 + 1} \frac{\varphi - \varphi_0}{M} \right]}}.$$

5-masala. Zarraning $U(r) = -\frac{\alpha}{r}$ maydondagi harakat integrallari topilsin

(Kepler masalasi).

Yechilishi. Ma'lumki, markaziy maydonda Lagranj funksiya

$$L = \frac{1}{2} m \dot{r}^2 - U(r) \quad \text{va} \quad \vec{F} = -\frac{\partial U}{\partial \vec{r}} = -\frac{\partial U}{\partial r} \frac{\partial \vec{r}}{\partial \vec{r}} = -\frac{\partial U}{\partial r} \frac{\vec{r}}{r}.$$

Harakat tenglamasi esa $m\ddot{r} = -\frac{\partial U}{\partial \vec{r}}$

1-harakat integrali- energiya quyidagicha

$$E = \frac{1}{2} m \dot{r}^2 + U(r) = \text{const.}$$

Impuls momentidan vaqt bo'yicha hosila olsak:

$$\frac{d}{dt} \vec{M} = \frac{d}{dt} [\vec{r} \vec{p}] = m [\vec{r} \ddot{\vec{r}}] + m [\dot{\vec{r}} \dot{\vec{r}}] = m [\vec{r} \ddot{\vec{r}}] = -\left[\vec{r}, \frac{\partial U}{\partial r} \frac{\vec{r}}{r} \right] = -\left[\vec{r} \frac{\vec{r}}{r} \right] \frac{\partial U}{\partial r} = 0.$$

Demak, yana uchta harakat integralini topdik.

$$\vec{M} = \vec{M}_0 = \text{const.}$$

Oxirgi ifodani \vec{r} ga skalyar ko'paytirib $\vec{M}_0 \vec{r} = 0$ tenglamani olamiz. Bu tenglama \vec{M}_0 ga perpendikulyar bo'lgan va koordinata boshidan o'tuvchi tekislikning tenglamasidir.

Bu harakat integralidan tashqari Kepler masalasida o'ziga xos harakat integrali mavjuddir. Harakat tenglamasi:

$$m\ddot{\vec{r}} + \alpha \frac{\vec{r}}{r^3} = 0$$

Bu maydondagi harakat integrallari yuqoridagilarga ko'ra quyidagicha:

$$\vec{M} = m [\vec{r} \dot{\vec{r}}], \quad E = \frac{m \dot{r}^2}{2} - \frac{\alpha}{r}.$$

Harakat tenglamasining ikkala tomonini \vec{M} ga vektor ko'paytiraylik:

$$m [\dot{\vec{r}} \vec{M}] = -\frac{\alpha}{mr^3} [\vec{r} \vec{M}]$$

va uning chap tomonida to'la hasiladan foydalansak

$$\frac{d}{dt} [\dot{\vec{r}} \vec{M}] = -\frac{\alpha}{r^3} (\vec{r} (\vec{r} \dot{\vec{r}}) - \dot{\vec{r}} r^2) = -\frac{\alpha}{r^3} (\vec{r} \cdot r \dot{\vec{r}} - \dot{\vec{r}} r^2) = \alpha \left(\frac{\dot{\vec{r}} r - \dot{r} \vec{r}}{r^2} \right) = \alpha \frac{d}{dt} \left(\frac{\vec{r}}{r} \right).$$

Demak,

$$\vec{e} = \left[\dot{\vec{r}} \vec{M} \right] - \frac{\alpha \vec{r}}{r} = \text{const.} \quad (2.14)$$

Mazkur harakat integralini birinchi bo'lib Laplas topganligi sababli bu vektorni *Laplas vektori* deb atashadi.

6-masala. Kepler masalasini parabolik koordinatalarda yeching.

Yechilishi. Parabolik koordinatalar dekart koordinatalar bilan quyidagicha bog'lanishdadir:

$$x = \sqrt{\xi \eta} \cos \varphi, \quad y = \sqrt{\xi \eta} \sin \varphi, \quad z = \frac{1}{2} (\xi - \eta), \quad 0 < \xi, \quad \eta < \infty,$$

$$r = \frac{1}{2} (\xi + \eta), \quad \dot{r}^2 = v^2 = \frac{1}{4} (\xi + \eta) \left(\frac{\dot{\xi}^2}{\xi} + \frac{\dot{\eta}^2}{\eta} \right) = \xi \eta \dot{\varphi}^2.$$

Masalani yechishda birinchi harakat integralidan foydalanamiz, ya'ni energiyaning saqlanish qonunidan:

$$E = \frac{m}{8}(\xi + \eta) \left(\frac{\dot{\xi}^2}{\xi} + \frac{\dot{\eta}^2}{\eta} \right) + \frac{m}{2} \xi \eta \dot{\phi}^2 - \frac{2\alpha}{\xi + \eta}$$

Impuls momentining z o'qiga proeksiyasi $M_z = m\xi\eta\dot{\phi}$.

Bu ifodalarni zarraning parabolik koordinatalardagi Lagranj funksiyasidan topsa ham bo'ladi:

$$L = \frac{m}{8}(\xi + \eta) \left(\frac{\dot{\xi}^2}{\xi} + \frac{\dot{\eta}^2}{\eta} \right) + \frac{m}{2} \xi \eta \dot{\phi}^2 + \frac{2\alpha}{\xi + \eta}$$

Laplas vektori (2.14) ning z o'qiga proeksiyasini topamiz:

$$\frac{m}{8}(\xi + \eta) \left(\frac{\dot{\eta}^2}{\eta} \xi - \frac{\dot{\xi}^2}{\xi} \eta \right) + \frac{m}{2} \xi \eta \dot{\phi}^2 (\xi - \eta) - \frac{\alpha(\xi - \eta)}{\xi + \eta} = C.$$

Energiya va mazkur ifodalardan $\dot{\phi}$ larni tushirib qoldirib, tenglamalarni ξ^2 va η^2 larga nisbatan yechib chiqamiz:

$$\frac{\dot{\xi}}{\xi\sqrt{W_1}} = \frac{2}{(\xi + \eta)}, \quad \frac{\dot{\eta}}{\eta\sqrt{W_2}} = \frac{2}{(\xi + \eta)},$$

$$W_1(\xi) = \frac{2}{m} \left(E + \frac{\alpha - c}{\xi} - \frac{M_z^2}{2m\xi^2} \right), \quad W_2(\eta) = \frac{2}{m} \left(E + \frac{\alpha + c}{\eta} - \frac{M_z^2}{2m\eta^2} \right).$$

Yuqoridagi tenglamalardan ikkita o'zgaruvchilari ajralgan tenglamalarni olamiz:

$$\frac{\dot{\xi}}{\xi\sqrt{W_1}} = \frac{\dot{\eta}}{\eta\sqrt{W_2}}, \quad \frac{\dot{\xi}}{\sqrt{W_1}} + \frac{\dot{\eta}}{\sqrt{W_2}} = 2$$

M_z va yuqoridagi ξ^2 hamda η^2 lar aniqlangan tenglamalardan uchinchi tenglamani topamiz:

$$\dot{\phi} = \frac{M_z}{2m} \left[\frac{\dot{\xi}}{\xi\sqrt{W_1}} + \frac{\dot{\eta}}{\eta\sqrt{W_2}} \right].$$

Oxirgi uchta tenglamalar odatda qiyinchiliksiz integrallanadi.

7-masala.
$$U = \frac{1}{2} m \omega^2 r^2.$$

Yechilishi.
$$E = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\phi}^2) + \frac{1}{2} m \omega^2 r^2, \quad M_0 = m r^2 \dot{\phi},$$

$$U_{eff} = \frac{M^2}{2mr^2} + \frac{m\omega^2 r^2}{2}, \quad \dot{r}^2 = \frac{2}{m} [E - U_{eff}(r)] \quad (2.15)$$

$\dot{r} = 0$ shartidan ($E = U$) r koordinata bo'yicha harakat holatlarini topishimiz mumkin (burilish nuqtalari): $b < r < a$

$$a = \frac{1}{2} (\sqrt{C_1^2 + 2C_2^2} + \sqrt{C_1^2 - 2C_2^2}), \quad b = \frac{1}{2} (\sqrt{C_1^2 + 2C_2^2} - \sqrt{C_1^2 - 2C_2^2})$$

$$C_1 = \sqrt{\frac{2E}{m\omega^2}}, \quad C_2 = \sqrt{\frac{M}{m\omega}}. \quad (2.16)$$

harakat tenglamalarini (2.12), (2.13) tenglamalardan topiladi. Bu tenglamalardan trayektoriya tenglamasini topish uchun boshlang'ich shartlarni qo'llaylik:

$$r(0) = r_0, \quad v(0) = v_0$$

U holda (2.13) dan

$$\varphi = C_2^2 \int_a^r \frac{dr}{r^2 \sqrt{C_1^2 - \frac{C_2^2}{r^2} - r^2}} + \varphi_m,$$

bu yerda φ_m $r(\varphi_m) = b$ shartdan topiladi.

Yuqoridagi integralni $u = \frac{1}{r}$ almashtirish orqali integrallasak:

$$r^2 = \frac{2C_2^4}{C_1^2 + \sqrt{C_1^4 - 4C_2^4} \cos 2(\varphi - \varphi_m)} \quad (2.17)$$

(2.16) ifodadan foydalanib (2.17) ni qaytadan yozib olsak trayektoriya yarim o'qlari a va b lardan iborat ellips ekanini ko'rish qiyin emas:

$$r^2 = \frac{a^2 b^2}{a^2 \cos^2(\varphi - \varphi_m) + b^2 \sin^2(\varphi - \varphi_m)} \quad (2.18)$$

(2.14) tenglamani integrallasak:

$$r^2 = \frac{1}{2} [C_1^2 - \sqrt{C_1^4 - 4C_2^4} \cos 2\omega(t - t_m)], \quad (2.19)$$

bu yerda t_m $r = b$ nuqtadan $t = 0$ momentga eng yaqin vaqt momentida o'tish vaqtidir. (2.19) ifodani (2.18) ga monand ravishda yozib olaylik:

$$r^2 = b^2 \cos^2 \omega(t-t_m) + a^2 \sin^2 \omega(t-t_m) \quad (2.20)$$

(2.20) va (2.14) dan $\varphi(t) = \arctg \left[\frac{a}{b} \operatorname{tg} \omega(t-t_m) \right] + \varphi_m$

2.4-misollar.

1. Markazga eng yaqin va eng uzoq nuqtalar koordinatasi topilsin.

a) $U = \frac{a}{r}$.

b) $U = \frac{1}{r^4}$.

d) $U = \frac{r^2 + \frac{1}{r^2}}{2}$, $m = 1$.

e) $U = -U_0 \left(1 + \frac{r^2}{c^2} \right)^{-2}$.

2. Harakat tenglamasi integrallansin.

a) $U = \frac{1}{2r^2}$, $m = 1$.

b) $U = \frac{1}{2} \left(\frac{1}{r^2} - \frac{1}{r} \right)$,

$m = 1$.

d) $U = -U_0 \left(\frac{r_0}{r} \right)^2 \ln \frac{r}{r_0}$, $E = 0$, $z = 0$.

III. KICHIK TEBRANISHLAR

Mexanik sistemaning *barqaror (turg'un) muvozanat holati* yaqinidagi harakatini odatda *kichik tebranishlar* deb nomlanadi. Bunday harakatlarning eng soddasi erkinlik darajasi birga teng bo'lgan holi (bir o'lchovli erkin harakat) quyidagi Lagranj funksiyasi bilan xarakterlanadi:

$$L = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2} \quad (3.1)$$

Bu Lagranj funksiyaga mos harakat tenglamasi esa, ushbu ko'rinishga ega

$$\ddot{x} + \omega^2 x = 0 \quad (3.2)$$

Bu tenglamadan ko'rinadiki, sistema barqaror muvozanat holati yaqinida garmonik tebranma harakat qiladi (garmonik ossilyator, chiziqli garmonik ossilyator modeli atamaları ham ishlatiladi):

$$x = C_1 \cos \omega t + C_2 \sin \omega t, \quad (3.3)$$

$$x = a \cos(\omega t + \alpha) \quad (3.3')$$

a - tebranishlar amplitudasi, $a = \sqrt{C_1^2 + C_2^2}$, α - boshlang'ich faza,

$$\operatorname{tg} \alpha = -\frac{C_2}{C_1}, \quad \omega = \sqrt{\frac{k}{m}} \text{ - chastota (siklik chastota).}$$

Sistemaning energiyasi $E = \frac{1}{2} m \omega^2 a^2$

(3.4)

tebranish amplitudasi a ning kvadratiga proporsionaldir.

Ko'pincha (3.3) ga mos bog'lanishni quyidagi

$$x = \operatorname{Re} \{ A e^{i\omega t} \}, \quad A = a e^{i\alpha} \quad (3.5)$$

kompleks ifodaning xaqiqiy qismi sifatida olish qulay bo'ladi. A ni *kompleks amplituda* deyiladi.

Mexanik sistemaning tashqi maydon ta'siridagi tebranishlari *majburiy tebranishlar* deyiladi. Bu holdagi tebranishlar kichik bo'lganligi sababli, tashqi

maydon kuchsiz deb faraz qilinadi, aks holda tashqi maydon katta siljishlarga olib kelishi mumkin. Bir o' lchamli majburiy harakat uchun:

$$L = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2} + xF(t) \quad (3.6)$$

va harakat tenglamasi
$$\ddot{x} + \omega^2 x = \frac{1}{m} F(t)$$

(3.7)

dan iborat bo'ladi. Jumladan, tashqi kuchning ko'rinishi $F(t) = f \cos(\gamma t + \beta)$ bo'lgan holda majburiy tebranishlar koordinatasining o'zgarishi esa:

$$x = a \cos(\omega t + \alpha) + \frac{f}{m(\omega^2 - \gamma^2)} \cos(\gamma t + \beta) \quad (3.8)$$

ifoda bilan aniqlanadi. Bu erda γ - majbur etuvchi kuchning chastotasi.

Rezonans holida $\omega \approx \gamma$ kichik tebranishlar uchun:

$$x = a \cos(\omega t + \alpha) + \frac{f}{2m\omega} t \sin(\omega t + \beta) \quad (3.9)$$

Erkinlik darajasi ko'p (S ga teng deydik) bo'lgan sistemaning tebranishi quyidagicha xarakterlanadi:

$$L = \frac{1}{2} \sum_{i,k} (m_{i,k} \dot{x}_i \dot{x}_k - k_{i,k} x_i x_k) \quad (3.10)$$

$$\sum_k (m_{i,k} \ddot{x}_k + k_{i,k} x_k) = 0 \quad (3.11)$$

Bular o'zgarimas koeffitsiyentli S ta chiziqli bir jinsli tenglamalar sistemasidan iboratdir. Bu yerda x_i - barqaror muvozanat holatidan kichik og'ishlar (siljishlar).

Bunday tenglamalarni yechishning umumiy qoidasiga asosan (3.11) tenglamalar quyidagi

$$\sum_k (-\omega^2 m_{i,k} + k_{i,k}) A_k = 0 \quad (3.12)$$

algebraik tenglamalar sistemasiga keltirib olinadi. Bu sistema noldan farqli yechimga ega bo'lishi uchun uning aniqlovchisi (xarakteristik tenglama)

$$|k_{i,k} - \omega^2 m_{i,k}| = 0 \quad (3.13)$$

bo'lishi kerak. Umumiy holda bu tenglama S ta haqiqiy musbat $\omega_\alpha^2 (\alpha = 1, 2, \dots, S)$ ildizga ega (odatda xususiy chastotalar deb nomlanadi).

Umumiy holda (3.11) ning echimi

$$x_k = \sum_{\alpha} \Delta_{k,\alpha} C_{\alpha} e^{i\omega_{\alpha} t} = \sum_{\alpha} \Delta_{k,\alpha} \theta_{\alpha}, \quad \theta_{\alpha} = \text{Re} \{ C_{\alpha} e^{i\omega_{\alpha} t} \} \quad (3.14)$$

ko'rinishda bo'ladi. Bu yerda $\Delta_{k,\alpha}$ - (3.13) aniqlovchining ω_{α} ga mos minori.

Shunday qilib, sistema koordinatalaridan har birining vaqt bo'yicha o'zgarishi ixtiyoriy amplituda va fazali, lekin aniq chastotali S ta $\theta_1, \theta_2, \dots, \theta_S$ oddiy davriy tebranishlarning qo'shilishidan iborat ekan. Har biri faqat birgina oddiy tebranishda bo'la oluvchi umumlashgan koordinatalar *normal koordinatalar* deyiladi. Oddiy davriy tebranishlarni esa, *normal tebranishlar* deyiladi. Ta'rifga binoan normal tebranishlar uchun:

$$\ddot{\theta}_{\alpha} + \omega_{\alpha}^2 \theta_{\alpha} = 0 \quad (3.15)$$

So'navchi tebranishlar tenglamasi:

$$\ddot{x} + 2\lambda \dot{x} + \omega_0^2 x = 0 \quad (3.16)$$

λ - so'nish koeffitsienti (λT - so'nishning logarifmik dekrementi).

ω_0 - ishqalanish bo'lmagandagi erkin tebranishlar chastotasi.

(3.16) tenglamaning echimi quyidagichadir:

$$x = C_1 e^{r_1 t} + C_2 e^{r_2 t}, \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2} \quad (3.17)$$

Bu yerda uch hol bo'lishi mumkin:

$$1) \quad \lambda < \omega_0, \quad x = a e^{-\lambda t} \cos(\omega t + \alpha) \quad (3.18)$$

$\omega = \sqrt{\omega_0^2 - \lambda^2}$ - "chastotali" so'navchi tebranishlar.

$$2) \quad \lambda > \omega_0, \quad x = C_1 e^{-\left(\lambda - \sqrt{\lambda^2 - \omega_0^2}\right)t} + C_2 e^{-\left(\lambda + \sqrt{\lambda^2 - \omega_0^2}\right)t} \quad (3.19)$$

$$3) \quad \lambda = \omega_0, \quad x = (C_1 + C_2 t) e^{-\lambda t} \quad (3.20)$$

Oxirgi ikki holda, ya'ni so'nish koeffitsiyenti yetarli ravishda katta bo'lganda harakat nodavriy so'navchi harakatdan iborat bo'ladi. So'navchi

harakatni xarakterlovchi umumlashgan ishqalanish kuchlari tezlikning kvadratiga proporsional bo'lgan dissipativ funksiyalar- F orqali aniqlanadilar:

$$f_{\text{umuk}} = -\frac{\partial F}{\partial \dot{x}_i} \quad (3.21)$$

Bu kuchlar Lagranj tenglamasining o'ng tomonida ishtirok etishlari kerak:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} = \frac{\partial L}{\partial x_i} - \frac{\partial F}{\partial \dot{x}_i} \quad (3.22)$$

Ishqalanish kuchlari ta'sirida energiyaning o'zgarishi (ajralishi, dissipasiyasi) dissipativ funksiya orqali quyidagicha aniqlanadi:

$$\frac{dE}{dt} = \dot{E} = -2F \quad (3.23)$$

§3.1. Berilgan Lagranj funksiyasiga oid erkin tebranma harakatning barqaror muvozanat holati va chastotasini topish

1-masala. Lagranj funksiyasi $L = \frac{\dot{x}^2}{2} + \sin x$ dan iborat sistemaning kichik tebranishlar chastotasini toping.

Yechilishi. Bu masalani yechishdan oldin bir o'lchamli harakatga mos chastotani topishning umumiy metodini ko'rsataylik. Aytaylik,

$$L = \frac{f(q)\dot{q}^2}{2} - U(q) \quad (3.24)$$

bo'lsin, $q = q_0$ nuqta barqaror muvozanat nuqtasi bo'lsin. U holda $U'(q = q_0) = 0$ bo'ladi, binobarin nazariyaga binoan bu nuqtda $U''(q = q_0) > 0$ bo'lishi kerak. Natijada, barqaror muvozanat atrofida Lagranj funksiyasi

$$L = \frac{f(q_0) \cdot \dot{q}^2}{2} - \frac{U''(q_0)}{2} q^2$$

ko'rinishga keladi. Harakat tenglamasi esa

$$f(q_0)\ddot{q} + U''(q_0)q = 0$$

ko'rinishda yoziladi. Erkin tebranishning chastotasi esa ushbu formula bilan aniqlanadi

$$\omega = \sqrt{\frac{U''(q_0)}{f(q_0)}} \quad (3.25)$$

Bizning masala uchun $U = -\sin x$, $U' = -\cos x = 0$,

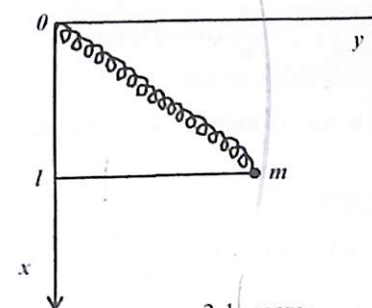
shartidan $x_0 = \frac{\pi}{2}$. $U'' = \sin x \Rightarrow U''\left(\frac{\pi}{2}\right) = 1 > 0$ ga tengligidan bu nuqta

barqaror muvozanat holidir. (3.24) ifodaga binoan $f(x) = 1$ ga teng, yani

$$f(x_0) = 1. \text{ Demak, chastota } \omega = \sqrt{\frac{U''(x_0)}{f(x_0)}} = 1.$$

2-masala. Massasi m bo'lgan zarraning prujina ta'sirida gorizontol sterjen bo'ylab tebranma harakatining barqaror muvozanat holati topilsin.

Prujining bir uchi silliq sterjenã ulangan, ikkinchi uchi esa sterjendan l masofada yotuvchi O nuqtaga mahkamlangan. Prujining kuch ta'sir etmagan holdagi uzunligi a , bikrligi k deb olinsin.



3.1.-rasm

Yechilishi. Koordinata sistemasini 3.1.-rasmda ko'rsatilgandek tanlab olsak, zarraning kinetik va potensial energiyalari quyidagicha

$$T = \frac{m\dot{y}^2}{2}, \quad U = \frac{k}{2}(\sqrt{l^2 + y^2} - a)^2$$

bo'ladi. Barqaror muvozanat holatini topish uchun U dan olingan birinchi va ikkinchi tartibli hosilalarni hisoblab olaylik,

$$\frac{\partial U}{\partial y} = k(\sqrt{l^2 + y^2} - a) \cdot \frac{y}{\sqrt{l^2 + y^2}},$$

$$\frac{\partial^2 U}{\partial y^2} = \frac{k}{\sqrt{l^2 + y^2}} \left\{ \frac{y^2}{\sqrt{l^2 + y^2}} + \left(\sqrt{l^2 + y^2} - a \right) \cdot \frac{l^2}{l^2 + y^2} \right\}$$

$\frac{\partial U}{\partial y} = 0$ ekanligidan uchta muvozanat holatini topamiz:

$$y_{\text{muv}1} = 0 \quad (a < l) \quad y_{\text{muv}2,3} = \pm \sqrt{a^2 - l^2} \quad (a > l)$$

Har bir muvozanat holatining barqarorlik shartini tekshirib chiqaylik. Ma'lum ediki, barqaror (turg'un) muvozanat holida ikkinchi tartibli hosila musbat

$\left(\frac{\partial^2 U}{\partial y^2} \right)_{y_{\text{muv}}} > 0$ bo'lishi kerak. Birinchi holat barqaror holat chunki

$\left(\frac{\partial^2 U}{\partial y^2} \right)_{y_{\text{muv}1}} = \frac{k}{l}(l-a) > 0$ agarda $l > a$ bo'lsa. Qolgan ikki holat $l < a$ dagina

mavjud bo'lib, barqaror holatlaridir, chunki

$$\left(\frac{\partial^2 U}{\partial y^2} \right)_{y_{\text{muv}2,3}} = \frac{k}{a^2}(a^2 - l^2) > 0$$

Demak, sistema $y_{\text{muv}1} = 0$ ($a < l$) nuqta, $a > l$ da esa $y_{\text{muv}2,3} = \pm \sqrt{a^2 - l^2}$ nuqtalar atrofida

$$\omega_1 = \sqrt{\frac{k}{ml}(l-a)} \quad \text{va} \quad \omega_{2,3} = \sqrt{\frac{k}{ma^2}(a^2 - l^2)}$$

chastotalar bilan tebranma harakat qiladi. Lagranj tenglamasining $y = y_{\text{muv}}$ atrofidagi yechimi (3.3) ga ko'ra:

$$y(t) = a \cos(\omega_1 t + \alpha)$$

2-usul. Shu masalani umumlashgan koordinata φ ni kiritish yo'li bilan yechishni ko'rsataylik:

$$l = a \cos \varphi, \quad y = \frac{l \sin \varphi}{\cos \varphi}, \quad L = \frac{m(\varphi)\dot{\varphi}^2}{2} - U(\varphi) = \frac{ml^2\dot{\varphi}^2}{2\cos^4 \varphi} - \frac{k}{2} \left(\frac{l}{\cos \varphi} - a \right)^2$$

Barqaror muvozanat holati

$$\frac{\partial U}{\partial \varphi} = -k \left(\frac{l}{\cos \varphi} - a \right) \frac{l \sin \varphi}{\cos^2 \varphi} = 0$$

ifodadan aniqlanadi: $\varphi_1 = 0$, $l > a$ da ; $\cos \varphi_{2,3} = \frac{l}{a}$, $l < a$ da

Ikkinchi tartibli hosilalarni hisoblasak:

$$\left(\frac{\partial^2 U}{\partial \varphi^2} \right)_{\varphi} = k \left(\frac{l \sin \varphi}{\cos^2 \varphi} \right)^2 + kl \left(\frac{l}{\cos \varphi} - a \right) \left(\frac{1}{\cos \varphi} + \frac{2 \sin^2 \varphi}{\cos^3 \varphi} \right)$$

$$\left(\frac{\partial^2 U}{\partial \varphi^2} \right)_{\varphi=0} = kl(l-a), \quad \left(\frac{\partial^2 U}{\partial \varphi^2} \right)_{\varphi=\varphi_{2,3}} = k(a^2 - l^2) \frac{a^2}{l^2}$$

Chastota esa quyidagicha aniqlanadi:

$$\omega^2 = \frac{U''(\varphi_{\text{muv}})}{m(\varphi_{\text{muv}})}, \quad U''(\varphi_{\text{muv}}) > 0 \quad \text{va u yuqoridagi ko'rinishga ega.}$$

3-masala. $L = \frac{\dot{x}^2 - x^2 - 1}{2x}$ ga mos chastota topilsin.

Yechilishi. Bu ifodani quyidagicha yozib olaylik,

$$L = \frac{\dot{x}^2}{2x} - \frac{x^2 + 1}{2x}, \quad f(x) = \frac{2}{x}, \quad U(x) = \frac{x^2 + 1}{2x}, \quad U'(x) = \frac{x^2 - 1}{2x^2}$$

Bu yerdan $x_0 = 1$ nuqta barqaror muvozanat nuqtasi bo'ladi:

$$U''(x) = \frac{1}{x^3}, \quad U''(x_0 = 1) = 1, \quad f(x) = \frac{1}{x}, \quad f(x_0 = 1) = 1 \text{ dir,}$$

$$\text{Demak,} \quad \omega = \sqrt{\frac{U''(x_0)}{f(x_0)}} = 1.$$

4-masala. Quyidagi qonun bo'yicha $U = -\frac{\ln r}{r^2}$ o'zaro ta'sirlashayotgan ikki zarra aylanma harakatining tebranish chastotasini toping (zarraning massasi m ga teng).

Yechilishi. Masalani qutb koordinat sistemasida qaraylik. Bu sistemaning keltirilgan massasi $\frac{m}{2}$ ga teng. U holda sistemaning to'la energiyasi

$$E = T + U = \frac{m}{4} (r^2 \dot{\varphi}^2 + \dot{r}^2) - \frac{\ln r}{r^2}$$

Masalaning qo'yilishidan ayonki, siklik koordinata φ ga mos umumlashgan impuls $P_\varphi = \frac{m}{2}(r^2 \dot{\varphi}^2)$ doimiydir (saqlanuvchi). To'la energiyani qayta yozib olaylik: $E = \frac{m\dot{r}^2}{4} + \left(\frac{P_\varphi^2}{mr^2} - \frac{\ln r}{r^2} \right)$. Bu ifodaning ikkinchi hadi potentsial

energiya rolini oynaydi (effektiv potentsial) $U(r) = \frac{P_\varphi^2}{mr^2} - \frac{\ln r}{r^2}$,

Bu yerdan hosilalarni topib olaylik:

$$U'(r) = \frac{2}{r^3} \left(\ln r - \frac{1}{2} - \frac{P_\varphi^2}{m} \right), \quad U''(r) = \frac{6}{r^4} \left(-\ln r + \frac{5}{6} + \frac{P_\varphi^2}{m} \right)$$

Barqaror muvozanat nuqtasi r_0 esa, $U'(r) = 0$ ifodadandan topiladi, hamda ikkinchi tartibli hosilaning bu nuqtadagi qiymati musbat ekanligi ko'rinib turibdi:

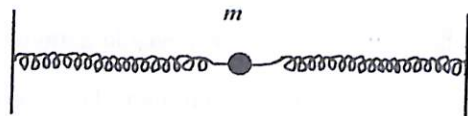
$$r_0 = e^{\frac{1 + \frac{P_\varphi^2}{m}}{2}}, \quad U''(r_0) = 2e^{-\frac{4P_\varphi^2}{m} - 2} > 0.$$

Kinetik energiyaning ifodasidan esa $f(r_0) = \frac{m}{2}$ ekanligi ko'rinib turibdi. Demak,

$$\omega = \sqrt{\frac{U''(r_0)}{f(r_0)}} = \frac{2}{\sqrt{m}} e^{-\left(\frac{2P_\varphi^2}{m} + 1\right)}$$

5-masala. 3.2.-rasmda ko'rsatilgan zarraning bo'ylama harakatining chastotasini toping. Prujinalarning bikrligi k va l ga teng.

Yechilishi. Bikrligi k va l ga teng prujinalarning potentsial energiyalari mos ravishda $\frac{kx^2}{2}$ va $\frac{lx^2}{2}$ ga teng.



3.2.-rasm.

Bu erda x - zarraning barqaror muvozanat nuqtasidan og'ish masofasi. Chizmadagi sistemaning Lagranj funksiyasini tuzaylik:

$$L = \frac{m\dot{x}^2}{2} - \frac{(k+l)x^2}{2} = T - U.$$

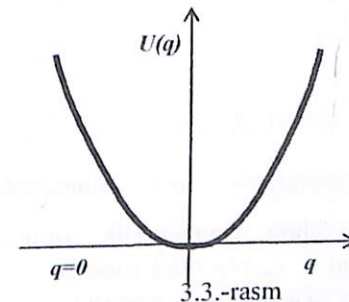
Bu erdan $\omega = \sqrt{\frac{k+l}{m}}$.

6-masala. Zarraning $U(q, k) = \frac{kq^2}{2} + \frac{cq^3}{4}$, $c > 0$ potentsial maydondagi harakati tahlil qilinsin.

Yechilishi. Barqaror muvozanat holati $kq + cq^3 = 0$ tenglamadan topiladi. Aksariyat real hollarda o'zaro ta'sir energiyasi boshqaruvchi parametrlar deb nomlanuvchi qo'shimcha o'zgaruvchilarga bog'liq bo'ladi. Bizning holda k shunday parametrdir.

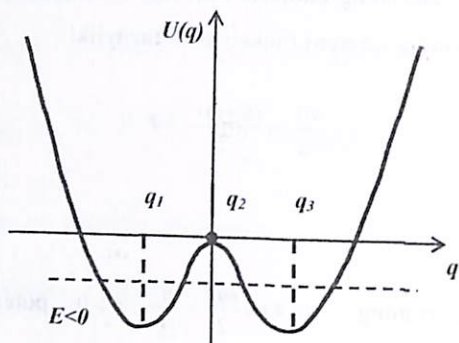
a) Agarda $k > 0$ bo'lsa, yolg'iz yechim $q = 0$, turg'un holat bo'ladi (3.3.-rasm).

$$\left(\frac{\delta^2 U}{\delta q^2} \right)_{q=0} = k$$



b) $k < 0$ bo'lsa, uchta yechim mavjud (3.4.-rasm):

$$q_{1,3} = \mp \left(\frac{|k|}{c} \right)^{\frac{1}{2}}, \quad \left(\frac{\delta^2 U}{\delta q^2} \right)_{q=q_{1,3}} = 2|k|, \quad q_2 = 0, \quad \left(\frac{\delta^2 U}{\delta q^2} \right)_{q_2} = |k|$$



3.4.-rasm

Bu holda ikki simmetrik turg'un holat $q_{1,3} = 0$ mavjud. $q_2 = 0$ nuqta esa $k > 0$ holda turg'un edi, endi noturg'un muvozanat holatiga aylanib qoldi, ya'ni parameter k musbat qiymatlardan manfiyga tomon o'zgarib borganda $q_2 = 0$ holat turg'un dan noturg'un holatga aylanadi. Ushbu noturg'unlikni simmetriyani buzuvchi noturg'unlik deyish mumkin, chunki zarra yo chap, yo o'ng tomondagi potensial chuqurlikka tushib qoladi. Turg'un holat $q = q_3$ atrofida $E < 0$ holdagi yechim (3.26) ga binoan:

$$q(t) = q_3 + a \cos(\omega_0 t + \alpha), \quad \omega_0 = \sqrt{\frac{2|k|}{m}}$$

Binobarin o'zaro ta'sir energiyasi $U(q, k)$ - simmetrik funksiyasidir:

$U(q, k) = U(-q, k)$ ammo yechim simmetriklik xususiyatiga ega emas. Sistemaning muvozanat holati $q_{muv}(k)$ ning parametr $U(q, k)$ ning kritik nuqtasi $k = 0$ dan o'tishda o'zgarishi *bifurkasiya* (ikkilanish) deyiladi.

7-masalasi. Radiusi a bo'lgan aylana vertikal tekislikda markazidan o'tgan o'q atrofida Ω burchak tezlik bilan tekis aylanyapti. Shu aylana bo'ylab harakatlanayotgan zarraning turg'un holat atrofidagi harakati chastotasi topilsin.

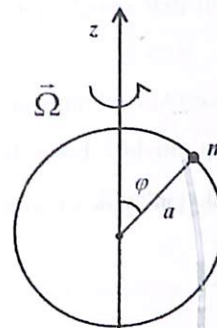
Yechilishi. Aylana bilan bog'liq bo'lgan noinersional sanoq sistemasining Z o'qini aylanish o'qi bilan ustma-ust joylashtiraylik. Zarraning aylana

tekisligidagi harakatini yoritish uchun umumlashgan koordinata sifatida qutb burchagi φ ni tanlaylik. Zarraning kinetik energiyasi

$$T = \frac{m}{2} a^2 \dot{\varphi}^2.$$

Potensial energiya ikki qismdan iborat: birinchisi-og'irlik maydonidagi potensial energiya, ikkinchisi-markazdan qochma energiya

$$U = m g a \cos \varphi - \frac{1}{2} m [\bar{a} \bar{\Omega}]^2 = m g a \cos \varphi - \frac{1}{2} m a^2 \Omega^2 \sin^2 \varphi.$$



3.5.-rasm.

Muvozanat holatlari

$$U' = -\sin \varphi (m g a + m a^2 \Omega^2 \cos \varphi) = 0$$

shartdan topiladi: $\varphi_1 = 0, \quad \varphi_2 = \pi, \quad \varphi_{3,4} = \arccos\left(-\frac{g}{a \Omega^2}\right)$

$$\left(\frac{\partial^2 U}{\partial \varphi^2}\right)_{\varphi=\varphi_1} = -(m g a + m a^2 \Omega^2) < 0$$

demak φ_1 - noturg'un holat. Qolgan nuqtalarda:

$$\left(\frac{\partial^2 U}{\partial \varphi^2}\right)_{\varphi=\varphi_2} = (m g a - m a^2 \Omega^2) > 0, \quad \Omega^2 < \frac{g}{a}$$

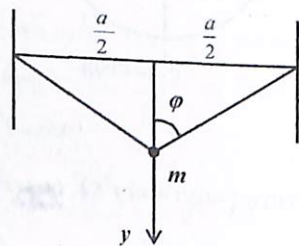
$$\left(\frac{\partial^2 U}{\partial \varphi^2}\right)_{\varphi=\varphi_{3,4}} = m a^2 \Omega^2 - m \left(\frac{g}{\Omega^2}\right)^2 > 0, \quad \Omega^2 > \frac{g}{a}$$

Demak, oxirgi ikki holat turg'un muvozanat holatlari ekan va bu holatlar atrofida tebranma harakatning chastotalari quyidagicha:

$$\omega_{2,3}^2 = \frac{g}{a} - \Omega^2, \quad \omega_{3,4}^2 = \Omega^2 - \left(-\frac{g}{a\Omega^2}\right)^2.$$

8-masala. Bir hil balandlikda turgan oralig'i a bo'lgan ikki parallel sterjen ustidan uzunligi $2a$ ga teng elastik ip tashlangan va uning uchlari m massali sharcha bilan birgalikda biriktirilgan. Muvozanat holatida ip teng tomonli uchburchak hosil qiladi deb sharchaning vertikal tebranish chastotasi topilsin.

Echilishi. Sharchaning vertikal tebranishini xarakterlash uchun sharcha bilan birlashgan ip uchlarning bir-biri bilan hosil qilgan 2φ burchakni umumlashgan koordinata sifatida tanlaylik va sharchaning Lagranj funksiyasini tuzaylik:



3.6.-rasm

$$L = T - U, \quad T = \frac{m}{2} \left[\frac{d}{dt} \left(\frac{a}{2} \text{ctg} \varphi \right) \right]^2 = \frac{1}{8} \frac{m a^2 \dot{\varphi}^2}{\sin^4 \varphi}.$$

Potensial energiya ikki qismdan iborat. Birinchisi-kichik tebranishga xos elastiklik koeffitsiyenti (bikrlilik) ga bog'liq bo'lgan kichik og'ishlarning potensial energiyasi:

$$U_1 = \frac{1}{2} k (\delta y)^2 = \frac{k}{2} \left[\left(\frac{a}{2 \sin \varphi} - \frac{a}{2 \sin \varphi} \right) - a \right]^2 = \frac{k}{2} \left(\frac{a}{\sin \varphi} - a \right)^2$$

Ikkinchisi esa, og'irlik maydoni ta'siridagi potensial energiya:

$$U_2 = mgy = -mg \frac{a}{2 \sin \varphi} \cos \varphi = -\frac{m}{2} g a \text{ctg} \varphi$$

Shunday qilib, potensial energiya ushbu ko'rinishni oladi:

$$U(\varphi) = \frac{a}{2} m g \text{ctg} \varphi + \frac{k}{2} \left(\frac{a}{\sin \varphi} - a \right)^2$$

Barqaror muvozanat holati esa

$$\frac{\partial U}{\partial \varphi} = \frac{m g a}{2 \sin^2 \varphi} - k a^2 \left(\frac{1}{\sin \varphi} - 1 \right) \frac{\cos \varphi}{\sin^2 \varphi} = 0$$

shartidan topiladi. Bu ifoda faqatgina $\varphi = \frac{\pi}{6}$ da nolga teng bo'lib, noma'lum

bo'lgan koeffitsient k ni aniqlashga imkon beradi: $k = \frac{m g}{a \sqrt{3}}$

Bu ifodalardan foydalanib ikkinchi tartibli hosilaning qiymatini topamiz:

$$\left(\frac{\partial^2 U}{\partial \varphi^2}\right)_{\varphi=\frac{\pi}{6}} = \frac{14}{\sqrt{3}} m g a > 0$$

Demak, sharcha $\varphi = \frac{\pi}{6}$ holat atrofida $\omega = \frac{7}{2\sqrt{3}} \frac{g}{a}$ chastota bilan

vertikal tebradi.

3.1. – Misollar.

1. Berilgan Lagranj funksiyasiga mos kichik erkin tebranishlarning chastotasini toping.

a) $L = \dot{x}^2 - x^2 e^x$.

v) $L = \frac{\dot{x}^2}{2} - \frac{x}{\ln x}$.

d) $L = \frac{m\dot{x}^2}{2} - (V \cos \alpha x - Fx)$.

b) $L = chx(\dot{x}^2 - 1)$.

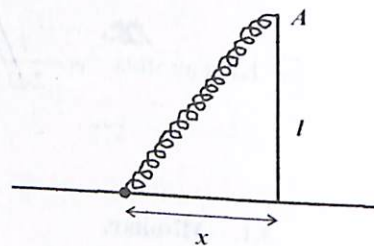
g) $L = \frac{m\dot{x}^2}{2} - \frac{kx^2}{2}$.

e) $L = \frac{m\dot{x}^2}{2} - \left(\frac{kx^2}{2} - F_0 x \right)$, $F_0 = const$.

2. Turli izotop atomlaridan tarkib topgan ikkita ikki atomli molekula tebranish chastotalari ω va ω' ning nisbatini toping, atomlar massasi mos ravishda m_1, m_2 va m'_1, m'_2 ga teng.

3. 3.1.-rasmدا berilgan sistemaning chastotasi topilsin.

4. To'g'ri chiziq bo'yicha harakatlana oluvchi va bir uchi A nuqtada bo'lgan prujining ikkinchi uchiga birlashtirilgan m massali nuqtaning (3.7-rasm) tebranish chastotasi va barqaror muvozanat holati aniqlansin. Prujining tinch holatdagi uzunligi l ga teng bo'lib, F kuch ta'sirida tortiladi ($x \ll l, U = F \delta l$ deb qaralsin).

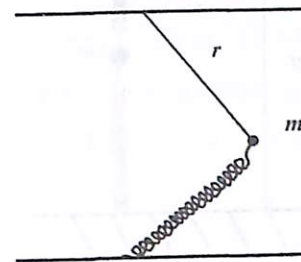


3.7.-rasm

5. 3.8.-rasmدا ko'rsatilgan sistemaning barqaror muvozanat holati va chastotasi topilsin. Muvozanat holatida prujinaga F kuch ta'sir etadi, va uning uzunligi l ga

teng. Ko'rsatma: $\delta l = \sqrt{[(\vec{r} + \vec{l}) - \vec{r}]^2} - l$ va bu vektorlar orasidagi φ burchak uchun

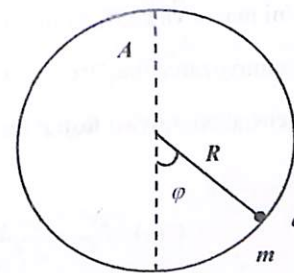
$\cos \varphi \equiv 1 - \frac{\varphi^2}{2}$ deb olinsin.



3.8.-rasm

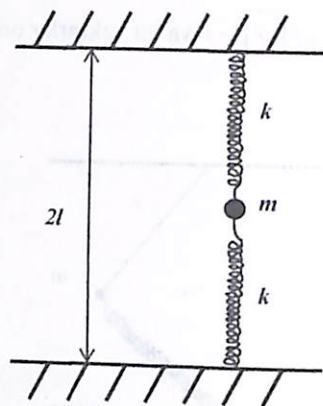
6. Noturg'un muvozanat nuqtasi atrofida zarra koordinatasining vaqtga bog'liqligini toping. Lagranj funksiyasini tuzing.

7. Massasi m zaryadi e bo'lgan zarra radiusi R bo'lgan vertikal vaziyatdagi aylana bo'ylab og'irlik kuchi ta'sirida harakatlanadi. Aylananing eng quyi nuqtasida zaryadi e ga teng bo'lgan aynan shunday zarra turibdi. Muvozanat holati va tebranish chastotasi topilsin. (3.9.-rasm).



3.9.-rasm.

8. 3.10.-rasmدا ko'rsatilgan sistemaning og'irlik kuchi ta'siridagi erkin tebranishi topilsin. Lagranj funksiyasi tuzilsin. Muvozanat nuqtasi, chastotasi topilsin.



3.10.-rasm.

9. Kichik tebranishlarning amplitudasi va fazasini

a) koordinata, tezlik hamda chastotaning,

b) koordinata va energiyaning funksiyasi sifatida aniqlang.

§3.2. Majburiy tebranishlar.

1-masala. Kichik doimiy tashqi kuch F_0 ta'sirida muvozanat holatining $x_0 = 0$ siljishini toping. Uni massa va chastota orqali ifodalang.

Yechilishi. Sistema muvozanat nuqtasi $x_0 = 0$ atrofida kichik tebranishlar bo'lgani uchun uning potensial energiyasi tashqi kuch ta'sir etguncha quyidagi ko'rinishda bo'ladi:

$$U(x) = \frac{kx^2}{2} = \frac{m\omega^2 x^2}{2}$$

Mavjud tashqi kuch ta'sirida potensial energiya o'zgaradi:

$$U(x) = \frac{m\omega^2 x^2}{2} - F_0 x$$

Muvozanat nuqtasi x_0 esa shu potensial energiyadan olingan hosilaning nolga tengligidan topiladi:

$$U'(x) = m\omega^2 x_0' - F_0 = 0, \text{ ya'ni } x_0' = \frac{F_0}{m\omega^2}$$

Demak, o'zgarmas kuch tebranishning muvozanat holatini siljitadi.

2-masala. Boshlang'ich $t=0$ momentda sistema muvozanat holatida $x=0$ tinch turgan $\dot{x}=0$ deb olib, $F(t) = F_0 \cos \alpha t$ kuch ta'sirida sistemaning majburiy tebranishini aniqlang. Erkin tebranish chastotasi ω ga teng.

Yechilishi. Sistemaning tashqi kuch ta'siridagi majburiy tebranishi (3.7) tenglamaga mos ravishda kechadi:

$$\ddot{x} + \omega^2 x = \frac{1}{m} F_0 \cos \alpha t$$

Bu tenglama bir jinsli bo'lmagan 2-tartibli oddiy differensial tenglamadir. Uning umumiy yechimi shu tenglamaga mos bir jinsli tenglamaning (3.2) umumiy yechimi x_0 va shu tenglamaning biror xususiy x_1 yechimlari yig'indisidan iboratdir:

$$x = x_0 + x_1$$

Ma'lumki, (3.2) ifodaga mos ravishda $x_0 = C_1 \cos \omega t + C_2 \sin \omega t$. Xususiy yechim x_1 ni esa tashqi kuchga mos ravishda $x_1 = b \cos \alpha t$ holda qidiraylik. Bu ifodani yuqoridagi tenglamaga olib borib qo'yish va qisqartirish natijasida:

$$b = \frac{F_0}{m} \frac{1}{(\omega^2 - \alpha^2)}, \text{ } x = x_0 + x_1 = C_1 \cos \omega t + C_2 \sin \omega t + \frac{F_0}{m} \frac{1}{(\omega^2 - \alpha^2)} \cos \alpha t.$$

C_1 va C_2 koeffitsientlarni boshlang'ich shartlardan $t=0, x=0, \dot{x}=0$ foydalanib topiladi. Avval $t=0$ da $x=0$ shartdan $C_1 = \frac{F_0}{m} \frac{1}{(\omega^2 - \alpha^2)}$ ekanligi, so'ngra $t=0$ da $\dot{x}=0$ shartdan esa $C_2 = 0$ ekanligi kelib chiqadi. Natijada ushbu echimni olamiz:

$$x = \frac{F_0}{m} \frac{1}{(\omega^2 - \alpha^2)} (\cos \alpha t - \cos \omega t)$$

3-masala. Tashqi kuch $F = F_0 \cos(\omega + \Delta)t$, ($\omega \neq \Delta$) ta'sirida tebranish amplitudasining vaqtga bog'liq ravishda o'zgarishini toping, ω - erkin tebranish chastotasi $t=0$, $x=0$, $\dot{x}=0$.

Yechilishi. 2-masalaning yechimidan foydalansak:

$$x = \frac{F_0}{m[\omega^2 - (\omega + \Delta)^2]} [\cos(\omega + \Delta)t - \cos \omega t]$$

Lekin ($\omega \neq \Delta$) ekanligidan $\omega^2 - (\omega + \Delta)^2 \approx -2\omega\Delta$

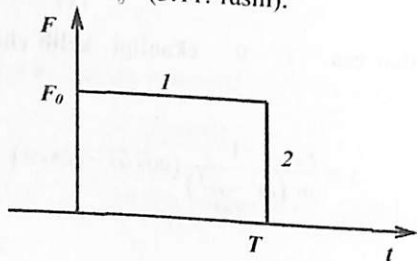
$$x = \frac{F_0}{2m\omega\Delta} [\cos(\omega + \Delta)t - \cos \omega t]$$

O'rtta qavs ichidagi ifodani trigonometrik almashtirishlar natijasida sinuslar ko'paytmasiga keltirib yozib olsak bo'ladi. Natijada

$$x = \frac{F_0}{m\omega\Delta} \sin \frac{\Delta t}{2} \cdot \sin \left(\omega + \frac{\Delta}{2} \right) t$$

Demak, tebranish amplitudasi vaqt o'tishi bilan chastotasi $\frac{\Delta}{2}$ (yoki tebranish davri $\frac{4\pi}{\Delta}$) ga teng bo'lgan holda davriy o'zgaradi. Uning maksimal qiymati esa $A_{\max} = \frac{F_0}{m\omega\Delta}$

4-masala. Sistema tebranishining tashqi F kuch ta'siridagi keyingi amplitudasi topilsin. Sistema $t=0$ momentgacha muvozanat holatida tinch turgan $x=0$, $\dot{x}=0$, F kuch esa quyidagi qonun bo'yicha o'zgaradi: $t < 0$ va $t > T$ da $F=0$, $0 < t < T$ da $F = F_0$ (3.11.-rasm).



3.11.-rasm.

Yechilishi: $t < T$ vaqtidagi, yani $F = F_0$ kuch ta'sir etib turgan vaqtdagi (1-sohadagi) tenglamaning xususiy yechimini $x_1^{(1)} = \frac{F_0}{m\omega^2}$ (3.7) shaklida qidiramiz. Umumiy yechim esa

$$x^{(1)} = x_0^{(1)} + x_1^{(1)} = C_1 \cos \omega t + C_2 \sin \omega t + \frac{F_0}{m\omega^2}$$

boshlang'ich shartlar asosida quyidagicha yozib olinadi:

$$x^{(1)} = \frac{F_0}{m\omega^2} (1 - \cos \omega t)$$

Bu 1-sohadagi umumiy yechimdir. Kuch ta'sir etmaganda (2-soha) $t > T$ esa, tebranish erkin.

$$x^{(2)} = C_1 \cos \omega(t - T) + C_2 \sin \omega(t - T)$$

Umumiy nazariyaga ko'ra bu ikki yechimlar (tebranish) $t = T$ momentda (chegarada) bir-biriga teng bo'lishi kerak, shu jumladan ularning hosilalari ham:

$$x^{(1)}(t = T) = x^{(2)}(t = T), \quad \dot{x}^{(1)}(t = T) = \dot{x}^{(2)}(t = T)$$

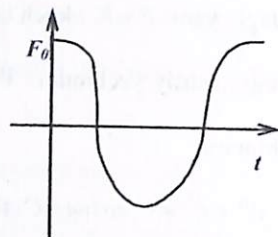
Mazkur «chegaraviy» shartlaridan noma'lum koeffitsientlar aniqlanadi

$$C_1 = \frac{F_0}{m\omega^2} (1 - \cos \omega T), \quad C_2 = \frac{F_0 \sin \omega T}{m\omega^2}$$

Sistemaning kuch ta'siridan keyingi amplitudasi esa quyidagicha bo'ladi

$$a = \sqrt{C_1^2 + C_2^2} = \frac{2F_0}{m\omega^2} \sin \frac{\omega T}{2}$$

5-masala. Oldingi masaladagi kabi, ammo $t=0$ dan $T = \frac{2\pi}{\omega}$ gacha $F(t) = F_0 \cos \omega t$ qonun bilan o'zgaruvchi kuch ta'siri uchun yechilsin (3.12.-rasm).



3.12.-rasm.

Yechilishi. Bu masalani ham huddi oldingi holdagidek yechish mumkin. Lekin biz hozir (3.7) ga mos yechimning boshqacha ko'rishidan foydalanamiz. Ma'lumki, (3.7) tenglamani ixtiyoriy tashqi kuch $F(t)$ uchun umumiy holda integrallash mumkin. Buning uchun uni $\xi = \dot{x} + i\omega x$ mavhum funsiya yordamida ushbu ko'rinishda yozib olinadi:

$$\frac{d}{dt}(\dot{x} + i\omega x) - i\omega(\dot{x} + i\omega x) = \frac{1}{m} F(t), \quad \frac{d\xi}{dt} - i\omega\xi = \frac{1}{m} F(t).$$

Bu bir jinsli bo'lmagan birinchi tartibli tenglamaning yechimini $\xi = A(t)e^{i\omega t}$

holda qidiriladi va vaqtga bo'liq o'zgarmasning hosiasi uchun ushbu ifoda olinadi:

$$\dot{A}(t) = \frac{1}{m} F(t)e^{-i\omega t}$$

Uni integrallab keng ishlatiladigan ifodani olamiz:

$$\xi = e^{i\omega t} \left\{ \int_0^t F(t)e^{-i\omega t} \frac{dt}{m} + \xi_0 \right\} \quad (3.26)$$

$\xi = \dot{x} + i\omega x$, $\xi_0 = \xi(t=0)$. Tebranish $x(t)$ esa (3.26) ifodaning mavhum

qismini $i\omega$ ga bo'lish natijasida topiladi. Amplitudasi esa, $|\xi|^2 = \frac{2}{m} E = \omega^2 a^2$

dan aniqlanadi. Maslada berilgan $F(t) = F_0 \cos \omega t = F_0 \frac{e^{i\omega t} + e^{-i\omega t}}{2}$ kuchni

(3.26) ga olib borib qo'ysak, va integrallasak ($t > T$ da sistema $x=0$ atrofida erkin tebranadi):

$$\begin{aligned} \xi &= \frac{F_0}{2m} e^{i\omega t} \left\{ \int_0^T e^{-i\omega t} (e^{-i\omega t} + e^{i\omega t}) dt \right\} = \frac{F_0}{2m} e^{i\omega t} \int_0^T (1 + e^{-2i\omega t}) dt = \\ &= \frac{F_0}{2m} e^{i\omega t} \left[T + \frac{1}{2i\omega} (1 - e^{-2i\omega T}) \right] \end{aligned}$$

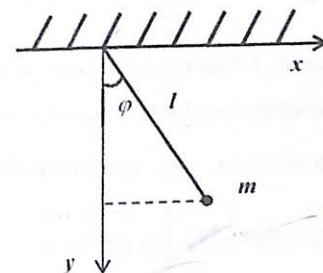
Bu erdan

$$|\xi|^2 = \frac{F_0^2}{16m^2\omega^2} [4\omega^2 T^2 + 4\omega T \sin 2\omega T + 4\sin^2 \omega T]$$

$T = \frac{2\pi}{\omega}$ ekanini hisobga olsak, amplituda $a = \frac{F_0\pi}{m\omega^2}$ ga teng.

6-masala. Uzunligi $l=2$ massasi $m=1$ bo'lgan mayatnik so'nuvchi tebranishining chastotasi va so'nish koeffitsiyentini toping. Ishqalanish natijasida energiyaning yutilishi burchak tezlikni kvadratining ikkilanganiga tengdir. (Og'irlik kuchi tezlanishi $g=1$ deb olinsin, 3.13-rasm).

Yechilishi. Erkin tebranayotgan mayatnikning Lagranj funksiyasini tuzaylik: $L = T - U$, $T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$



3.13.-rasm.

Qutb koordinata sistemasida: $x = l \sin j$, $y = l \cos j$

$$T = \frac{ml^2}{2} (\dot{j}^2 \cos^2 j + \dot{j}^2 \sin^2 j) = \frac{ml^2}{2} \dot{j}^2$$

Chizmadan ko'rinib turibdiki, potensial energiya $U = mgl(1 - \cos j)$ ga teng. Bu ifodadan ko'rinib turibdiki, $j_{\text{max}} = 0$ kichik tebranishlarning barqaror muvozanat nuqtasidir. Bu nuqta atrofida potensial funksiyani qatorga qo'yish natijasidagi $U = \frac{mgj^2}{2}$ ni olamiz (3.24 ga qaralsin). Demak, erkin tebranayotgan mayatnikning Lagranj funksiyasi quyidagichadir:

$$L = \frac{ml^2}{2} \dot{j}^2 - \frac{mgj^2}{2}$$

Ammo, masalaning shartiga ko'ra ishqalanish natijasida energiyaning yutilishi $E = -2j\&$ ga teng. Demak, (3.23) ifodaga asosan dissipativ funksiya $F = -j\&$ ga teng. Shunday qilib, (3.22) ga asosan harakat tenglamasini tuzamiz:

$$\ddot{\varphi} + 2 \frac{\dot{\varphi}}{ml^2} + \frac{g\varphi}{l} = 0$$

Masalaning berilishiga ko'ra $m = 1$, $l = 2$, $g = 1$ desak,

$$\ddot{\varphi} + 2 \frac{\dot{\varphi}}{4} + \frac{\varphi}{2} = 0$$

Bu tenglamani (3.16) tenglama bilan solishtirsak $\lambda = \frac{1}{4}$ va $\omega_0^2 = \frac{1}{2}$ ni topamiz.

Demak, so'nuvchi tebranishlarning birinchi holini olamiz, ya'ni $\lambda < \omega_0$

.So'nuvchi tebranishlarning "chastotasi" esa $\omega = \sqrt{\omega_0^2 - \lambda^2} = \frac{\sqrt{7}}{4}$ ga tengdir.

7-masala. So'nuvchan tebranayotgan zarraning boshlang'ich shartlar $t=0$ da $x=x_0$, $\dot{x}=v_0$ asosidagi harakatni aniqlang. So'nish koeffitsiyenti l ishqalanish bo'lmagandagi chastota ω_0 dan kichik deb oling.

Echilishi. Harakat tenglamasi (3.16) ga binoan $\ddot{x} = 2\lambda\dot{x} + \omega_0^2 x = 0$.

Bu tenglamaning yechimi (3.17) ga asosan quyidagichadir:

$$x = C_1 e^{r_1 t} + C_2 e^{r_2 t}, \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega_0^2}$$

Bizning hol $\lambda < \omega_0$ bo'lgani uchun:

$$r_{1,2} = -\lambda \pm i\sqrt{\omega_0^2 - \lambda^2}$$

$$x = e^{-\lambda t} (C_1 \cos \sqrt{\omega_0^2 - \lambda^2} t + C_2 \sin \sqrt{\omega_0^2 - \lambda^2} t)$$

Bu erdan esa,

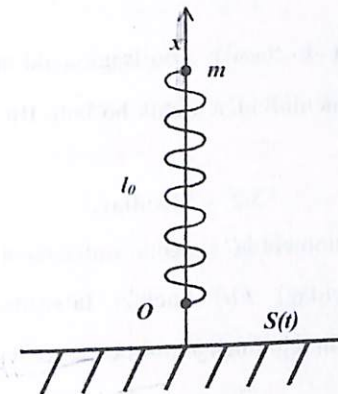
$$v = \dot{x} = -\lambda x + e^{-\lambda t} \sqrt{\omega_0^2 - \lambda^2} (C_1 \sin \sqrt{\omega_0^2 - \lambda^2} t + C_2 \cos \sqrt{\omega_0^2 - \lambda^2} t)$$

Endi boshlang'ich shartlardan noma'lum koeffitsiyentlar C_1 va C_2 ni topamiz:

$$C_1 = x_0, \quad C_2 = \frac{v_0 + \lambda x_0}{\sqrt{\omega_0^2 - \lambda^2}}$$

Demak,
$$x = e^{-\lambda t} \left(x_0 \cos \sqrt{\omega_0^2 - \lambda^2} t + \frac{v_0 + \lambda x_0}{\sqrt{\omega_0^2 - \lambda^2}} \sin \sqrt{\omega_0^2 - \lambda^2} t \right)$$

8-masala. Prujina bilan birlashtirilgan ikki zarra vertikal chiziq bo'yicha harakatlana oladi. Bularidan biri $S(t) = S_0 \cos \omega t$ qonuni bilan tebranadi. Qaysi shartlarda ikkinchi zarraning majburiy tebranish amplitudasi birinchiniki S_0 dan kichik bo'ladi?



3.14.-rasm

Yechilishi. Zarraning Lagranj funksiyasini tuzaylik (koordinata boshi 1-zarrada joylashgan noinersional sanoq sistemasida).

$$L = \frac{m}{2} \dot{x}^2 - \frac{k}{2} (x - l_0)^2 - mgx - m\dot{S}\dot{x}$$

bu erda uchinchi had og'irlik maydonidagi potensial energiya bo'lsa, to'rtinchi had (3.6) ga mos tashqi kuch bilan bog'liq: $F(t)x = m\ddot{S}(t)x$.

Harakat tenglamasi: $m\ddot{x} = -k(x - l_0) - mg + mS_0\omega^2 \cos \omega t$

(1)

$$\ddot{x} = -\omega_0^2(x - l_0) + S_0\omega^2 \cos \omega t - g, \quad \omega_0^2 = \frac{k}{m}$$

Zarra $x_{\text{stav}} = l_0 - \frac{g}{\omega_0^2}$ holat atrofida tebranma xarakat qiladi. Shuning uchun

(1) tenglamaning yechimi nazariyaga ga asosan: $x = x_{\text{stav}} + x_0 + x_1$, $x_0 = a \cos(\omega_0 t + \alpha)$ - erkin tebranishga mos, a va α lar boshlang'ich shartlardan aniqlanadi.

$$x_1 = \frac{\omega^2 S_0}{\omega_0^2 - \omega^2} \cos \omega t - \text{majbur etuvchi kuch ta'siridagi xususiy yechim.}$$

Demak, $x = l_0 - \frac{mg}{k} + a \cos(\omega_0 t + \alpha) + \frac{\omega^2 S_0}{\omega_0^2 - \omega^2} \cos \omega t$

Agarda $\omega_0^2 > 2\omega^2$ ($-k > 2m\omega^2$) bo'lsagina ikkinchi zarraning majburiy tebranish amplitudasi birinchi nikidan kichik bo'ladi. Bu sistema amartizatorning modeli hisoblanadi.

3.2. - Misollar.

1. Boshlang'ich $t = 0$ momentda sistema muvozanat holatida ($x = 0$) tinch turibdi ($\dot{x} = 0$) deb, quyidagi $F(t)$ kuchlar ta'sirida sistemaning majburiy tebranishini aniqlang. Erkin tebranish chastotasi ω ga teng.

a) $F(t) = F_0$.

v) $F(t) = F_0 t$.

d) $F(t) = F_0 e^{-\alpha t} \cos \beta t$.

b) $F(t) = F_0 \sin \alpha t$.

g) $F(t) = F_0 e^{-\alpha t}$

e) $F(t) = F_0 e^{-\alpha t} \sin \beta t$

2. Quyidagi tashqi kuchlar $F(t)$ ta'sirida tebranish amplitudasining vaqtga bog'liq

a) $F(t) = a \cos \omega t$, b) $F(t) = a \sin \omega t$, v) $F(t) = a \sin(\omega + \Delta)t$

ravishda o'zgarishi topilsin. Erkin tebranish chastotasi ω , $t = 0$, $x = 0$.

3. $t = 0$ momentda muvozanat holatida tinch turgan ($x = 0$, $\dot{x} = 0$), sistema tebranishining $F(t)$ kuch ta'siridagi keyingi amplitudasi topilsin. $F(t)$ kuchlarning o'zgarishi quyidagichadir:

a) $t < 0$ da $F = 0$, $0 < t < T$ da $F = \frac{t}{T} F_0$, $t > T$ da $F = F_0$.

b) $t < 0$ da $F = 0$, $0 < t < T$ da $F = F_0 t$, $t > T$ da $F = 0$.

v) $t < 0$ da $F = 0$, $0 < t < T$ da $F = F_0$, $t > T$ da $F = -\frac{F_0 t}{T}$.

g) $F = F_0 \sin \omega t$, $0 < t < T = \frac{2\pi}{\omega}$ da, va $F = 0$, $T < t < \infty$ da.

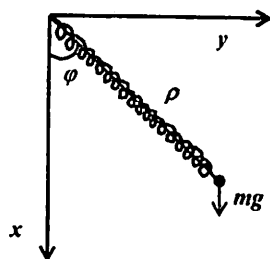
4. 6-masalani $\dot{E} = -4(l\dot{\varphi})^2$ holi uchun yeching.

5. 3.2.-chizmada ko'rsatilgan sistemaning so'nish koeffitsiyenti va chastotasini $k = 5$, $l = 4$ va har bir prujinada energiya yo'qotilishi tezlik kvadratining ikkilanganiga teng bo'lgan holda toping.

6. 7-masalani a) $\lambda > \omega_0$ va b) $\lambda = \omega_0$ hollar uchun yeching.

§.3.3 Ko'p erkinlik darajasiga ega sistemaning tebranishi.

1-masala. Prujinaga osilgan massasi m ga teng zarraning og'irlik kuchi ta'sirida tebranishining barqaror muvozanat holati topilsin. Prujinaning kuch ta'sir etmagandagi uzunligi a ga bikrligi k ga teng (3.15.-rasm).



3.15.-rasm.

Yechilishi. Koordinata sistemasini (3.15.-rasm) dagidek tanlab olamiz, mustaqil koordinatalar sifatida qutb koordinatalari ρ va φ ni olaylik. U holda sistemaning potensial energiyasi:

$$U = mg\rho \cos\varphi + k \frac{(\rho - a)^2}{2}$$

Barqaror muvozanat holatini topish uchun potensial energiyadan olingan hosilalarini topaylik:

$$\frac{\partial U}{\partial \rho} = -mg \cos\varphi + k \frac{(\rho - a)}{1}, \quad \frac{\partial U}{\partial \varphi} = mg\rho \sin\varphi$$

$$\frac{\partial^2 U}{\partial \rho^2} = k, \quad \frac{\partial^2 U}{\partial \rho^2} = mg \cos\varphi, \quad \frac{\partial^2 U}{\partial \rho \partial \varphi} = mg \sin\varphi$$

Birinchi tartibli hosilalarni nolga tenglash natijasida

$$\rho_{\text{muv}} = a \pm \frac{mg}{k}, \quad \varphi_{\text{muv}} = \{0, \pi\}$$

ni topamiz. Bulardan $\rho_{\text{muv}} = a + \frac{mg}{k}, \quad \varphi_{\text{muv}} = 0$ barqaror muvozanat nuqtalari bo'ladi. Chunki ikkinchi tartibli hosilalar

$$\left(\frac{\partial^2 U}{\partial \rho^2}\right)_{\text{muv}} = k > 0, \quad \left(\frac{\partial^2 U}{\partial \varphi^2}\right)_{\text{muv}} = mg\rho_{\text{muv}} > 0$$

quyidagi tengsizlikni qanoatlantiradi:

$$\left(\frac{\partial^2 U}{\partial \rho^2}\right)_{\text{muv}} \left(\frac{\partial^2 U}{\partial \varphi^2}\right)_{\text{muv}} - \left(\frac{\partial^2 U}{\partial \rho \partial \varphi}\right)_{\text{muv}}^2 > 0 \quad (3.27)$$

Oxirgi tengsizlik potensial energiyaning shu holatda minimumi borligini anglatadi. Muvozanat holatining boshqalari $\rho_{\text{muv}} = a - \frac{mg}{k}, \quad \varphi_{\text{muv}} = \pi$ barqaror emas (noturg'un) muvozanat holatlari ekanligini ham shu tengsizlikdan ko'rsa bo'ladi.

2-masala. Quyidagi Lagranj funksiyasi yordamida aniqlangan sistemaning xususiy chastotasi tipilsin:

$$L = \frac{1}{2}(x\dot{x}^2 + y\dot{y}^2) - \left(\frac{1}{xy} + x + y\right)$$

Yechilishi. Potensial energiyaning ko'rinishidan barqaror muvozanat holatini topaylik

$$\frac{\partial U}{\partial x} = 1 - \frac{1}{x^2 y}, \quad \frac{\partial U}{\partial y} = 1 - \frac{1}{xy^2}, \quad x_{\text{muv}} = 1, \quad y_{\text{muv}} = 1$$

$$\left(\frac{\partial^2 U}{\partial x^2}\right)_{\text{muv}} = \left(\frac{\partial^2 U}{\partial y^2}\right)_{\text{muv}} = 2, \quad \left(\frac{\partial^2 U}{\partial x \partial y}\right)_{\text{muv}} = 1$$

Demak, $x_{\text{muv}} = 1, \quad y_{\text{muv}} = 1$ holatlar sistemaning barqaror muvozanat holati ekan (3.27 ga qarang). Endi berilgan Lagranj funksiyasini (3.10) ga muvofiq ravishda yozib olaylik. Avval kinetik energiyaning koordinatalar orqali yozilgan ifodasini muvozanat holatidan og'ish $\xi_1 = x - 1, \quad \xi_2 = y - 1$ lar orqali yozib olaylik. Buning uchun kinetik energiya ifodasida $x = 1, \quad y = 1$ va $\dot{\xi}_1 = \dot{x}, \quad \dot{\xi}_2 = \dot{y}$ almashtirishlar bajarish yetarlidir. Endi potensial energiya ifodasini ham shu ko'rinishga keltiraylik. Buning uchun potensial energiya ifodasini muvozanat holati $x = 1, \quad y = 1$, atrofida qatorga yoyib, ta'rifga binoan ikkinchi tartibli hadlarini saqlab qolsak yetarli. O'zgarmasdan iborat bo'lgan qismini, Lagranj funksiyasining xususiyatlariga binoan, hisobga olmasa bo'ladi.

$$U_{muv} = 3, \quad U'_{x\ muv} = U'_{y\ muv} = 0, \quad U''_{xx\ muv} = U''_{yy\ muv} = 2, \quad U''_{xy\ muv} = U''_{yx\ muv} = 1$$

Demak,

$$U(\xi_1, \xi_2) = \frac{2[(x-1)^2 + (y-1)^2]}{2} + \frac{(1+1)(x-1)(y-1)}{2} = \xi_1^2 + \xi_2^2 + \xi_1 \xi_2$$

Shunday qilib, Lagranj funksiyasini quyidagi holga keltirib oladik:

$$L = \frac{\xi_1^2 + \xi_2^2}{2} - (\xi_1^2 + \xi_2^2 + \xi_1 \xi_2)$$

Harakat tenglamalari (1.3) ga binoan quyidagicha

$$\ddot{\xi}_1 + 2\xi_1 + \xi_2 = 0, \quad \ddot{\xi}_2 + 2\xi_2 + \xi_1 = 0.$$

Bu tenglamalarning yechimini $\xi_1 = C_1 e^{i\omega t}$, $\xi_2 = C_2 e^{i\omega t}$ holda qidiramiz.

Bularni tenglamaga qo'ysak

$$C_1(2 - \omega^2) + C_2 = 0, \quad C_1 + C_2(2 - \omega^2) = 0$$

ni olamiz. (3.13) ga binoan oxirgi tenglamadagi koeffitsientlardan tuzilgan determinant nolga teng bo'lishi kerak:

$$\begin{vmatrix} 2 - \omega^2 & 1 \\ 1 & 2 - \omega^2 \end{vmatrix} = (2 - \omega^2)^2 - 1 = (1 - \omega^2)(3 - \omega^2) = 0$$

Bu yerdan tebranishlarning xususiy chastotasi $\omega_1 = \sqrt{1}$, $\omega_2 = \sqrt{3}$ ekanligi kelib chiqadi.

3-masala. Quyidagi Lagranj funksiyasi bilan

$$L = \frac{\dot{x}^2 + \dot{y}^2}{2} - \frac{x^2 - xy + y^2}{2}$$

aniqlangan sistemaning normal tebranishi va chastotasini toping.

Yechilishi. Harakat tenglamalarini (1.3) ga asosanib tuzamiz

$$\ddot{x} + \frac{y}{2} + x - \frac{y}{2} = 0, \quad \ddot{y} + \frac{x}{2} + y - \frac{x}{2} = 0$$

Bu tenglamalarning yechimi $x_1 = C_1 e^{i\omega t}$, $x_2 = C_2 e^{i\omega t}$ holda qidiramiz. Bu ifodalarni yuqoridagi tenglamalarga olib borib qo'ysak:

$$\begin{aligned} C_1(1 - \omega^2) + \frac{C_2(1 + \omega^2)}{2} &= 0 \\ -\frac{C_1(1 + \omega^2)}{2} + C_2(1 - \omega^2) &= 0 \end{aligned} \quad (3.28)$$

Nazariyaga binoan, (3.13) ga mos determinant nolga teng bo'lgandagina noldan farqli yechim topiladi:

$$\begin{vmatrix} 1 - \omega^2 & -\frac{(1 + \omega^2)}{2} \\ -\frac{(1 + \omega^2)}{2} & 1 - \omega^2 \end{vmatrix} = (1 - \omega^2)^2 - \frac{(1 + \omega^2)^2}{4} = (1 - 3\omega^2) \frac{(3 - \omega^2)}{4} = 0$$

Bu yerdan xususiy chastotalar $\omega_1 = \sqrt{3}$, $\omega_2 = \frac{1}{\sqrt{3}}$ ga tengligini topamiz.

Shunday qilib, normal tebranishlar (3.14) ga binoan

$$Q_1 = \text{Re}\{a_1 e^{i\sqrt{3}t}\}, \quad Q_2 = \text{Re}\left\{a_2 e^{\frac{t}{\sqrt{3}}}\right\}$$

Ma'lumki, (3.14) ga binoan

$$x = C_{11}Q_1 + C_{12}Q_2, \quad y = C_{21}Q_1 + C_{22}Q_2.$$

Bu koeffitsiyentlarni topish uchun (3.28) tenglamaga avval $\omega_1^2 = 3$ ni, keyin $\omega_2^2 = \frac{1}{3}$ ni qo'yamiz. Natijada $\omega = \omega_1$ da $C_{11} = -C_{21}$ va $\omega = \omega_2$ da $C_{12} = C_{22}$ ni topamiz.

Bu bilan biz x va y larni normal koordinatalar Q_1 va Q_2 lar orqali yozib olishimiz mumkin. Lekin C_{11} va C_{22} koeffitsiyentlarni shunday tanlash kerakki, normal koordinatalar orqali yozilgan Lagranj funksiyasi ta'rifga binoan harmonik ossillyatornikiga o'xshash bo'lishi kerak:

$$L = \frac{\dot{Q}_1^2 + \dot{Q}_2^2}{2} - \frac{\omega_1^2 Q_1^2 + \omega_2^2 Q_2^2}{2} \quad (3.29)$$

Yuqoridagilarga binoan:

$$x = C_{11}Q_1 + C_{12}Q_2, \quad y = -C_{21}Q_1 + C_{22}Q_2$$

Bularni berilgan Lagranj funksiyasidagi kinetik va potensial energiya ifodalariga qo'yib, (3.29) ko'rinishini talab qilsak:

$$C_{11} = -C_{22} = 1, \quad C_{12} = -C_{21} = \frac{1}{\sqrt{3}}$$

larni olamiz. Shuni ko'rsataylik. Kinetik energiya ifodasiga

$$\dot{x} = C_{11}\dot{Q}_1, \quad \dot{y} = -C_{12}\dot{Q}_1 = C_{21}\dot{Q}_1$$

larni olib borib qo'ysak, $\frac{C_{11}^2\dot{Q}_1^2}{2}$ ko'rinishidagi ifodani olamiz. Lekin, $C_{11}^2 = 1$

bo'lishi kerak, ya'ni $C_{11} = -C_{21} = 1$ ekan. Xuddi yuqoridagidek

$$\dot{x} = C_{22}\dot{Q}_2, \quad \dot{y} = -C_{12}\dot{Q}_2 = C_{22}\dot{Q}_2$$

larni kinetik energiya ifodasiga olib borib qo'yish natijasida $\frac{3C_{22}^2\dot{Q}_2^2}{2}$ ifodani

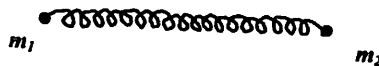
olamiz, ya'ni $C_{22}^2 = \frac{1}{3}$ bo'lishi kerak.

$$\text{Demak, } C_{22} = C_{12} = \frac{1}{\sqrt{3}} \text{ ekan.}$$

Shunday qilib, normal tebranishlar quyidagicha aniqlanar ekan:

$$x = \left(Q_1 + \frac{Q_2}{\sqrt{3}} \right), \quad y = \left(-Q_1 + \frac{Q_2}{\sqrt{3}} \right)$$

4-masala. Massalari m_1 va m_2 ga teng va bir-biri bilan bikrligi k ga, uzunligi esa a teng bo'lgan prujina bilan bog'langan sistemaning tebranishi topilsin (ikki molekula modeli).



3.16.-rasm

Yechilishi. Dekart koordinatalarining Ox o'qini prujina bo'ylab joylab, sistemaning Logranj funksiyani yozishimiz mumkin.

$$L = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 - \frac{k}{2}(x_2 - x_1 - a)^2$$

Potensial energiyaning ifodasidan ko'rinib turibdiki, uning minimal qiymati faqatgina $x_{2, \text{max}} - x_{1, \text{min}} = a$ da emas, balki $x_2 - x_1 = a$ ni qanoatlantirgan hamma nuqtalarda mavjud bo'ladi. Shuning uchun bu barqaror muvozanat nuqtasini chetlantirilgan deb bo'lmaydi. Shunga qaramasdan, masalani yechishda yuqorida ishlatilgan metoddan foydalanaveramiz. Muvozanat holatidan og'ish masofasini kiritaylik:

$$\xi_1 = x_1 - x_{1, \text{muv}}, \quad \xi_2 = x_2 - x_{2, \text{muv}}$$

Lagranj funksiyasini bular orqali yozib olaylik:

$$L = \frac{m_1\dot{\xi}_1^2}{2} + \frac{m_2\dot{\xi}_2^2}{2} - \frac{k(\xi_2 - \xi_1)^2}{2}$$

harakat tenglamasi bunday bo'ladi:

$$\ddot{\xi}_1 + \frac{k\xi_1}{m_1} - \frac{k\xi_2}{m_1} = 0, \quad \ddot{\xi}_2 + \frac{k\xi_2}{m_2} - \frac{k\xi_1}{m_2} = 0$$

$\xi_1 = C_1 e^{i\omega t}$, $\xi_2 = C_2 e^{i\omega t}$ ni tenglamaga qo'yish natijasida

$$\left(-\omega^2 + \frac{k}{m_1} \right) C_1 - C_2 \frac{k}{m_1} = 0, \quad -\frac{k}{m_2} C_1 + \left(-\omega^2 + \frac{k}{m_2} \right) C_2 = 0$$

Koeffisientlardan tuzilgan determinant esa,

$$\begin{vmatrix} -\omega^2 + \frac{k}{m_1} & -\frac{k}{m_1} \\ -\frac{k}{m_2} & -\omega^2 + \frac{k}{m_2} \end{vmatrix} = \left(-\omega^2 + \frac{k}{m_1} \right) \left(-\omega^2 + \frac{k}{m_2} \right) - \frac{kk}{m_2 m_1} = \\ = \omega^4 - \omega^2 \left(\frac{k}{m_1} + \frac{k}{m_2} \right) + \omega^2 \left(\omega^2 - \frac{k}{m} \right) = 0$$

bo'lishi kerak. Bu erda $m = \left(\frac{1}{m_1} + \frac{1}{m_2} \right)^{-1} = \frac{m_1 m_2}{m_1 + m_2}$ - keltirilgan massa.

Natijada ushbu chastotalarni topamiz: $\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{k}{m}}$

Endi ω_1 va ω_2 larni xarakteristik tenglamaga olib borib qo'yamiz va

$C_{11} = C_{21}, \quad -C_{22} = C_{12} \frac{m_2}{m_1}$ ni olamiz. Normal tebranishlar esa,

$$\xi_1 = \frac{1}{\sqrt{m}} \left(Q_1 + \sqrt{\frac{m_2}{m_1}} Q_2 \right), \quad \xi_2 = \frac{1}{\sqrt{m}} \left(Q_1 - \sqrt{\frac{m_1}{m_2}} Q_2 \right)$$

ko'rinishda bo'ladi. Agarda x_1 va x_2 koordinatalarga o'tsak,

$$x_1 = x_{1\text{mv}} + \frac{1}{\sqrt{m}} \left(Q_1 + \sqrt{\frac{m_2}{m_1}} Q_2 \right), \quad x_2 = x_{2\text{mv}} + \frac{1}{\sqrt{m}} \left(Q_1 - \sqrt{\frac{m_1}{m_2}} Q_2 \right)$$

5-masala. Berilgan Lagranj va dissipativ funksiyalariga mos so'nuvchi harakatning so'nish koeffitsienti topilsin:

$$L = \frac{\dot{x}^2 + \dot{y}^2}{2}, \quad F = \frac{\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2}{2}$$

Yechilishi: Harakat tenglamasini (3.22) ga mos ravishda tuzamiz

$$\ddot{x} + \dot{x} + \frac{\dot{y}}{2} = 0, \quad \ddot{y} + \dot{y} + \frac{\dot{x}}{2} = 0$$

harakat tenglamalaridan ko'rinib turibdiki, so'nuvchi harakat nodavriy.

Yechimni $x = C_1 e^r, \quad y = C_2 e^r$ ko'rinishida qidiramiz:

$$C_1(1+r) + \frac{C_2}{2} = 0, \quad C_2(1+r) + \frac{C_1}{2} = 0$$

ni olamiz. Bu yerdan r ni topish uchun yuqoridagi tenglamalar sistemasining determinanti nolga teng bo'lishi kerak

$$\begin{vmatrix} 1+r & \frac{1}{2} \\ \frac{1}{2} & 1+r \end{vmatrix} = (1+r)^2 - \frac{1}{4} = \left(\frac{1}{2} + r\right) \left(\frac{3}{2} + r\right) = 0$$

Demak, $\lambda_1 = -\frac{1}{2}, \quad \lambda_2 = -\frac{3}{2}$ ga teng ekan. Bu koeffitsiyentlar aslida

harakat tezligining so'nish koeffitsiyentlaridir, chunki bu harakat nodavriy harakatdir (garmonik funksiyalarga bog'liqligi yo'q).

6-masala. 1.4.-rasmdagi yassi qo'shaloq mayatnikning barqaror muvozanat holati atrofidagi harakati o'rganilsin.

Yechilishi. 1.4-mashqlarning 2v misolining javobidan foydalanib $a = l_1, \quad b = l_2$ bu sistemaning Lagranj funksiyasini qaytadan yozib olaylik:

$$L = \frac{m_1 + m_2}{2} l_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2} l_2^2 \dot{\varphi}_2^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 + \varphi_2) +$$

$$+ (m_1 + m_2) g l_1 \cos \varphi_1 + m_2 l_2 \cos \varphi_2 = T - U$$

Potensial energiyaning ko'rinishidan ayon bo'lib turibdiki turg'un nuqtalar $\varphi_1 = 0, \quad \varphi_2 = 0$. Kichik tebranishlar uchun ($\varphi_{1,2} \ll 1$)

$$\cos \varphi_{1,2} = 1 - \frac{\varphi_{1,2}^2}{2}, \quad \cos(\varphi_1 - \varphi_2) \cong 1$$

deb olsak bo'ladi. U holda

$$L = \frac{m_1 + m_2}{2} l_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2} l_2^2 \dot{\varphi}_2^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 - \frac{m_1 + m_2}{2} g l_1 \varphi_1^2 - \frac{m_2}{2} g l_2 \varphi_2^2$$

(1)

Bu ifodadan harakat tenglamasini tuzsak:

$$(m_1 + m_2) l_1 \ddot{\varphi}_1 + m_2 l_2 \ddot{\varphi}_2 + (m_1 + m_2) g \varphi_1 = 0,$$

$$l_1 \ddot{\varphi}_1 + l_2 \ddot{\varphi}_2 + g \varphi_2 = 0$$

Bu tenglamada $\varphi_1 = C_1 e^{i\omega t}$, $\varphi_2 = C_2 e^{i\omega t}$ almashtirish bajarsak:

$$C_1(m_1 + m_2)(g - l_1\omega^2) - C_2\omega^2 m_2 l_2 = 0,$$

$$-C_1\omega^2 l_1 + C_2(g - l_2\omega^2) = 0$$

Bu karakteristik tenglamalarning yechimi (3.13 dan)

$$\omega_{1,2}^2 = \frac{g}{2m_1 l_1 l_2} \left\{ (m_1 + m_2)(l_1 + l_2) \pm \sqrt{(m_1 + m_2) [(m_1 + m_2)(l_1 + l_2)^2 - 4m_1 l_1 l_2]} \right\} \quad (2)$$

a) Agarda (2) ifodada $m_1 \rightarrow \infty$ desak quyidagini olamiz

$$\omega_{1,2}^2 = \frac{g}{l_1 l_2} \left\{ (l_1 + l_2) \pm \sqrt{(l_1 + l_2)^2 - 4l_1 l_2} \right\}$$

Ya'ni, xususiy chastotalar bir-biri bilan o'zaro ta'sirlashmayotgan ikki

mayatnikning tebranma harakatining chastotalarini olamiz $\omega_{1,2}^2 = \frac{g}{l_{1,2}}$

b) Masalani soddalashtirish uchun (2) ifodada $m_1 = m_2$, $l_1 = l_2$ desak:

$$L = \frac{m}{2} l^2 \left[2\dot{\varphi}_1^2 + 2\dot{\varphi}_1\dot{\varphi}_2 + \dot{\varphi}_2^2 - \frac{g}{l} (2\varphi_1^2 + \varphi_2^2) \right]$$

Bu holdagi harakat tenglamalaridan ($\varphi_{1,2} = 0$ turg'un) quyidagi xususiy chastotalar topiladi:

$$\omega_{1,2}^2 = \frac{g}{l} (2 \pm \sqrt{2}) = \omega_0^2 (2 \pm \sqrt{2}).$$

Mazkur natijani (2) ifodadan ham olish mumkin.

3.3. – Misollar.

1. Quyidagi Lagranj funksiyasi yordamida aniqlangan sistemaning barqaror muvozanat holati va xususiy chastotasini toping:

a) $L = \frac{\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2}{2} - (x^3 - y^3 + 3xy).$

b) $L = \frac{\dot{x}^2 + \dot{y}^2 + \dot{x}\dot{y}}{2} - \left(\ln(xy) + \frac{1}{x} + \frac{1}{y} \right).$

2. Quyidagi Lagranj funksiyasi bilan aniqlangan sistemaning normal tebranishi va chastotasini toping:

a) $L = \frac{2\dot{x}^2 + 2\dot{x}\dot{y} + \dot{y}^2}{2} - \frac{3x^2 + 2y^2}{2}$

b) $L = \frac{\dot{x}^2 + \frac{2\dot{x}\dot{y}}{5} + \dot{y}^2 + z^2}{2} - \frac{x^2 + y^2 + \frac{2yz}{5} + z^2}{2}$

v) $L = \frac{\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2 + \dot{z}^2}{2} - \frac{x^2 + xy + y^2 + 2z^2 + yz}{2}$

g) $L = \frac{\dot{x}^2 + \dot{y}^2}{2} - \frac{\omega_{01}^2 x^2 + \omega_{02}^2 y^2}{2} + \alpha xy$

3. 3.10.-rasmdagi sistemening tebranishini toping.

4. Quyidagi hollarda n ta zarradan iborat sistemaning tebranma harakatining erkinlik darajasi va xarakteri topilsin:

a) fazodagi,

b) tekislikdagi,

v) chiziq bo'yicha.

5. Quyidagi Lagranj va dissipativ funksiyalariga mos so'nish koeffitsiyentini toping

$$L = \frac{\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2}{2}, \quad F = \frac{\dot{x}^2 + 2\dot{y}^2}{2}.$$

6. Harakat tenglamasini tuzing:

a) $L = \frac{\dot{x}^2 + 2\dot{y}^2}{2} - \frac{3x^2 + 5y^2}{2}, \quad F = \frac{\dot{x}^2 - 3\dot{x}\dot{y} + 2\dot{y}^2}{2}$

b) $L = \frac{4\dot{x}^2 + 3\dot{x}\dot{y} + \dot{y}^2}{2}, \quad F = \frac{\dot{x}^2 + 3\dot{y}^2}{2}$

v) $L = \frac{1}{2} (x\dot{x}^2 + y\dot{y}^2) - \frac{1}{2} \left(\frac{1}{xy} + x + y \right), \quad F = \frac{1}{2} \left(\frac{\dot{x}^2}{x} + \frac{\dot{y}^2}{y} \right)$

IV. KANONIK TENGLAMALAR.

Mexanik sistema holatini umumlashgan koordinata va tezliklar orqali aniqlash (Lagranj formalizmi) bunday sistemalarni yoritishning yagona usuli emas. Ba'zan mexanik sistemalar holatini umumlashgan koordinata q_i va umumlashgan impulslar p_i orqali ifodalash qulayroq bo'ladi (Gamilton formalizmi). Bu usulda mexanik sistemaning holati Gamilton funksiyasi

$$H(p, q) = \sum p_i \dot{q}_i - L(\dot{q}, q, t) \quad (4.1)$$

bilan xarakterlanadi. Harakat tenglamasi esa $-S$ (S -erkinlik darajasi) ta birinchi tartibli tenglamalar sistemasidan iboratdir:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (4.2)$$

Bu tenglamalar-Gamilton tenglamalari yoki kanonik tenglamalar deb ataladi. (4.1) munosabatdan ko'rinib turibdiki, Gamilton funksiyasi H Lagranj funksiyasi L olgan orttirma $L = L_0 + L'$ ga teskari ishorali orttirma oladi (bu holda p, q va \dot{q}, q lar o'zgarimas deb hisoblanadi):

$$(H')_{p, q} = -(L')_{q, \dot{q}} \quad (4.3)$$

Dinamik kattaliklar-umumlashgan koordinata, impuls, hamda vaqtning ixtiyoriy funksiyasi $f(q, p, t)$ ning vaqt davomida o'zgarishi quyidagicha aniqlanadi:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\} \quad (4.4)$$

Bu yerda

$$\{H, f\} = \sum_k \left(\frac{\partial H}{\partial p_k} \frac{\partial f}{\partial q_k} - \frac{\partial H}{\partial q_k} \frac{\partial f}{\partial p_k} \right) \quad (4.5)$$

operator kattalik *Puasson qavslari* deb ataladi. Agarda ixtiyoriy $f(q, p, t)$ funksiya harakat integrali (harakat davomida saqlanuvchi kattalik) bo'lsa, u holda

$$\frac{df}{dt} + \{H, f\} = 0. \quad (4.6)$$

Shu jumladan, vaqtga oshkora bog'liqlik yo'q bo'lsa

$$\{H, f\} = 0. \quad (4.7)$$

Gamilton tenglamalari Puasson qavslari orqali juda ham chiroyli simmetrik ravishda ifodalanadi

$$\dot{p}_k = \{H, p_k\}, \quad \dot{q}_k = \{H, q_k\}. \quad (4.8)$$

Dinamik kattaliklarning funksiyasi bo'lgan uchta ixtiyoriy f, g, h kattaliklarning Puasson qavslari orasida Yakobi ayniyati o'rinlidir:

$$\{f\{g, h\}\} + \{g\{h, f\}\} + \{h\{f, g\}\} = 0. \quad (4.9)$$

§4.1. Berilgan Gamilton funksiyalariga mos Gamilton tenglamalarini tuzish va ularni integrallash.

1-masala. Ushbu Gamilton funksiyasiga mos Gamilton tenglamasini tuzing:

$$H = \sqrt{1+p^2} + r$$

Yechilishi:

$$\dot{r} = \frac{\partial H}{\partial p} = \left(\frac{1}{2}\right)(1+p^2)^{-\frac{1}{2}} 2p = \frac{p}{\sqrt{1+p^2}}, \quad \dot{p} = -\frac{\partial H}{\partial r} = -1.$$

Demak $\dot{r} = \frac{p}{\sqrt{1+p^2}}, \quad \dot{p} = -1.$

2-masala.
$$H = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2 \sin^2 \theta} + \sin \theta.$$

Yechilishi:

$(q, \rightarrow r, \theta; p, \rightarrow p_r, p_\theta), \quad \dot{r} = \frac{\partial H}{\partial p_r} = p_r, \quad \dot{\theta} = -\frac{\partial H}{\partial p_\theta} = \frac{p_\theta^2}{r^3 \sin^2 \theta}.$

$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{r^2 \sin^2 \theta}, \quad \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\theta^2 \cos \theta}{r^2 \sin^3 \theta} - \cos \theta.$

3-masala.
$$H = p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \theta} + U(r, \theta, \varphi).$$

Yechilishi: Bu gal erkinlik darajasi 3 ga teng, demak, uch juft tenglama topiladi

$\dot{r} = 2p_r, \quad \dot{\theta} = \frac{2}{r^3} \left(\frac{p_\theta^2 + p_\varphi^2}{\sin^2 \theta} \right) - \frac{\partial U}{\partial r}, \quad \dot{\varphi} = \frac{2p_\varphi}{r^2},$

$\dot{p}_\theta = \frac{2p_\theta \cos \theta}{r^2 \sin^3 \theta} - \frac{\partial U}{\partial \theta}, \quad \dot{p}_\varphi = \frac{2p_\varphi}{r^2 \sin^2 \theta}, \quad \dot{p}_\varphi = -\frac{\partial U}{\partial \varphi}.$

4.1.-Misollar.

1 Gamilton tenglamasini tuzing.

a) $H = \frac{(p-r)^2}{2},$ b) $H = \frac{p_x p_y}{tx^2 y},$ d) $H = \frac{p_u^2 + p_v^2}{2(u^2 + v^2)} + u.$

2. 4.2.-misollarning 3-punktida berilgan Gamilton funksilariga mos Gamilton tenglamalarini tuzing

§4.2. Lagranj va Gamilton funksiyalari orasidagi bog'lanish.

Gamilton formalizmi doirasida harakat tenglamalarini yechish avvalida Gamilton funksiyalarini tuzish amaliyotini aniqlab olish saruratdir. Keling fizik sistemaning Lagranj funksiyasi ma'lum bo'lganda, uning Gamilton funksiyasini topish yo'lini ko'rsataylik. Eng qulayi Gamilton funksiyasining aniqlanish formulasidan (4.1) foydalanishdir (yoki umumlashgan energiya):

$$H(q_1, q_2, \dots, q_n; p_1, p_2, \dots, p_n, t) = \sum_{j=1}^n p_j \dot{q}_j - L(q_1, q_2, \dots, q_n; \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t).$$

Bu ifodaning o'ng tomonida bizga "kerakli" bo'lgan o'zgaruvchilar \bar{q}, \bar{p}, t dan tashqari "ortiqcha" o'zgaruvchi \dot{q}_j lar ham bor. Bu "ortiqcha" o'zgaruvchilarni "kerakli" lari orqali ifodalash uchun, ya'ni $\dot{q}_j = \dot{q}_j(\bar{q}_0, \bar{p}_0, t)$ bog'lanishni topish uchun umumlashgan impulsning aniqlanish ifodasidan foydalanamiz:

$$p_j = \frac{\partial L(\bar{q}, \dot{\bar{q}}, t)}{\partial \dot{q}_j} = p_j(q_1, q_2, \dots, q_n; \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t) \quad (j=1, 2, \dots, n).$$

Hosil bo'lgan bu algebraik tenglamalardan topilgan \dot{q}_j larni Gamilton funksiyasining o'ng tomoniga olib-borib qo'yish natijasida izlangan ifodani olamiz.

Ba'zan, teskari masalani, ya'ni ma'lum bo'lgan Gamilton funksiyasidan Lagranj funksiyasini topish masalasini hal qilishga to'g'ri keladi:

$$L(q_1, q_2, \dots, q_n; \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t) = \sum_{j=1}^n p_j \dot{q}_j - H(q_1, q_2, \dots, q_n; p_1, p_2, \dots, p_n, t),$$

Bu ifodaning o'ng tomonidagi "ortiqcha" o'zgaruvchi p_j larni "kerakli" o'zgaruvchilar orqali ifodalash $p_j(\bar{q}, \dot{\bar{q}}, t)$ uchun ushbu Gamilton tenglamasidan (4.2) foydalanamiz:

$$\dot{q}_j = \frac{\partial H(\bar{q}; \bar{p}, t)}{\partial p_j} = \dot{q}_j(q_1, q_2, \dots, q_n; p_1, p_2, \dots, p_n, t) \quad (j=1, 2, \dots, n).$$

Umumlashgan impulslarning ifodalarini $p_j(\bar{q}, \dot{\bar{q}}, t)$ yuqoridagi Lagranj funksiyasi formulasiga olib-borib qo'yish va o'xshash hadlarni soddalashtirib izlangan ifodani olamiz

1-masala. Ushbu Lagranj funksiyasiga mos Gamilton funksiyasini toping:

$$L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}}$$

Yechilishi. (4.1) ifodaga asosan

$$H = pv - L(v),$$

bu yerda

$$p = \frac{\partial L}{\partial v} = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Oxirgi ifodada tezlik v ni impuls p orqali ifodalab, Gamilton funksiyasining formulasiga olib borib qo'lamiz:

$$v = \frac{pc}{\sqrt{p^2 + m^2c^2}}, \quad \sqrt{1 - \frac{v^2}{c^2}} = \frac{mc}{\sqrt{p^2 + m^2c^2}}$$

Demak,

$$H = pv - L = \frac{p^2c}{\sqrt{p^2 + m^2c^2}} + \frac{m^2c^3}{\sqrt{p^2 + m^2c^2}} = \frac{c(p^2 + m^2c^2)}{\sqrt{p^2 + m^2c^2}} = c\sqrt{p^2 + m^2c^2}$$

2-masala. Berilgan $L = I^2 \left(\dot{\phi}^2 + \frac{\dot{\theta}^2}{2} + \dot{\phi}\dot{\theta} \right) + 3I \cos \phi$ dan Gamilton funksiyasini tuzing.

Yechilishi. Bu Lagranj funksiyasi erkinlik darajasi ikkiga teng mexanik sistemani xarakterlaydi $q_i \rightarrow \phi, \theta, \dot{q}_i \rightarrow \dot{\phi}, \dot{\theta}$

Izlanayotgan Gamilton funksiyasi quyidagicha bo'lishi kerak.

$$H(P_\phi, P_\theta, \phi, \theta) = \sum P_i \dot{q}_i - L(\phi, \theta, \dot{\phi}, \dot{\theta})$$

Avval, umumlashgan impulslarni topib olaylik:

$$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = I^2 (2\dot{\phi} + \dot{\theta}), \quad P_\theta = \frac{\partial L}{\partial \dot{\theta}} = I^2 (\dot{\phi} + \dot{\theta})$$

$$\text{Bu ifodalardan } \dot{\phi} = \frac{P_\phi - P_\theta}{I^2}, \quad \dot{\theta} = \frac{2P_\theta - P_\phi}{I^2}$$

Bularni Gamilton funksiyasining ifodasiga olib borib qo'ysak:

$$H = P_\phi \dot{\phi} + P_\theta \dot{\theta} - L(\phi, \theta, \dot{\phi}, \dot{\theta}) = P_\phi \frac{P_\phi - P_\theta}{I^2} + P_\theta \frac{2P_\theta - P_\phi}{I^2} -$$

$$\begin{aligned} -I^2 \left[\frac{(P_\phi - P_\theta)^2}{I^4} + \frac{(2P_\theta - P_\phi)^2}{2I^4} + \frac{(P_\phi - P_\theta)(2P_\theta - P_\phi)}{I^4} \right] - 3I \cos \phi = \\ = \frac{1}{I^2} \left(\frac{P_\phi^2}{2} + P_\theta^2 + P_\phi P_\theta \right) - 3I \cos \phi \end{aligned}$$

3-masala. Massasi m ga teng zarrachaning sferik koordinat sistemasidagi Gamilton funksiyasini yozing.

Yechilishi. Ma'lumki, bitta zarrachaning sferik koordinat sistemasidagi Lagranj funksiyasi quyidagi ko'rinishdadir:

$$L = m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - U(r, \theta, \phi)$$

Har bir umumlashgan koordinataga mos impulslarni hisoblab olaylik:

$$P_r = m\dot{r}, \quad P_\theta = mr^2\dot{\theta}, \quad P_\phi = mr^2 \sin^2 \theta \dot{\phi}$$

Biz bu ifodalar orqali, yana yuqoridagi masalalarni yechgandek Gamilton funksiyasini topishimiz mumkin. Hozir esa boshqacha usul ishlatamiz. Mexanik sistemaning energiyasini Lagranj funksiyasi orqali ifodasini eslaylik:

$$E = \sum \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) + U(r, \theta, \phi)$$

Gamilton funksiyasini topish uchun esa, energiyani impulslar orqali yo'zish kifoyadir:

$$H = \frac{1}{2} m \left(\frac{P_r^2}{m^2} + \frac{r^2 P_\theta^2}{m^2 r^4} + \frac{r^2 \sin^2 \theta \cdot P_\phi^2}{m^2 r^4 \sin^4 \theta} \right) + U(r, \theta, \phi)$$

Demak,

$$H = \frac{1}{2m} \left(P_r^2 + \frac{P_\theta^2}{r^2} + \frac{P_\phi^2}{r^2 \sin^2 \theta} \right) + U(r, \theta, \phi)$$

4-masala. Ushbu Gamilton funksiyasiga mos Lagranj funksiyasini toping:

$$H = \frac{P_r^2}{2\theta^2} + \frac{P_\phi^2}{2r^2 \sin \theta} + r$$

Yechilishi. Yuqoridagi (4.1) va (4.2) ifodalardan

$$L = \sum P_i \dot{q}_i - H = \sum P_i \frac{\partial H}{\partial P_i} - H,$$

lekin o'ng tomonda hamma umumlashgan impulslarni tezliklar orqali ifodalab olish kerak. Bizning hol uchun (4.2) ifodadan tezliklarni topamiz:

$$\dot{r} = \frac{P_r}{\theta^2}, \quad \dot{\theta} = \frac{P_\theta}{r^2 \sin \theta}.$$

Bularni Lagranj funksiyasining ifodasiga olib borib qo'yamiz:

$$L = \sum P_i \frac{\partial H}{\partial P_i} - H(r, \theta, P_r, P_\theta) = \dot{r} \theta^2 \dot{r} + \dot{\theta} \cdot r^2 \sin \theta \cdot \dot{\theta} - \left[\frac{(\dot{r} \theta^2)^2}{2\theta^2} + \frac{(\dot{\theta} \cdot r^2 \sin \theta)^2}{2r^2 \sin^2 \theta} + r \right] =$$

$$= \frac{\theta^2 \dot{r}^2}{2} + \frac{r^2 \sin \theta \cdot \dot{\theta}^2}{2} - r.$$

Demak,

$$L = \frac{\theta \cdot \dot{r}^2}{2} + \frac{r^2 \sin \theta \cdot \dot{\theta}^2}{2} - r.$$

5-masala. $H = \frac{p_x^2 t}{2} + p_x p_y.$

Lagranj funksiyasi topilsin.

Yechilishi. Gamilton tenglamasini tuzamiz

$$\dot{x} = p_x t + p_x, \quad \dot{y} = p_y,$$

ya'ni, undan

$$p_x = \dot{y}, \quad p_y = \dot{x} - \dot{y}t.$$

$$L = \dot{x}\dot{y} + (\dot{x} - \dot{y}t)\dot{y} - \frac{\dot{y}^2 t}{2} - \dot{y}(\dot{x} - \dot{y}t) = \dot{x}\dot{y} - \frac{\dot{y}^2 t}{2}.$$

6-masala. Quyidagi sodda fizik sistemalar uchun Gamilton funksiyalarini tuzing va Gamilton formalizmida sistemalar harakatini tahlil qiling:

a) erkin zarra;

b) chiziqli garmonik ossilyator;

v) markaziy maydondagi zarra

Yechilishi. Ushbu algoritm asosida ish tutamiz:

• avval Lagranj funksiyalarini topamiz

$$L(\bar{q}, \dot{\bar{q}}, t) = T(\bar{q}, \dot{\bar{q}}, t) - U(\bar{q}, t) \quad (\text{bu sistemalar uchun Lagranj}$$

funksiyalari §1.4 bo'limning 10-masalasida berilgan);

• so'ngira, umumlashgan impulsning aniqlanish formulasidan umumlashgan tezliklarni umumlashgan impuls va koordinatalar orqali ifodalab olamiz;

• topilgan ifodalarni Gamilton funksiyasi aniqlanishi formulasi (4.1) ga olib-borib qo'yamiz.

a) Erkin zarraning Lagranj funksiyasi dekart koordinatalarida ushbu ko'rinishga

ega $L = (m/2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ (erkinlik darajasi $n=3$ ga teng va $q_1 = x,$

$q_2 = y, q_3 = z$). Mos umumlashgan impulslarni topamiz:

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z},$$

(bizning holda umumlashgan impulslar odatdagi kinematik impulslarga mos keladi), shuning uchun, $\dot{x} = p_x/m, \dot{y} = p_y/m, \dot{z} = p_z/m.$

Topilgan bu tezlik ifodalarini Gamilton funksiyasi ifodasiga olib-borib qo'yamiz:

$$H(x, y, z, p_x, p_y, p_z) = \dot{x}p_x + \dot{y}p_y + \dot{z}p_z - (m/2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) =$$

$$= (p_x^2 + p_y^2 + p_z^2)/m - (p_x^2 + p_y^2 + p_z^2)/(2m) = \frac{p_x^2 + p_y^2 + p_z^2}{2m} = \frac{p^2}{2m}.$$

Erkin zarra Gamilton funksiyasi shundan iboratdir.

Bu ifodalarning vector ko'rinishlari ham keng qo'llaniladi:

$$L = mv^2/2 = m(\bar{v}\bar{v})/2 = m(\bar{r}\ddot{\bar{r}})/2 \rightarrow$$

$$\bar{p} = \partial L / \partial \dot{\bar{r}} = m\dot{\bar{r}} \rightarrow \dot{\bar{r}} = \bar{p}/m \rightarrow$$

$$\rightarrow H(\bar{r}, \bar{p}) = (\bar{p}\dot{\bar{r}}) - m\dot{\bar{r}}^2/2 = p^2/m - p^2/2m = p^2/m.$$

Mazkur Gamilton funksiyasida koordinatalar ishtirok etmayotganligi uchun impulsning barcha komponentalari - harakat integrallaridir. Gamilton tenglamalari esa ushbu ko'rinishni oladi:

$$\dot{\bar{p}} = -\partial H / \partial \bar{r} \quad \dot{\bar{r}} = \partial H / \partial \bar{p} = \bar{p}/m.$$

Bu tenglamalarning yachimi osongina topiladi:

$$\vec{p} = \vec{p}_0, \quad \vec{r} = \vec{p}_0 t / m + \vec{r}_0.$$

b) Chiziqli garmonik ossillyatorning Lagranj funksiyasi quyidagicha

$$L(x, \dot{x}) = m\dot{x}^2/2 - m\omega_0^2 x^2/2.$$

Shuning uchun,

$$p = \partial L / \partial \dot{x} = m\dot{x} \quad \dot{x} = p/m.$$

Gamilton funksiyasi osongina topiladi

$$H(x, p) = p\dot{x}(p) - L(x, \dot{x}(p)) = p^2/2m + m\omega_0^2 x^2/2$$

va uning ifodasi garmonik ossillyatorning to'la energiyasi E bilan ustma-ust tushadi. Gamilton formalizmda harakat tenglamasi ko'rinishi ham osongina topiladi:

$$\dot{p} = -\partial H / \partial x = -m\omega_0^2 x, \quad \dot{x} = \partial H / \partial p = p/m.$$

Bu tenglamalarning ikkinchisini vaqt bo'yicha differensiallab, topilgan \dot{p} ni birinchisining chap tomoniga qo'yib, garmonik ossillyatorning standart tenglamasini olamiz:

$$\ddot{x} + \omega_0^2 x = 0$$

a) v) Markaziy maydondagi zarraning Lagranj funksiyasi (erkinlik darajasi ikkiga teng: $q_1 = \rho$, $q_2 = \varphi$ - qutb koordinat sistemasi) ushbu ko'rinishga ega:

$$L(\rho, \varphi, \dot{\rho}, \dot{\varphi}) = m\dot{\rho}^2/2 + m\rho^2\dot{\varphi}^2/2 - U(\rho).$$

Shuni ta'kidlash lozimki, Lagranj formalizmda koordinata φ siklikdir, chunki u Lagranj funksiyasi ifodasida ishtirok etmaydi.

Bu koordinatalar ρ va φ ga mos keluvchi umumlashgan impulsni aniqlaylik:

$$p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho}, \quad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi}.$$

Umumlashgan tezliklarni impuls va koordinatalar orqali ifodalab olaylik

$$\dot{\rho} = \frac{p_\rho}{m}, \quad \dot{\varphi} = \frac{p_\varphi}{m\rho^2}.$$

Bu ifodalarni Gamilton funksiyasi aniqlangan (4.1) formulaga qo'yib, uni

$H = H(\rho, \varphi, p_\rho, p_\varphi)$ ko'rinishga keltirib olamiz:

$$H = p_\rho \frac{p_\rho}{m} + p_\varphi \frac{p_\varphi}{m\rho^2} - \frac{m}{2} \left(\frac{p_\rho}{m} \right)^2 - \frac{m\rho^2}{2} \left(\frac{p_\varphi}{m\rho^2} \right)^2 + U(\rho) = \frac{p_\rho^2}{2m} + \frac{p_\varphi^2}{2m\rho^2} + U(\rho).$$

Ko'rinish turibdiki koordinata φ Gamilton formalizmda ham siklik koordinata ekan. shuning uchun unga mos umumlashgan impuls p_φ (uning fizik ma'nosi $M_z!$) harakat integralidir.

Bu natija Gamilton tenglamalaridan to'g'ridan to'g'ri kelib chiqadi va bu holat tenglamalarni yechishni soddalashtiradi

$$\dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = 0 \rightarrow p_\varphi = p_{\varphi 0} = \text{const}; \quad \dot{\rho} = -\frac{\partial H}{\partial \rho} = \frac{p_\varphi^2}{m\rho^3} - \frac{\partial U}{\partial \rho};$$

$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{m\rho^2} = \frac{p_{\varphi 0}}{m\rho^2}; \quad \dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m}.$$

Yuqoridagi hamma hollarda vaqt t ham "siklikdir" natijada umumlashgan energiya (Gamilton funksiyasi) saqlanuvchi kattalik bo'ladi

$$H(\vec{q}, \vec{p}) = E = \text{const}.$$

Ushbu holat markaziy maydondagi harakatni tahlil qilishda juda qo'l keladi:

$$\dot{\rho}^2 = \frac{2}{m} \left[E - U(\rho) - \frac{p_{\varphi 0}^2}{2m\rho^2} \right].$$

7-masala. Erkin zarraning $\bar{\Omega}(t)$ burchak tezlik bilan aylanayotgan sanoq sistemasidagi Gamilton funksiyasi va tenglamasini tuzing.

Yechilishi. Bunday sanoq sistemasidagi zarraning tezliklari $\vec{v} + [\bar{\Omega}\vec{r}]$ shaklda aniqlangani uchun Lagranj funksiyasi ushbu ko'rinishda bo'ladi

$$L(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2} m(\dot{\vec{r}} + [\bar{\Omega}\vec{r}])^2 - U(r).$$

Bu yerdan umumlashgan impulsni topamiz:

$$\bar{p} = \frac{\partial L}{\partial \bar{r}} = m(\dot{\bar{r}} + [\bar{\Omega} \bar{r}]).$$

Binobarin Gamilton funksiyasi (4.1) dan aniqlanadi:

$$H(\bar{r}, \bar{p}, t) = \frac{1}{2m} (\bar{p} - m[\bar{\Omega} \bar{r}])^2 - \frac{m}{2} [\bar{\Omega} \bar{r}]^2 + U(r) = \frac{\bar{p}^2}{2m} - \bar{\Omega}[\bar{r} \bar{p}] + U(r).$$

Bu yerdan Gamilton tenglamasi osonlikcha topiladi:

$$\dot{x}' = \frac{\partial H}{\partial p_k} = \frac{p_k}{m} - [\bar{\Omega} \bar{r}]_k, \quad \dot{p}'_k = -\frac{\partial H}{\partial x_k} = [\bar{p} \bar{\Omega}]_k - \frac{\partial U}{\partial x_k}.$$

8-masala. Elektromagnit maydondagi erkin zarraning Gamilton funksiyasi va harakat tenglamalarini yozing.

Yechilishi. §1.6. dagi (1.15.-1.17.) ifodalarga asosan:

$$L(x, \dot{x}, t) = \frac{1}{2} m \dot{x}^2 - e\varphi(x, t) + \frac{e}{c} \dot{x} \bar{A}(x, t), \quad p_i = m\dot{x}_i + \frac{e}{c} A_i(x, t),$$

$$H(x, p, t) = p_i \dot{x}_i - L = \left(m\dot{x}_i + \frac{e}{c} A_i \right) \cdot \dot{x}_i - \frac{1}{2} m \dot{x}_i^2 + e \cdot \varphi - \frac{e}{c} \dot{x}_i A_i = \\ = \frac{1}{2} m \dot{x}_i^2 + e\varphi = \frac{1}{2m} \left(p_i - \frac{e}{c} A_i \right)^2 + e\varphi.$$

Demak,

$$H(x, p, t) = \frac{1}{2m} \left(p_i - \frac{e}{c} \bar{A}_i \right)^2 + e\varphi.$$

Harakat tenglamalari (4.8) dan topiladi:

$$\dot{x}_i = \{H, x_i\} = \frac{\partial H}{\partial p_i} = \frac{1}{m} \left(\bar{p}_i - \frac{e}{c} \bar{A}_i \right) = v_i,$$

$$\dot{p}_i = \{H, p_i\} = -\frac{\partial H}{\partial x_i} = -e \frac{\partial \varphi}{\partial x_i} + \frac{e}{c} v_j \cdot \frac{\partial A_j}{\partial x_i}.$$

Bu oltita tenglamalar uchta ikkinchi tartibli tenglamalarga ekvivalentdir.

Haqiqatdan ham:

$$m\ddot{x}_i = \dot{p}_i - \frac{e}{c} \frac{d}{dt} (A_i) = \dot{p}_i - \frac{e}{c} \left(\frac{\partial A_i}{\partial t} + \frac{\partial A_j}{\partial x_j} \dot{x}_j \right) = -e \left(\frac{\partial \varphi}{\partial x_i} + \frac{1}{c} \frac{\partial A_i}{\partial t} \right) +$$

$$+ \frac{e}{c} \left(\frac{\partial A_i}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \dot{x}_j = eE_i + \frac{e}{c} \varepsilon_{ijk} \dot{x}_j B_k, \quad E_i = -\frac{\partial \varphi}{\partial x_i} - \frac{1}{c} \frac{\partial A_i}{\partial t}, \quad \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} = \varepsilon_{ijk} B_k$$

Biz buyerda Gamilton funksiyasi bilan adashtirib yubormaslik uchun maydon kuchlanganligini B deb belgiladik. Tenglamani vektor ko'rinishda yozib qo'yaylik:

$$m\ddot{\bar{r}} = e\bar{E} + \frac{e}{c} [\dot{\bar{r}} \cdot \bar{B}].$$

4.2.-Misollar.

1. Quyidagi hollarda massasi m ga teng zarraning Gamilton funksiyasi tuzilsin

- silindrik koordinat sistemada,
- dekart koordinat sistemasida,
- garmonik ossilyator uchun (bikrligi k),
- matematik mayatnik uchun (uzunligi l).

2. Quyida berilgan L ga mos Gamilton funksiyalarini toping:

a) $L = \frac{xy^2}{2} + \ln \dot{x}$

b) $L = \frac{m}{2} \sum_a \dot{v}_a^2 - \frac{m^2}{2\mu} \left(\sum_a \dot{v}_a^2 \right)^2 - U.$

v) $L = \frac{\dot{x}^2}{x} + xy^2 + x$

g) $L = \frac{\dot{x}^2 + \dot{y}^2}{x}$

d) $L = \frac{i\dot{v}}{2} + \frac{1}{\sqrt{vu}}$

e) $L = \frac{\dot{r}^2 + r^2 \dot{v}^2}{2} + \frac{1}{r}$

j) $L = \frac{\dot{x}^2 + \dot{y}^2}{2} + xy - jx$

z) $L = \frac{m}{2} (\dot{r}^2 + \omega^2 r^2 \sin^2 \theta) - mgr \cos \theta.$

i) $L = \frac{(m_1 + m_2) \dot{r}^2}{2} + \frac{(\dot{\varphi}^2 - 2\dot{r}\dot{\varphi} \sin \varphi) m_2}{2} - m_2 \sin \varphi$

k) $L = \frac{\dot{x}^2}{2} - \frac{x^2 \omega^2}{2} - \alpha x^3 + \beta x \dot{x}^2.$

l) $L = -mc^2 \sqrt{1 - \beta^2} - e\varphi + \sum_i \frac{\dot{x}_i A_i}{e}, \quad \beta = \frac{\dot{x}}{c} = \frac{v}{c}.$

$$m) L = \frac{1}{2} \left\{ I \left[\dot{\theta}^2 + (\dot{\phi} + \Omega)^2 \sin^2 \theta \right] + I_3 \left[\dot{\psi} + (\dot{\phi} + \Omega) \cos \theta \right]^2 + 3\lambda^2 (I - I_3) \sin^2 \theta \sin^2 \phi \right\}.$$

3. Quyidagi Gamilton funksiyalariga mos Lagranj funksiyalarini toping:

$$a) H = \ln p_x + p_x + \frac{p_y^2}{2}$$

$$b) H = \sqrt{1 + p^2} + r.$$

$$v) H = \frac{p_x^2}{2} + \frac{p_y^2}{2r^2 \sin^2 \nu} + \sin \nu$$

$$g) H = \frac{p_u^2 + p_v^2}{2(u^2 + v^2)} + (u + v^2).$$

$$d) H = \frac{p_x p_y}{\ln x^2} + x^2 y^2.$$

$$e) H = \frac{(p - r^2)^2}{2} + \frac{1}{r}.$$

$$j) H = 2p_\phi p_\theta + \frac{1}{\sqrt{\phi \theta}}.$$

$$z) H = \frac{1}{2m} \left(\frac{p_x^2 + p_y^2}{r^2} \right) + \frac{1}{(r^2 + 1)}.$$

$$i) H = \frac{p^2 + w_0^2 x^2}{2} + \lambda \frac{(p^2 + w_0^2 x^2)^2}{4}.$$

$$k) H = \frac{\bar{p}^2}{2m} - \bar{p} \bar{a}.$$

4. $L = \frac{1}{2} \dot{\phi}^2 - \omega^2 (1 - \cos \phi)$ da umumlashgan koordinatani $q = 2 \sin \frac{\phi}{2}$ deb H ni toping.

§4.3. Puasson qavslari.

Puasson qavslari, ilgari ta'kidlab o'tganimizdek, differensial operatoridir va uning nazariy fizika ta'limidagi roli katta ahamiyatga egadir. Bunday operatorlar bilan ishlash, ularning qiymatlarini hisoblab topish, amaliy jihatdan juda muhimdir. Asosiy fizik kattaliklardan bo'lgan koordinata, impuls va impuls momentlari komponentalaridan tuzilgan Puasson qavslarini hisoblashni bilish muhimdir. Ayniqsa, ularning vektor xarakterga ekanligi, vektorlar va tenzorlar analizi borasidagi bilimlar ko'nikmasini qo'llashni taqozo etishi talabalarda jiddiy qiyinchiliklar tug'dirishi mumkin. Albatta, dekart komponentalardan olingan uzundan - uzoq hosilalardan iborat Puasson qavsining qiymatlarini

hisoblay olish mumkin, ammo tenzor belgilashlar ko'rinishidagi ifodalarni topish bir muncha qiyindek tuyulsa ham, bu usul afzalligi yaqqol namoyon bo'ladi. Shuning uchun ham, avval dekart komponentalardan an'anviy hosilalarni topib yechish usuliga masalalar ko'ramiz, so'ngra egri chiziqli koordinatalarda operator ko'rinishlarini qo'llash, vektorlar analizi formulalaridan foydalanish, tenzor ifodalar ustida amal bajarish kabi usullarga o'tib boramiz. Muhimi, bunday hisoblashni amalga oshirayotganda Puasson qavslari hossalardan foydalanish natijaga tezroq eltishini anglashdir. Jumladan, ushbu fundamental Puasson qavslarining qiymatlari asqotadi:

$$\{p_i, q_k\} = \delta_{ik}, \quad \{p_i, p_k\} = \{q_i, q_k\} = 0,$$

bu yerda δ_{ik} Kroneker simvoli bo'lib, uning qiymati indeksleri bir xil bo'lganda birga teng, aks holda nolga teng.

1-masala. Moddiy nuqta impulsi bilan impuls momentining dekart komponentalaridan tuzilgan Puasson qavslarini hisoblang.

Yechilishi. Bunday qavslarning soni ko'p bo'lgani uchun dastavval, masalan $\{M_x, p_y\}$ ni hisoblashni ko'raylik va ba'zi xulosalarga kelish bilan tanishib chiqaylik. Buni (4.5) ifodaga asosan konkret ko'rinishda yozib olaylik:

$$\{M_x, p_y\} = \sum_i \left(\frac{\partial M_x}{\partial p_i} \frac{\partial p_y}{\partial q_i} - \frac{\partial M_x}{\partial q_i} \frac{\partial p_y}{\partial p_i} \right).$$

Uch o'lchovli fazoda yig'indi alohidi alohidi $i = 3$ ta qiymatni qabul qiladi: $i = x, y, z$

Bizning hol uchun $q_i \rightarrow x, y, z, p_j \rightarrow p_x, p_y, p_z$

Ya'ni,

$$p_x = m\dot{x}, p_y = m\dot{y}, p_z = m\dot{z}, M_x = yp_z - zp_y, M_y = zp_x - xp_z, M_z = xp_y - yp_x.$$

Natijada, berilgan ifodada 6 ta haddan iborat bo'ladi va har bir hosila olish amalini bajarish har doim ham juda engil emas:

$$\{M_x, p_y\} = \frac{\partial M_x}{\partial p_x} \frac{\partial p_y}{\partial x} - \frac{\partial M_x}{\partial x} \frac{\partial p_y}{\partial p_x} + \frac{\partial M_x}{\partial p_y} \frac{\partial p_y}{\partial y} - \frac{\partial M_x}{\partial y} \frac{\partial p_y}{\partial p_y} + \frac{\partial M_x}{\partial p_z} \frac{\partial p_y}{\partial z} - \frac{\partial M_x}{\partial z} \frac{\partial p_y}{\partial p_z}.$$

Ammo, dinamik kattaliklar bo'lgan impuls va koordinatalar bir-biriga bog'liq bo'lmagan (mustaqil) o'zgaruvchilar ekanligidan va xususiy hosilalar xossasidan foydalanib ushbularni yozishimiz mumkin:

$$\frac{\partial p_i}{\partial q_k} = \frac{\partial q_i}{\partial p_k} = 0, \quad \frac{\partial p_i}{\partial p_k} = \frac{\partial q_i}{\partial q_k} = \delta_{ik}, \quad q_i \rightarrow x, y, z, \quad p_i \rightarrow p_x, p_y, p_z.$$

Natijada

$$\{M_x, p_y\} = \frac{\partial M_x}{\partial p_x} \frac{\partial p_y}{\partial x} - \frac{\partial M_x}{\partial x} \frac{\partial p_y}{\partial p_x} + \frac{\partial M_x}{\partial p_y} \frac{\partial p_y}{\partial y} - \frac{\partial M_x}{\partial y} \frac{\partial p_y}{\partial p_y} + \frac{\partial M_x}{\partial p_z} \frac{\partial p_y}{\partial z} - \frac{\partial M_x}{\partial z} \frac{\partial p_y}{\partial p_z} = -\frac{\partial M_x}{\partial y} \delta_{yy} = -p_z.$$

Demak, $\{M_x, p_y\} = -p_z.$

Xuddi shuningdek, $\{M_x, p_z\} = p_y, \quad \{M_x, p_x\} = 0$

ekanligini hisoblab topish qiyin emas. Qolgan Poisson qavslarining noldan farqli qiymatlarini shu ifodalarda x, y, z larni siklik almashtirish natijasida hosil bo'lishini ko'rish qiyin emas.

Bu ifodalarni butunlay antisimmetrik tenzor ε_{123} orqali yozib olishimiz mumkin (Ilovaga qarang)

$$\{M_i, p_j\} = -\varepsilon_{ijk} p_k,$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \varepsilon_{123} \text{ - siklik almashtirilganda.} \\ -1 & \varepsilon_{213} \text{ - siklik almashtirilganda.} \\ 0 & ij k \text{ larning boshqa qiymatlarida.} \end{cases}$$

Mazkur formula orqali barcha shu kabi qavslarning qiymatini topish mumkin, masalan, $\{M_y, p_x\}$ ni yuqoridagi ifodadan topaylik:

$$\{M_y, p_x\} = -\varepsilon_{yxz} p_z = \varepsilon_{zyx} p_z = p_z.$$

Shuni ta'kidlash lozimki, berilgan ifodani xususiy hosilalar xossasidan foydalanib qisqaroq yo'l bilan hisoblash mumkin edi:

$$\{M_x, p_y\} = \sum_i \left(\frac{\partial M_x}{\partial p_i} \frac{\partial p_y}{\partial q_i} - \frac{\partial M_x}{\partial q_i} \frac{\partial p_y}{\partial p_i} \right) = \sum_i \left(\frac{\partial M_x}{\partial p_i} \delta_{iy} - \frac{\partial M_x}{\partial q_i} \frac{\partial p_y}{\partial p_i} \right) = -\frac{\partial M_x}{\partial y} \frac{\partial p_y}{\partial p_y} = -\frac{\partial M_x}{\partial y} = -p_z.$$

2-usul. Yuqorida biz Poisson qavslari natijasini to'g'ridan-to'g'ri hosilalarni hisoblash yo'li bilan topdik. Ko'p hollarda bunday masalani natijasi ma'lum bo'lgan soddarok ko'rinishdagi Poisson qavslariga (masalan, fundamental qavslar) keltirib olish yo'li bilan ham hal qilsa bo'ladi. Bunday amallarni Poisson qavslarining xossalariidan foydalanib bajariladi:

$$\{M_x, p_y\} = \{yp_z - zp_y, p_y\} = \{yp_z, p_y\} - \{zp_y, p_y\} = y\{p_z, p_y\} + p_z\{y, p_y\} -$$

$$-z\{p_y, p_y\} - p_y\{z, p_y\} = y \cdot 0 + p_z \cdot (-1) - z \cdot 0 - p_y \cdot 0 = -p_z.$$

Yana o'sha natija. Talabalarining ko'zi son indeksli komponentalarga o'rganishi uchun bu natijani tenzor indekslar bilan qaytadan yozib ko'rsataylik:

$$q_i \rightarrow x_1, x_2, x_3, \quad p_i \rightarrow p_1, p_2, p_3$$

$$M_1 = x_2 p_3 - x_3 p_2, \quad M_2 = x_3 p_1 - x_1 p_3, \quad M_3 = x_1 p_2 - x_2 p_1$$

$$\{M_1, p_2\} = \{x_2 p_3 - x_3 p_2, p_2\} = \{x_2 p_3, p_2\} - \{x_3 p_2, p_2\} = x_2 \{p_3, p_2\} + p_3 \{x_2, p_2\} -$$

$$-x_3 \{p_2, p_2\} - p_2 \{x_3, p_2\} = x_2 \cdot 0 + p_3 \cdot (-\delta_{32}) - x_3 \cdot 0 - p_2 \cdot (-\delta_{32}) = -p_3.$$

3-usul. Vektorlar va tenzorlar analizidan ma'lumki (ilovaga qarang) impuls momenti komponentalarini ikki vektorning vektor ko'paytmasi shaklida butunlay antisimmetrik tenzor orqali yozib olish mumkin:

$$M_i = (\vec{M})_i = [\vec{r} \vec{p}]_i = -\varepsilon_{ijk} x_j p_k = -\varepsilon_{ilm} x_l p_m.$$

Impuls momenti va impuls komponentalari orasidagi Poisson qavslari qiymatlari Poisson qavslari xossalari va fundamental Poisson qavslari qiymatlarini inobatga olgan holda osongina topiladi:

$$\{M_i, p_j\} = \{\varepsilon_{ilk} x_l p_k, p_j\} = \varepsilon_{ilk} \{x_l p_k, p_j\} = \varepsilon_{ilk} [x_l \{p_k, p_j\} + p_k \{x_l, p_j\}] =$$

$$= \varepsilon_{ilk} [x_l \cdot 0 + p_k (-\delta_{lj})] = -\varepsilon_{ilk} p_k \delta_{lj} = -\varepsilon_{ijk} p_k.$$

Pirovardida, yana o'sha yuqoridagi natijani oldik:

$$\{M_i, p_j\} = -\varepsilon_{ijk} p_k.$$

2- *masala.* Moddiy nuqtaning impuls momentlari dekart komponentlaridan tuzilgan Puasson qavslari hisoblansin.

Yechilishi. Xususan, $\{M_x, M_y\}$ ni to'g'ridan-to'g'ri hosilalarni hisoblash yo'li bilan topishni ko'raylik

$$\begin{aligned} \{M_x, M_y\} &= \frac{\partial M_x}{\partial p_x} \frac{\partial M_y}{\partial x} - \frac{\partial M_x}{\partial x} \frac{\partial M_y}{\partial p_x} + \frac{\partial M_x}{\partial p_y} \frac{\partial M_y}{\partial y} - \frac{\partial M_x}{\partial y} \frac{\partial M_y}{\partial p_y} + \frac{\partial M_x}{\partial p_z} \frac{\partial M_y}{\partial z} - \frac{\partial M_x}{\partial z} \frac{\partial M_y}{\partial p_z} = \\ &= 0 \cdot \frac{\partial M_y}{\partial x} - 0 \cdot \frac{\partial M_y}{\partial p_x} + \frac{\partial M_x}{\partial p_y} \cdot 0 - \frac{\partial M_x}{\partial y} \cdot 0 + \frac{\partial M_x}{\partial p_z} \frac{\partial M_y}{\partial z} - \frac{\partial M_x}{\partial z} \frac{\partial M_y}{\partial p_z} = yp_x - xp_y = -M_z. \end{aligned}$$

Demak $\{M_x, M_y\} = -M_z.$

Xuddi shuningdek $\{M_y, M_z\} = -M_x, \quad \{M_z, M_x\} = -M_y.$

Bir xil komponentalarning bir-biri bilan Puasson qavslari nolga aylanib ketadi, chunki har bir ayriluvchi hadlar bir-biriga tengdir:

$$\{M_x, M_x\} = \{M_y, M_y\} = \{M_z, M_z\} = 0.$$

Bu natijalarni antisimmetrik tenzor orqali umumlashtirib yozib olish mumkin. Ixtiyoriy ikki komponentalarning Puasson qavsi uchinchisi orqali ifodalanadi

$$\{M_i, M_j\} = -\varepsilon_{ijk} M_k.$$

2-usul. Bu ifodaning qiymatini fundamental qavslarga keltirib olish qiyin ish emas va faqatgina ikkita had noldan farqli ifodaga ega, ularning yig'indisi esa impuls momentining uchinchi komponentasiga tengdir:

$$\begin{aligned} \{M_1, M_2\} &= \{x_2 p_3 - x_3 p_2, x_1 p_1 - x_1 p_1\} = \{x_2 p_3, x_3 p_1 - x_1 p_1\} - \{x_3 p_2, x_3 p_1 - x_1 p_1\} = \\ &= \{x_2 p_3, x_3 p_1\} - \{x_2 p_3, x_1 p_1\} - \{x_3 p_2, x_3 p_1\} + \{x_3 p_2, x_1 p_1\} = x_2 \{p_3, x_3 p_1\} + p_3 \{x_2, x_3 p_1\} - \\ &- x_2 \{p_3, x_1 p_1\} - p_3 \{x_2, x_1 p_1\} - x_3 \{p_2, x_3 p_1\} - p_2 \{x_3, x_3 p_1\} + x_3 \{p_2, x_1 p_1\} + p_2 \{x_3, x_1 p_1\} = \\ &= x_2 x_3 \{p_3, p_1\} + x_2 p_1 \{p_3, x_3\} + p_3 x_3 \{x_2, p_1\} + p_3 p_1 \{x_2, x_3\} - x_2 x_1 \{p_3, p_1\} - x_2 p_3 \{p_3, x_1\} - \\ &- p_3 x_1 \{x_2, p_1\} - p_3 p_3 \{x_2, x_1\} - x_3 x_3 \{p_2, p_1\} - x_3 p_1 \{p_2, x_3\} - p_2 x_3 \{x_3, p_1\} - p_2 p_1 \{x_3, x_3\} + \\ &+ x_3 x_1 \{p_2, p_1\} + x_3 p_3 \{p_2, x_1\} + p_2 x_1 \{x_3, p_1\} + p_2 p_3 \{x_3, x_1\} = x_2 p_1 \{p_3, x_3\} + p_2 x_1 \{x_3, p_3\} = \\ &= x_2 p_1 \cdot \delta_{33} + p_2 x_1 \cdot (-\delta_{33}) = x_2 p_1 - p_2 x_1 = -M_3. \end{aligned}$$

3-usul. Mazkur ifodani tenzor belgilashlar asosida keltirib chiqaraylik. Bizga ma'lumki:

$$M_i = [\vec{r}, \vec{p}]_i = \varepsilon_{ikl} x_k p_l, \quad M_j = [\vec{r}, \vec{p}]_j = \varepsilon_{jmn} x_m p_n.$$

U holda izlanayotgan ifoda ushbu ko'rinishni oladi:

$$\{M_i, M_j\} = \{\varepsilon_{ikl} x_k p_l, \varepsilon_{jmn} x_m p_n\} = \varepsilon_{ikl} \varepsilon_{jmn} \{x_k p_l, x_m p_n\}.$$

Bu erda doimiy Puasson qavsi ichidan tashqariga chiqarib yozildi va bu qavsni

ko'paytmalardan olingan Puasson qavslari xossalariidan foydalanib yozib olaylik:

$$\{x_k p_l, x_m p_n\} = x_k \{p_l, x_m p_n\} + \{x_k, x_m p_n\} p_l = x_k x_m \{p_l, p_n\} + x_k \{p_l, x_m\} p_n +$$

$$+x_m \{x_k, p_n\} p_l + \{x_k, x_m\} p_n p_l = x_k x_m \cdot 0 + x_k \{p_l, x_m\} p_n +$$

$$+x_m \{x_k, p_n\} p_l + 0 \cdot p_n p_l = \delta_{ln} x_k p_n + (-\delta_{kn}) x_m p_l$$

Bu natijani oldingi ifodaga qo'yib davom ettirsak va birlik antisimmetrik tenzor bilan ishlash qoidasini qo'llasak:

$$\{M_i, M_j\} = \varepsilon_{ikl} \varepsilon_{jmn} \{x_k p_l, x_m p_n\} = \varepsilon_{ikl} \varepsilon_{jmn} [\delta_{ln} x_k p_n + (-\delta_{kn}) x_m p_l] = \varepsilon_{ikln} \varepsilon_{jmn} x_k p_n - \varepsilon_{ikln} \varepsilon_{jmn} x_m p_l =$$

$$= \varepsilon_{ikn} \varepsilon_{jym} x_k p_n - \varepsilon_{ilm} \varepsilon_{jmn} x_m p_l = (\delta_{in} \delta_{jk} - \delta_{jn} \delta_{ik}) x_k p_n - (\delta_{im} \delta_{jl} - \delta_{ml} \delta_{ji}) x_m p_l = x_j p_i - \delta_{ij} x_k p_k -$$

$$-x_i p_j + \delta_{ij} x_m p_m = x_j p_i - x_i p_j = -(x_i p_j - x_j p_i) = -\varepsilon_{ijk} M_k.$$

Demak, $\{M_i, M_j\} = -\varepsilon_{ijk} M_k.$

3-masala. $\{M_x, M^2\}$ hisoblansin.

Yechilishi. Bu masalani ham Puasson qavslarining asosiy ifodasiga berilgan kattaliklarni o'rniga qo'yib, to'g'ridan- to'g'ri hosilalarni hisoblab toppish yo'li bilan yechishdan boshlaylik:

$$\{M_x, M^2\} = 0 - 0 + \frac{\partial M_x}{\partial p_x} \frac{\partial M^2}{\partial y} - \frac{\partial M_x}{\partial y} \frac{\partial M^2}{\partial p_x} + \frac{\partial M_x}{\partial p_z} \frac{\partial M^2}{\partial z} - \frac{\partial M_x}{\partial z} \frac{\partial M^2}{\partial p_z} = (-z) \frac{\partial}{\partial y} (M_x^2 + M_y^2 + M_z^2) -$$

$$-p_z \frac{\partial}{\partial p_y} (M_x^2 + M_y^2 + M_z^2) + y \frac{\partial}{\partial z} (M_x^2 + M_y^2 + M_z^2) - (-p_x) \frac{\partial}{\partial p_z} (M_x^2 + M_y^2 + M_z^2) =$$

$$= 2M_z (-p_x - xp_x) + 2M_y (yp_x - xp_y) = 2M_z M_y + 2M_y (-M_z) = 0.$$

Demak, $\{M_x, M^2\} = 0.$

Bu natijaga umumiyroq ko'rinish beraylik, uning uchun vektorlar skalyar ko'paytmasi ifodasidan foydalanamiz:

$$M^2 = (\vec{M}\vec{M}) = \sum M_k M_k = \sum M_n M_n.$$

Ko'paytmaning Puasson qavslarini hisoblash usuli, hamda oldingi masala natijalaridan foydalanib ushuni olamiz:

$$\{M_i, M^2\} = \{M_i, \sum M_k M_k\} = \sum \{M_i, M_k M_k\} = \sum [\{M_i, M_k\} M_k + \{M_i, M_k\} M_k] =$$

$$= 2 \sum (-\varepsilon_{ikn} M_j) M_k = 2 \sum (-\varepsilon_{ikn} M_k M_j) = 2 [\vec{M}\vec{M}] = 0.$$

Ya'ni, impuls momenti komponentalarining uni kvadrati bilan hosil qilgan Puasson qavslari qiymati nolga teng ekan:

$$\{M_i, M^2\} = 0.$$

Bu muhim natija nazariy fizikaning keyingi bo'limlarida keng qullaniladi.

4-masala. $\{(\vec{a} \vec{p}), (\vec{b} \vec{r})\}$ hisoblansin. \vec{a}, \vec{b} - doimiy vektorlar.

Yechilishi. Bu ifodani hisoblashdan oldin impuls va radius vektor dekart komponentalari orasidagi Puasson qavslari qiymatini topaylik:

$$\{p_k, q_j\} = \sum_i \left(\frac{\partial p_k}{\partial p_i} \frac{\partial q_j}{\partial q_i} - \frac{\partial p_k}{\partial q_i} \frac{\partial q_j}{\partial p_i} \right) = \sum_i \frac{\partial p_k}{\partial p_i} \delta_{ij} - 0 = \delta_{kj}.$$

Hamda, tenzorlar analizidagi Eynshteyn qoidasi, ya'ni qaytariluvchi indekslar bo'yicha yigindi amali ketadi deb kelishilganligi sababli skalyar ko'paytma ifodasidagi yig'indi alomati \sum ni tushirib qoldirishimiz mumkinligini nazarda tutamiz. Shunga qaramasdan, bundan buyon anglashuvmovchilik oldini olib, bu belgini ba'zan ishlatib turgan ma'qul, hamda belgi ostidagi yig'indi indeksini ham tushirib qoldirishimiz mumkin, albatta qanday indekslar bo'yicha yig'indini olish ayon bo'lsa. Ushbu ekvivalent ifodalarni yodda tuting:

$$(\vec{a} \vec{b}) = \sum_i a_i b_i = \sum_k a_k b_k = \sum_{i,k} a_i b_k \delta_{ik} = a_i b_i = a_n b_n = a_n b_m \delta_{nm}$$

Shularni e'tiborga olib va Puasson qavslari xossalariiga asoslanib berilgan ifodani hisoblaymiz:

$$\{(\vec{a} \vec{p}), (\vec{b} \vec{r})\} = \left\{ \sum_i a_i p_i, \sum_k b_k q_k \right\} = \sum_{i,k} a_i b_k \{p_i, q_k\} = \sum_{i,k} a_i b_k \delta_{ik} = \sum_i a_i b_i = (\vec{a} \vec{b}).$$

Shunday qilib, $\{(\vec{a} \vec{p}), (\vec{b} \vec{r})\} = (\vec{a} \vec{b}).$

5-masala. Kepler masalasida to'la energiya, impuls momenti va Laplas vektori saqlanuvchi kattalik ekanligini ko'rsating.

Yechilishi. Ikkinchi bobning §2.4 bandidagi 5-masalada Kepler masalasi Lagranj usuli bilan tahlil etilgan edi. Shu muammoni Gamilton usuli bilan yechilishini ko'raylik. Kepler masalasining Gamilton funksiyasi ushbu ko'rinishda:

$$H = \frac{p^2}{2m} - \frac{\alpha}{r}.$$

Vaqtga oshkora bog'liqlik yo'qligidan (4.4) ga ko'ra

$$\frac{dH}{dt} = \dot{H} = \frac{\partial H}{\partial t} + \{H, H\} = \frac{\partial H}{\partial t} = 0,$$

ya'ni, to'la energiya saqlanuvchi kattalik ekan: $E = const.$

Xuddi shuningdek (4.4) ga binoan impuls momentining o'zgarishini topaylik:

$$\dot{M}_i = \{H, M_i\} = \{H, [\vec{r} \vec{p}]\}_i = \{H, \varepsilon_{ijk} x_j p_k\} = \varepsilon_{ijk} x_j \{H, p_k\} +$$

$$+ \varepsilon_{ijk} p_k \{H, x_j\} = \varepsilon_{ijk} x_j \left\{ -\frac{\alpha}{r}, p_k \right\} + \varepsilon_{ijk} p_k \left\{ \frac{p^2}{2m}, x_j \right\} =$$

$$\varepsilon_{ijk} x_j (-\alpha x_k r^{-2}) + \varepsilon_{ijk} p_k \left(-\frac{1}{2m} 2p_j \right) =$$

$$= \varepsilon_{ijk} x_j x_k \frac{\alpha}{r^3} + \varepsilon_{ijk} p_j p_k m^{-1} = \frac{\alpha}{r^3} [\vec{r} \vec{r}]_i + m^{-1} [\vec{p} \vec{p}]_i = 0.$$

Demak, $(\dot{M})_i = 0.$

Ya'ni, impuls momenti ham saqlanuvchi kattalik ekan.

Endi, Laplas vektorining o'zgarishini ko'rsatamiz. Uning uchun Laplas vektorini quyidagi ko'rinishda yo'zib olaylik:

$$\vec{\varepsilon} = \frac{1}{\alpha} [\dot{\vec{r}} \vec{M}] - \frac{\vec{r}}{r}.$$

Uning tashkil etuvchisining (komponentalarining) o'zgarishini ko'raylik va hisoblab topaylik:

$$\dot{\varepsilon} = \{H, \vec{\varepsilon}\} = \left\{ H, \frac{1}{\alpha} [\dot{\vec{r}} \vec{M}] - \frac{\vec{r}}{r} \right\} \Rightarrow \frac{1}{\alpha} \{H, [\dot{\vec{r}} \vec{M}]\}_i - \left\{ H, \frac{x_i}{r} \right\} =$$

$$= \frac{1}{\alpha} \{H, \varepsilon_{ijk} \dot{r}_j M_k\} - \left\{ H, \frac{x_i}{r} \right\} = \frac{\varepsilon_{ijk}}{m\alpha} \{H, p_j\} M_k - \left\{ H, \frac{x_i}{r} \right\}.$$

Bu yerda $\vec{p} = m\dot{\vec{r}}$ va markaziy maydonda \vec{M} ning saqlanuvchi kattalik ekanligi hisobga olindi. Ana endi, Puasson qavslarining qiymatlarini hisoblaylik:

$$\dot{\varepsilon}_i = \frac{1}{m\alpha} \varepsilon_{ijk} M_k \left\{ \left(-\frac{\alpha}{r} \right), p_j \right\} - \left\{ \frac{p^2}{2m}, \frac{x_i}{r} \right\} =$$

$$= \frac{1}{m} \varepsilon_{ijk} M_k (x_j r^{-1-2}) - \frac{1}{2mr} \{p^2, x_i\} - \frac{1}{2m} x_i \left\{ p^2, \frac{1}{r} \right\} =$$

$$= -\frac{1}{mr^3} \varepsilon_{ijk} x_j M_k - \frac{1}{mr} p_i - \frac{1}{2m} x_i (2p_n \{p_n, r^{-1}\}) = -\frac{1}{mr^3} \varepsilon_{ijk} x_j M_k - \frac{1}{mr} p_i + \frac{x_i}{mr^3} x_n p_n =$$

$$= -\frac{1}{mr^3} \varepsilon_{ijk} x_j (\varepsilon_{knl} x_n p_l) - \frac{1}{mr} p_i + \frac{x_i}{mr^3} x_n p_n = -\frac{1}{mr^3} \varepsilon_{ijk} \varepsilon_{nlk} x_j x_n p_l - \frac{1}{mr} p_i + \frac{x_i}{mr^3} x_n p_n =$$

$$= -\frac{1}{mr^3} (\delta_{in} \delta_{jl} - \delta_{il} \delta_{jn}) x_j x_n p_l - \frac{1}{mr} p_i + \frac{x_i}{mr^3} x_n p_n = -\frac{1}{mr^3} [x_i x_l p_l - x_n x_n p_l] - \frac{1}{mr} p_i + \frac{x_i}{mr^3} x_n p_n = 0$$

Bu keltirib chiqarishda uchinchi rangli antisimmetrik tenzor xossalardan foydalandik (Ilovaga qarang).

Agarda, tenzor ko'rinishdagi ifodalardan odatda ishlatiladigan vektor shaklga o'tilsa bu natija yanada yaqqolroq ko'rinadi:

$$(\dot{\varepsilon})_i = \frac{1}{mr^3} [\vec{r} \vec{M}]_i + \left[\frac{\vec{r} (\vec{r} \vec{p})}{mr^3} - \frac{\vec{p}}{mr} \right]_i = -\frac{1}{mr^3} [\vec{r} (\vec{r} \vec{p}) - \vec{p} (\vec{r} \vec{r})]_i + \left[\frac{\vec{r} (\vec{r} \vec{p})}{mr^3} - \frac{\vec{p}}{mr} \right]_i = 0.$$

Demak, $(\dot{\varepsilon})_i = 0.$

Ya'ni, Laplas vektori ham saqlanuvchi kattalik ekan.

$$q_i = -\frac{\partial F_4}{\partial p_i}, \quad Q_i = -\frac{\partial F_4}{\partial P_i}, \quad H' = H + \frac{\partial F_4}{\partial t} \quad (4.18)$$

Biz, bundan buyon, vaqt t invariant qoladi deb hisoblaymiz. Kanonik almashtirishlar Puasson qavslarini ham invariant qoldiradi:

$$\{f, g\}_{p,q} = \{f, g\}_{P,Q} \quad (4.19)$$

1-masala. $F_1 = \sum q_i Q_i$ hosil qiluvchi funksiya yordamida sodir bo'ladigan kanonik almashtirishlarni toping.

Yechilishi. (4.14) ifodaga asosan:

$$p_i = \frac{\partial F_1}{\partial q_i} = Q_i, \quad P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i, \quad H' = H.$$

Shunday qilib, $P_i = -q_i$, $Q_i = p_i$ almashtirishlar, haqiqatdan ham Gamilton tenglamasining ko'rinishini o'zgartirmasligi yaqqol ko'rinib turibdi, chunki $H = H'$ va yuqoridagi almashtirishlarni Gamilton tenglamasiga olib borib qo'ysak, bas. Bu almashtirishlar koordinata va impulsning o'rinlarini almashtiradi (yangi koordinata eski impulsga teng, yangi impuls esa, eski koordinataga mos). Bu misol Gamilton tenglamalarida koordinata va impulsning tutgan o'rni bir xil ekanligining timsolidir. Bularning orasidagi farq faqatgina nomidadir xolos. Shuning uchun ham, biz ularning koordinata va impuls degan nomlarini shartli ravishda ekanligini nazarda tutmoqimiz lozim. Ko'pincha ularni *kanonik qo'shma kattaliklar* deyiladi.

2-masala. $F_2(q, P) = q^2 e^P$ hosil qiluvchi funksiya yordamida sodir bo'ladigan kanonik almashtirishlarga olib keladigan hosil qiluvchi funksiya $F_3(p, Q)$ topilsin.

Yechilishi: Hosil qiluvchi funksiyalar vaqtga oshkora bog'liq bo'lmaganligi uchun $H = H'$ va (4.16) tenglamalardan $F_2 = q^2 e^P$ ga mos kanonik almashtirishlarni topamiz:

$$p = \frac{\partial F_2}{\partial q} = 2qe^P, \quad Q = \frac{\partial F_2}{\partial P} = q^2 e^P.$$

Masalaning qo'yilishiga binoan shunday $F_3(p, Q)$ hosil qiluvchi funksiyani topish kerakki, unga mos kanonik almashtirishlar qam aynan yuqoridagi ifodaga teng bo'lsin. Bu hosil qiluvchi funksiyaga mos kanonik almashtirishlar (4.17) ifoda bilan aniqlanadi:

$$q = -\frac{\partial F_3}{\partial p}, \quad P = -\frac{\partial F_3}{\partial Q}.$$

Bu ifodalarning chap tomonlarini yuqoridagi tengliklardan aniqlab, so'ngra integrallab $F_3(p, Q)$ funksiyani topamiz:

$$q = \frac{2Q}{p} = -\frac{\partial F_3}{\partial p}, \quad F_3(p, Q) = -2Q \ln p + C(Q), \quad e^P = \frac{p^2}{4Q},$$

$$P = \ln\left(\frac{p^2}{4Q}\right) = -\frac{\partial F_3}{\partial Q}; \quad \ln\left(\frac{p^2}{4Q}\right) = 2 \ln p - \frac{\partial C(Q)}{\partial Q}, \quad \frac{\partial C(Q)}{\partial Q} = \ln(4Q).$$

Buyerdan bo'laklab integrallab, quyidagini olamiz

$$C(Q) = Q(\ln(4Q) - 1).$$

Demak,

$$F_3(p, Q) = -2Q \ln p + Q(\ln(4Q) - 1) = Q\left(1 + \ln\left(\frac{p^2}{4Q}\right)\right).$$

Shunday qilib,

$$F_2(q, P) = q^2 e^P \quad \text{va} \quad F_3(p, Q) = Q\left(1 + \ln\left(\frac{p^2}{4Q}\right)\right)$$

hosil qiluvchi funksiyalar aynan bir xil kanonik almashtirishlarga olib keladi:

$$p = \ln\left(\frac{p}{2q}\right), \quad Q = \frac{qp}{2}.$$

3-masala. Ushbu hosil qiluvchi funksiya

$$F_1(q, Q) = m\omega q^2 \frac{\text{ctg} Q}{2}$$

yordamida sodir bo'ladigan kanonik almashtirishlar topilsin. m va ω larni doimiy kattaliklar deb, garmonik ossillyator uchun harakat tenglamasi yechilsin.

Yechilishi. Bu hosil qiluvchi funksiya uchun (4.14) tenglamalarga asosan:

$$p = \frac{\partial F_1}{\partial q} = m\omega q \operatorname{ctg} Q, \quad P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2\sin^2 Q}.$$

Bu yerdan albatta yangi P, Q larni eski q, p lar orqali ifodalash mumkin, lekin hozir eski q, p larni yangi Q, P lar orqali ifodalab, masalani yozishimiz qulayroq. Oxirgi ifodadan

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q.$$

Buni birinchisiga olib borib qo'ysak, $p = \sqrt{2m\omega P} \cos Q$ ni olamiz. Hosil qiluvchi funksiya F_1 vaqtga oshkora bog'liq bo'lmaganligi uchun $H' = H$ ya'ni, yangi Gamilton funksiyasi H' eski Gamilton funksiyasidan q, p larni yuqoridagi bog'lanishlarga mos Q, P orqali ifodalasak yetarli. Ma'lumki, garmonik ossilyatorning Gamilton funksiyasi massa m va ω chastota orqali quyidagicha ifodalanadi:

$$H = \frac{mq^2}{2} + \frac{kq^2}{2} = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}.$$

Yangi o'zgaruvchilarga o'tsak,

$$H' = \omega P \cos^2 Q + \omega P \sin^2 Q = \omega P.$$

Demak,

$$H' = \omega P.$$

Bu Gamilton funksiyasida Q siklik koordinataga aylanib qolganligi uchun, unga mos umumlashgan impuls P saqlanuvchi kattalik bo'ladi. (Kanonik almashtirishlarni qo'llashning afzalligi ham xuddi shundadir. Ba'zi masalalarda hamma koordinatalarni siklik koordinatalarga aylantirish mumkin). Gamilton funksiyasi vaqtga oshkora bog'liq bo'lmagani uchun $P = \frac{E}{\omega}$ doimiy kattalikdir, E -energiya. Umumlashgan koordinata Q ni aniqlovchi harakat tenglamasi esa juda sodda ko'rinishga ega:

$$\dot{Q} = \frac{\partial H}{\partial P} = \omega, \quad Q = \omega t + \alpha.$$

Buyerda α - boshlang'ich shartlardan aniqlanuvchi integrallash doimiysidir. Eski koordinata uchun esa, oxirgi ifodadan quyidagini olamiz:

$$q = \sqrt{\frac{2P}{m\omega}} \sin Q = \sqrt{\frac{2E}{\omega^2}} \sin(\omega t + \alpha).$$

4-masala. $F_2(q, P) = q \ln P$ hosil qiluvchi funksiya yordamida sodir bo'ladigan kanonik almashtirishlar topilsin.

Yechilishi. $H' = H$ chunki hosil qiluvchi funksiya vaqtga bog'liq emas.

Demak,

$$p = \frac{\partial F_2}{\partial q} = \ln P, \quad Q = \frac{\partial F_2}{\partial P} = \frac{q}{P}, \quad Q = qe^{-p}, \quad P = e^p.$$

5-masala. Bir jinsli og'irlik maydonida harakatlanayotgan zarraning Gamilton funksiyasi

$$H = \frac{p^2}{2m} - mgx$$

hosil qiluvchi funksiya

$$F_2(x, p', t) = xp' + mgt$$

ga mos kanonik almashtirishlar natijasida qanday ko'rinishni oladi?

Yechilishi: Hosil qiluvchi funksiya ko'rinishidan, bu yerda eski (x, p) o'zgaruvchilarga yangi (x', p') o'zgaruvchilar mos kelishi ayon bo'lib turibdi:

$$p = \frac{\partial F_2}{\partial x} = p' + mgt, \quad x' = \frac{\partial F_2}{\partial p'} = x, \Rightarrow$$

$$x = x', \quad p = p' + mgt, \quad H' = H + \frac{\partial F_2}{\partial t} = \frac{(p' + mgt)^2}{2m}.$$

6-masala. Lagranj tenglamasining nuqtaviy almashtirishlar

$$q_i = q_i(Q_1, Q_2, \dots, Q_s, t), \quad i = 1, 2, \dots, S$$

natijasida o'z ko'rinishini saqlab qolishi ko'rsatilsin.

Yechilishi: Avval, masalani yechimini $S=1$, erkinlik darajasi bir bo'lgan holda ko'rsataylik:

$$q = q(Q, t), \quad \frac{dq}{dt} = \frac{\partial q}{\partial t} + \frac{\partial q}{\partial Q} \frac{dQ}{dt} = \frac{\partial q}{\partial t} + \frac{\partial q}{\partial Q} \dot{Q}, \quad \dot{q} = \frac{\partial q}{\partial Q} \dot{Q} + \frac{\partial q}{\partial t}, \quad (1).$$

Bu ifodalardan ko'rinib turibdiki, umumlashgan tezıklarda umumlashgan kordinatalarga ham bog'liqlik paydo bo'ladi, ya'ni

$$L'(Q, \dot{Q}, t) = L(q(Q, t), \dot{q}(Q, \dot{Q}, t), t).$$

Eski o'zgaruvchilardagi Lagranj tenglamasini yozib olaylik va uning mos hadlari nuqtaviy almashtirishlarda qanday o'zgarishini ko'raylik:

$$\begin{aligned} \frac{d}{dt} \frac{\partial L'}{\partial \dot{q}} - \frac{\partial L'}{\partial q} = 0, \quad \frac{\partial L'}{\partial Q} = \frac{\partial L}{\partial q} \frac{\partial q}{\partial Q} + \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial Q}, \quad \frac{\partial \dot{q}}{\partial Q} = \frac{\partial}{\partial Q} (\dot{q}) = \frac{\partial}{\partial Q} \left(\frac{dq}{dt} \right) = \frac{d}{dt} \frac{\partial q}{\partial Q}, \\ \Rightarrow \frac{\partial L'}{\partial Q} = \frac{\partial L}{\partial q} \frac{\partial q}{\partial Q} + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \frac{\partial q}{\partial Q}, \quad \frac{\partial L'}{\partial \dot{Q}} = \frac{\partial L}{\partial q} \frac{\partial q}{\partial \dot{Q}} + \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial \dot{Q}} = \frac{\partial L}{\partial q} \frac{\partial q}{\partial \dot{Q}}. \end{aligned} \quad (2)$$

Oxirgi ifodaning ikkinchi hadi umumlashgan koordainatani bog'lanishida

umumlashgan tezlik qatnashmagani uchun tushib qoladi, chunki $\left(\frac{\partial q}{\partial \dot{Q}} = 0 \right)$.

Bu hadni (1) formula orqali ko'rinishini o'zgartirib olaylik:

$$\frac{\partial L'}{\partial \dot{Q}} = \frac{\partial L}{\partial \dot{q}} \frac{\partial}{\partial \dot{Q}} (\dot{q}) = \frac{\partial L}{\partial \dot{q}} \frac{\partial}{\partial \dot{Q}} \left[\frac{\partial q}{\partial Q} \dot{Q} + \frac{\partial q}{\partial t} \right] = \frac{\partial L}{\partial \dot{q}} \frac{\partial q}{\partial Q} + \frac{\partial L}{\partial \dot{q}} \frac{\partial}{\partial \dot{Q}} \frac{\partial q}{\partial t} = \frac{\partial L}{\partial \dot{q}} \frac{\partial q}{\partial Q}.$$

Bu yerdan

$$\frac{d}{dt} \frac{\partial L'}{\partial \dot{Q}} = \frac{d}{dt} \left[\frac{\partial L}{\partial \dot{q}} \frac{\partial q}{\partial Q} \right] = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \cdot \frac{\partial q}{\partial Q} + \frac{\partial L}{\partial \dot{q}} \cdot \frac{d}{dt} \frac{\partial q}{\partial Q} = \frac{\partial q}{\partial Q} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial Q}. \quad (3)$$

(1-3) formulalar asosida yangi o'zgaruvchilar bo'yicha Lagranj tenglamasini tuzaylik

$$\frac{d}{dt} \frac{\partial L'}{\partial \dot{Q}} - \frac{\partial L'}{\partial Q} = \frac{\partial q}{\partial Q} \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} + \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial Q} - \frac{\partial L}{\partial q} \frac{\partial q}{\partial Q} - \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial Q} = \frac{\partial q}{\partial Q} \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} \right]. \quad (4)$$

Shunday qilib, (4) ifodadan ko'rinib turibdiki, Lagranj tenglamasi eski o'zgaruvchilarga nisbatan qanday ko'rinishga ega bo'lsa, yangi o'zgaruvchilarda

ham shunday ko'rinishga ega ekan. Demak, Lagranj tenglamasi nuqtaviy almashtirishlarga nisbatan *invariantlik* xossasiga ega ekan.

Ko'p erkinlik darajasi uchun (1-4) formulalarni umumlashtirish muammoni hosil qilmaydi:

$$\begin{aligned} \dot{q}_k = \sum \frac{\partial q_k}{\partial Q_i} \dot{Q}_i + \frac{\partial q_k}{\partial t}, \\ \frac{d}{dt} \frac{\partial L'}{\partial \dot{Q}_i} - \frac{\partial L'}{\partial Q_i} = \sum_k \frac{\partial q_k}{\partial Q_i} \left[\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} \right]. \end{aligned}$$

7-masala. $F_2 = (q, P) = \sum f_i(q, t) P_i$ hosil qiluvchi funksiya yordamida sodir bo'ladigan almashtirishlar kanonik almashtirishlar ekanligini isbotlang.

Yechilishi. (4.1) formulalarga asosan F_2 funksiya uchun kanonik almashtirishlar tenglamasi quyidagicha

$$p_i = \sum \frac{\partial f_i}{\partial q_i} P_i, \quad Q_i = f_i(q, t), \quad H' = H + \sum \frac{\partial f_i}{\partial t} P_i.$$

Bu ifodalardan ko'rinib turibdiki, mazkur hosil qiluvchi funksiya nuqtaviy almashtirishlarga olib kelar ekan. Demak, *har qanday nuqtaviy almashtirishlar kanonik almashtirishlar* ekan. Bu almashtirishlarning kanonik ekanligini to'g'ridan-to'g'ri hisoblab ko'rsataylik, soddaroq bo'lishi uchun $i=1$ bo'lgan holni ko'raylik:

$$F_2 = f(q, t) P \Rightarrow p = \frac{\partial f}{\partial q} P, \quad Q = f(q, t), \quad H' = H + \frac{\partial f}{\partial t} P$$

Bizga kerak bo'ladigan hosilalarni ulardan foydalanib topaylik, chunki ular kanonik almashtirishlar tenglamalaridir:

$$\begin{aligned} \dot{p} = \frac{d}{dt} \left(\frac{\partial f}{\partial q} P \right) = \dot{p} \frac{\partial f}{\partial q} + P \frac{\partial^2 f}{\partial q \partial t}, \\ \frac{\partial H}{\partial q} = \frac{\partial}{\partial q} \left(H' - \frac{\partial f}{\partial t} P \right) = \frac{\partial H'}{\partial q} - \frac{\partial}{\partial q} \left(\frac{\partial f}{\partial t} P \right) = \frac{\partial H'}{\partial q} \frac{\partial Q}{\partial q} - P \frac{\partial^2 f}{\partial q \partial t} = \frac{\partial H'}{\partial q} \frac{\partial f}{\partial q} - P \frac{\partial^2 f}{\partial q \partial t}. \end{aligned}$$

Mazkur ifodalarning chap va o'ng tomon yig'indilari ushbu beradi

$$\left(\dot{p} + \frac{\partial H}{\partial q}\right) = \frac{\partial f}{\partial q} \left(\dot{p} + \frac{\partial H'}{\partial q}\right),$$

qavslar ichidagi ifodalar esa, eski va yangi o'zgaruvchilar orqali yozilgan Gamilton tenglamalaridir, demak, ularning ko'rinishlari invariant ekan. Shu bilan talab qilingan tasdiq isbotlandi.

Xuddi shuningdek, Gamilton tenglamalarining ikkinchi juftini ham tekshirib ko'rish mumkin:

$$\dot{Q} = \frac{d}{dt} f(q, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \frac{\partial q}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \frac{dq}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q} \dot{q},$$

$$\frac{\partial H'}{\partial p} = \frac{\partial}{\partial p} \left(H + \frac{\partial f}{\partial t} p \right) = \frac{\partial H}{\partial p} + \frac{\partial f}{\partial t} = \frac{\partial H}{\partial p} \frac{\partial p}{\partial p} + \frac{\partial f}{\partial t} = \frac{\partial H}{\partial p} \frac{\partial}{\partial p} \left(\frac{\partial f}{\partial q} p \right) + \frac{\partial f}{\partial t} = \frac{\partial H}{\partial p} \frac{\partial f}{\partial q} + \frac{\partial f}{\partial t}.$$

Bu ifodalardan, Gamilton tenglamalarining ikkinchi juftining ham invariant ekanligi yaqqol ko'rinib turibdi:

$$\dot{Q} - \frac{\partial H'}{\partial p} = \frac{\partial f}{\partial q} \left(\dot{q} - \frac{\partial H}{\partial p} \right).$$

8-masala. Erkin zarra Gamiltonianining

$$F_2(\vec{r}, \vec{p}', t) = \vec{r}\vec{p}' - f(\vec{r}, t)$$

hosil qiluvchi funksiyasi yordamida amalga oshiriladigan gradient (kalibrovka) almashtirishlariga nisbatan o'z ko'rinishini saqlab qoladigan (invariant) formasini ko'rsating.

Yechilishi: (4.16) ga asosan $x'_k = x_k, \quad p_k = p'_k - \frac{\partial f}{\partial x_k}.$

buyurda x_k, p_k lar radius-vektor va impulsning komponentalari, x', p' lar yangi koordinata va impulslar.

Yangi Gamilton funksiyasi $H'(\vec{r}', \vec{p}', t) = \frac{1}{2m} (\vec{p}' - \vec{\nabla} f)^2 - \frac{\partial f}{\partial t},$

chunki erkin zarraning Gamilton funksiyasi $H = \frac{p^2}{2m}$ vaqtga oshkora bog'liq emas.

Gamilton tenglamalari esa

$$\dot{x}' = \frac{1}{m} \left(p'_k - \frac{\partial f}{\partial x'_k} \right), \quad \dot{p}' = \frac{1}{m} \left(p'_i - \frac{\partial f}{\partial x'_i} \right) \frac{\partial^2 f}{\partial x'_i \partial x'_k} + \frac{\partial^2 f}{\partial x'_k \partial t}.$$

Binobarin, erkin zarraning gamiltoniani kalibrovka almashtirishiga nisbatan invariant emasligi yaqqol ko'rinib turibdi. Shunga qaramasdan, harakat tenglamasi $\ddot{x}'_k = 0$ o'z ko'rinishini saqlab qoladi. Mazkur Gamilton funksiyasining kalibrovka almashtirishlariga nisbatan invariant holga keltirish mumkin, uning uchun qo'shimcha kalibrovkalash maydonini quyidagi almashtirishlarni qanoatlantiradigan qilib kiritamiz.

$$H \rightarrow H + e\varphi(\vec{r}, t), \quad \vec{p} \rightarrow \vec{p} + \frac{e}{c} \vec{A}(\vec{r}, t).$$

φ, \vec{A} - mos ravishda *skalyar va vektor potentsiallar deyiladi* va elektr hamda magnit maydon kuchlanganliklari bilan bog'liqdirlar. Yangi koordinatalarga o'tilganda bu qo'shimcha funksiyalar (maydonlar) quyidagicha o'zgarishini talab qilinadi.

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \frac{e}{c} \vec{\nabla} f, \quad \varphi \rightarrow \varphi' = \varphi - \frac{1}{c} \frac{\partial f}{\partial t}.$$

Natijada gamiltonianning invariant qoladigan shakli quyidagi ko'rinishga ega bo'ladi:

$$H(\vec{r}', \vec{p}', t) = \frac{1}{2m} \left(\vec{p}' - \frac{e}{c} \vec{A}(\vec{r}', t) \right)^2 + e\varphi.$$

Shunday qilib, erkin zarraning Gamilton funksiyasi kalibrovka almashtirishlariga nisbatan shaklan invariant qolishini talab qilish, elektromagnit maydon potentsiallarini kiritish zaruratini taqozo etar ekan. Aynan shu xususiyat kvantlashgan maydon nazariyasi asosida elementar zarrachalar nazariyasini yaratishda muhim omil bo'ldi.

9-masala. §1.4 ning 9-masalasida ko'rilgan uch jism (1.6-rasm) masalasining Gamilton funksiyasi

$$F_2(\vec{r}_a, \vec{P}, \vec{p}_{12}, \vec{p}_{13}) = \vec{R}\vec{P} + (\vec{r}_2 - \vec{r}_1)\vec{p}_{12} + (\vec{r}_3 - \vec{r}_1)\vec{p}_{13}$$

hosil qiluvchi funksiya orqali topilishini ko'rsating. Bu yerda, eski koordinata va impuls \vec{r}_a, \vec{p}_a lardan yangi $\vec{x}, \vec{r}_{1,2}, \vec{r}_{1,3}, \vec{P}, \vec{p}_{1,2}, \vec{p}_{1,3}$ koordinata va impulsarga o'tiladi. \vec{R} esa inersiya markazi radius-vektori.

Yechilishi. Ixtiyoriy inersial sanoq sistemasida uchta zarradan iborat mexanik sistemaning Gamilton funksiyasi ushbu ko'rinishga ega:

$$H = \sum_a \frac{p_a^2}{2m_a} + U(\vec{r}_1, \vec{r}_2, \vec{r}_3)$$

Yuqoridagi hosil qiluvchi funksiya F_2 dan (4.16) ga ko'ra:

$$\vec{p}_1 = \frac{\partial F_2}{\partial \vec{r}_1} = \frac{m_1}{M} \vec{P} - \vec{p}_{12} - \vec{p}_{13}, \quad \vec{p}_2 = \frac{\partial F_2}{\partial \vec{r}_2} = \frac{m_2}{M} \vec{P} + \vec{p}_{12}, \quad \vec{p}_3 = \frac{\partial F_2}{\partial \vec{r}_3} = \frac{m_3}{M} \vec{P} + \vec{p}_{13},$$

$$\vec{x} = \frac{\partial F_2}{\partial \vec{P}} = \vec{R}, \quad \vec{r}_{12} = \frac{\partial F_2}{\partial \vec{p}_{12}} = \vec{r}_2 - \vec{r}_1, \quad \vec{r}_{13} = \frac{\partial F_2}{\partial \vec{p}_{13}} = \vec{r}_3 - \vec{r}_1, \quad M = m_1 + m_2 + m_3.$$

Yangi o'zgaruvchilar orqali Gamilton funksiyasini yozib oladigan bo'lsak:

$$H = \frac{P^2}{2M} + \frac{\vec{p}_{12}^2}{2m_{12}} + \frac{\vec{p}_{13}^2}{m_{13}} + \frac{\vec{p}_{12}\vec{p}_{13}}{m_1} + U(\vec{r}_{12}, \vec{r}_{13}).$$

Buyerda $m_{ab} - a$ va b zarralarning keltirilgan massasi, \vec{P} - to'la impuls.

Yuqoridagi ifoda sanoq sistemasining koordinata boshi m_1 zarrada tanlab olingan holdagi uch zarradan iborat sistema Gamilton funksiyasining ifodasidir. Agarda $m_1 \gg m_2, m_3$ bo'ladigan bo'lsa, bu ifodaning to'rtinchi hadi (potensial energiyaning mos hadi bilan birga) oldingi hadlarga nisbatan kichik (g'alayon) deb qaralishi mumkin:

$$H = \frac{P^2}{2M} + H_{12} + H_{13} + \Delta H, \quad H_{1n} = \frac{p_{1n}^2}{2m_{1n}} - \gamma \frac{m_1 m_n}{r_{1n}}, \quad \Delta H = \frac{\vec{p}_{12}\vec{p}_{13}}{m_1} - \gamma \frac{m_2 m_3}{|\vec{r}_{12} - \vec{r}_{13}|}.$$

Bu ifodalarni (1.12) formuladan ham to'g'ridan-to'g'ri topish mumkin.

4.4.-Misollar.

1. Ortogonal almashtirishlar $Q_i = \sum a_{ik} q_k$ kanonik almashtirishlar ekanligini ko'rsating.
2. $F_2 = \sum q_i P_i$ ga mos almashtirishlar qanday almashtirishlardir?
3. Ushbu nuqtaviy almashtirishlar $q_i = q_i(Q_1, Q_2, \dots, Q_S)$ $i = 1, 2, \dots, S$ Gamilton tenglamalarining ko'rinishini saqlab qolishini hisoblab ko'rsating.
4. Ushbu almashtirishlar $q_i = f_i(Q_1, Q_2, \dots, Q_S, t)$ natijasida energiya va umumlashgan impulsning o'zgarish qonuniyatlarini topilsin.
5. $Q = \ln\left(\frac{\sin p}{q}\right)$, $P = q \operatorname{ctg} p$ almashtirishlar kanonik ekanligini ko'rsating.
6. Ushbu kanonik almashtirishlarga $Q = \ln(1 + \sqrt{q} \cos p)$, $P = 2(1 + \sqrt{q} \cos p)\sqrt{q} \sin p$ mos hosil qiluvchi funksiya $F_2(P, Q)$ topilsin.
7. Ushbu almashtirishlar α va β ning qanday qiymatlarida kanonikdir? Hosil qiluvchi funksiya $F_2(p, Q)$ ni toping. $Q = q^\alpha \cos \beta p$, $P = q^\alpha \sin \beta p$.
8. a) $F_1 = \sum q_i Q_i$, b) $F_1 = \sum p_i P_i$, v) $F_1 = -\sum P_i Q_i$ hosil qiluvchi funksiyalarga mos almashtirishlar to'g'risida nima deyish mumkin?
9. Puasson qavslari kanonik almashtirishlarga nisbatan invariant ekanligini isbotlang.
10. Berilgan hosil qiluvchi funksiya ga mos kanonik almashtirishlar topilsin: a) chastotasi $\omega(t)$ bo'lgan garmonik ossillyator uchun Q va P o'zgaruvchilarda harakat tenglamalari topilsin

$$F_1(q, Q, t) = m\omega(t)q^2 \frac{\operatorname{ctg} Q}{2}.$$

- b) Tashqi $F(t)$ kuch tasiridagi garmonik ossillyator uchun Q va P o'zgaruvchilarda harakat tenglamalari topilsin

$$F_1(q, Q, t) = m\omega \left(q - \frac{F(t)}{m\omega^2} \right)^2 \frac{\operatorname{ctg} Q}{2}.$$

11. Quyidagi almashtirishlarning kanonik ekanligini isbotlang:

$$x = \frac{\sqrt{2P_1} \sin Q_1 + P_2}{\sqrt{m\omega}}, \quad p_x = \frac{\sqrt{m\omega}(\sqrt{2P_1} \cos Q_1 - Q_2)}{2},$$

$$y = \frac{\sqrt{2P_1} \cos Q_1 + Q_2}{\sqrt{m\omega}}, \quad p_y = \frac{\sqrt{m\omega}(-\sqrt{2P_1} \sin Q_1 + P_2)}{2}.$$

Agar, vektor potentsiali $\vec{A} = \left(-\frac{yH}{2}, \frac{xH}{2}, 0\right)$ bo'lsa magnit maydonidagi zarra

uchun yangi o'zgaruvchilarda Gamilton tenglamalari topilsin. Buyerda $\omega = \frac{eH}{mc}$

12. $F_2(q, P) = \alpha qP$ hosil qiluvchi funksiya bilan berilgan, kanonik almashtirishlar qanday manoga ega?

13. Quyidagi hosil hiluvchi funksiyalarga mos kanonik almashtirishlar topilsin:

a) $F_2(q, P) = P \ln q,$

b) $F_2(q, P) = \frac{Pq^2}{2},$

v) $F_2(q, P) = q\sqrt{2P},$

g) $F_2(\vec{r}, \vec{p}', t) = \vec{r}\vec{p}' - \vec{V}\vec{p}'t + m\vec{r}\vec{V}.$

d) $F_1 = \sum_{i,j=1}^n a_{ij} q_i Q_j;$

e) $F_2 = \sum_{j=1}^n q_j \ln P_j;$

j) $F_2 = \sum_{i,j=1}^n a_{ij} \ln(q_i t) \exp(b_j P_j);$

z) $F_3 = \sum_{i,j=1}^n f_{ij} p_i Q_j;$

14. Garmonik ossilyatorning Gamilton funksiyasi quyidagi hosil qiluvchi funksiyalarga mos ravishda qanday o'zgaradi? Kanonik almashtirishlarni toping.

a) $F_2(q, P) = \frac{iP^2 - im\omega q^2 + 2\sqrt{2m\omega}qP}{2},$

b) $F_1(q, Q) = \frac{m\omega q^2 \operatorname{tg} Q}{2},$

v) $F_1(q, Q) = -m\omega q^2 \operatorname{tg} \frac{Q + \omega t}{2}.$

15. $H = \frac{p^2}{2m}$ Gamilton funksiyasi quyidagi hosil qiluvchi funksiyalarga mos almashtirishlar natijasida qanday o'zgarishini toping.

a) $F_1(x, x', t) = \frac{m(x-x')^2}{2}$

b) $F_2(x, p', t) = \frac{xp' - p'^2 t}{2m}$

16. Uch jism masalasidagi Yakobi o'zgaruvchilari (§ 1.4. ning 9-masalasi, 1.7-rasm) quyidagi

$$F_2 = \vec{R}\vec{P} + (\vec{r}_3 - \vec{r}_1)\vec{p}_1 + \left[\vec{r}_2 - \frac{m_1\vec{r}_1 + m_3\vec{r}_3}{m_{13}}\right]\vec{p}_2.$$

hosil qiluvchi funksiya orqali aniqlanishini ko'rsating (9-masalaga qarang).

§4.5. Gamilton-Yakobi metodi.

Ma'lumki, ta'sir funksiyasini yuqori chegara koordinatasining funksiyasi deb qaralganda quyidagi bog'lanishlarni olish mumkin:

$$\frac{\partial S}{\partial t} = -H, \quad \frac{\partial S}{\partial q_i} = p_i, \quad dS = \sum p_i dq_i - H dt \quad (4.20)$$

Yuqoridagi bog'lanishlardan foydalanib, Gamilton funksiyasidagi impulsni $\frac{\partial S}{\partial q_i}$ lar bilan almashtirib quyidagi ifodani olish mumkin:

$$\frac{\partial S}{\partial t} + H(q_1, \dots, q_n, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_n}, t) = 0 \quad (4.21)$$

Bu tenglama xususiy hosilali birinchi tartibli tenglama bo'lib, Gamilton-Yakobi tenglamasi deb ataladi. Lagranj va Gamilton tenglamalari qatori Gamilton-Yakobi tenglamasi ham harakat tenglamalarini integrallashning asoslaridan biri hisoblanadi. Gamilton-Yakobi tenglamasining to'la integrali quyidagichadir:

$$S = f(t, q_1, \dots, q_n, \alpha_1, \dots, \alpha_n) + A \quad (4.22)$$

Buyerda $\alpha_1, \alpha_2, \dots, \alpha_n$ va A lar ixtiyoriy doimiylardir. Endi Gamilton-Yakobi tenglamasining to'la integrali bilan harakat tenglamalarining yechimlari orasidagi bog'lanishni topaylik. Buning uchun kanonik almashtirishlar orqali q va p o'zgaruvchilardan yangi o'zgaruvchilarga o'taylik. Hosil qiluvchi funksiya

sifatida (4.22) dagi $f(q, \alpha, t)$ funksiyani tanladik deylik, unda $\alpha_1, \alpha_2, \dots, \alpha_n$ kattaliklar yangi impulslar rolini o'ynaydi. Yangi koordinatalarni $\beta_1, \beta_2, \dots, \beta_n$ lar orqali belgilab olaylik. Hosil qiluvchi funksiya eski koordinatalar va yangi impulsarga bog'liq bo'lganligi uchun uning turi

$$F_2(q, P, t) \Rightarrow f(q, \alpha, t)$$

bo'ladi va (4.16) tenglamalardan foydalanamiz.

$$p_i = \frac{\partial F_2}{\partial q_i} = \frac{\partial f}{\partial q_i}, \quad Q_i = \beta_i = \frac{\partial F_2}{\partial P_i} = \frac{\partial f}{\partial \alpha_i}; \quad H' = H + \frac{\partial f}{\partial t} \quad (4.23)$$

Lekin $f(t, q, \alpha)$ funksiya Gamilton-Yakobi tenglamasini qanoatlantirgani uchun yangi Gamilton funksiyasi H' nolga aynan tenglashadi:

$$H' = H + \frac{\partial f}{\partial t} = H + \frac{\partial S}{\partial t} = 0 \quad (4.24)$$

Shuning uchun, yangi o'zgaruvchilar α, β uchun Gamilton tenglamasi quyidagi ko'rinishga keladi:

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i} = 0 \Rightarrow \dot{\beta}_i = 0, \quad \dot{P}_i = -\frac{\partial H'}{\partial Q_i} = 0 \Rightarrow \dot{\alpha}_i = 0, \quad \alpha_i = const, \quad \beta_i = const \quad (4.25)$$

Ikkinchi tomondan esa, n ta $\frac{\partial f}{\partial \alpha_i} = \beta_i$ tenglamalar, n ta koordinata q_i larni $2n$ ta α_i va β_i doimiylar orqali ifodalashga imkon beradi. Shunday qilib, biz harakat tenglamasining umumiy integralini topamiz. Demak, mexanik sistema harakatini Gamilton-Yakobi metodi bilan yechish uchun (4.22) ifoda bilan aniqlangan to'la integral topiladi, so'ngra uni α_i bo'yicha differensiallanadi, natijada n ta algebraik tenglamalar $\frac{\partial S}{\partial \alpha_i} = \beta_i$ hosil bo'ladi.

Impulslarning vaqtga bog'liqligi esa $P_i = \frac{\partial S}{\partial q_i}$ tenglama bilan aniqlanadi.

Agarda, Gamilton funksiyasi $H(q, P, t)$ vaqtga oshkora bog'liq bo'lmasa, Gamilton-Yakobi tenglamasi ancha sodda ko'rinishga keladi. Bu holda tasir funksiyasi quyidagi ko'rinishga keladi:

$$S = S_0(q) - Et \quad (4.26)$$

Bu yerda

$$S_0 = \int \sum p_i \delta q_i$$

qisqartirilgan ta'sir funksiyasidir.

O'z navbatida bu funksiya uchun Gamilton-Yakobi tenglamasi esa quyidagichadir:

$$H(q_1, q_2, \dots, q_n, \frac{\partial S_0}{\partial q_1}, \frac{\partial S_0}{\partial q_2}, \dots, \frac{\partial S_0}{\partial q_n}, t) = E \quad (4.27)$$

1-masala. Massasi m ga teng erkin moddiy nuqta harakati Gamilton-Yakobi metodi bilan yechilsin.

Yechilishi. Erkin zarraning Gamilton funksiyasi

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

Gamilton-Yakobi tenglamasi esa, quyidagicha (4.21 dan)

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left[\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 \right] = 0$$

Yuqoridagi Gamilton funksiyasi vaqtga oshkor bog'liq bo'lmaganligi uchun

$$\frac{\partial S}{\partial t} = -E$$

bo'ladi va energiya saqlanadi.

Demak, (4.26) ga asosan S ning vaqtga bog'liqligi $-Et$ ko'rinishda bo'ladi. Yana $\frac{\partial S}{\partial q_i} = P_i$ tenglamaga asosan, siklik koordinata S ning ifodasida oshkora ko'rinishida ishtirok etishi mumkin. Bizning

holda $E = const$ va x, y, z koordinatalar siklikdir. Shuning uchun, ta'sir funksiyasi

$$S = p_x x + p_y y + p_z z - Et$$

ko'rinishga ega. Bu ifodani Gamilton-Yakobi tenglamasiga olib borib qo'ysak:

$$E = \frac{p_x^2 + p_y^2 + p_z^2}{2m}, \quad \frac{\partial S}{\partial \alpha_1} = \frac{\partial S}{\partial p_x} = x - \frac{p_x t}{m} \equiv x_0,$$

$$\frac{\partial S}{\partial \alpha_2} = \frac{\partial S}{\partial p_y} = y - \frac{p_y t}{m} \equiv y_0, \quad \frac{\partial S}{\partial \alpha_3} = \frac{\partial S}{\partial p_z} = z - \frac{p_z t}{m} \equiv z_0,$$

Bu yerdan

$$x = x_0 + \frac{p_x t}{m}, \quad y = y_0 + \frac{p_y t}{m}, \quad z = z_0 + \frac{p_z t}{m}$$

larni topamiz. Demak, erkin zarra chiziqli harakat qiladi.

2-masala. Lagranj funksiyasi $L = \frac{\dot{x}^2}{2} - \frac{x^2}{2}$ bo'lgan garmonik ossillyator

masalasi Gamilton-Yakobi metodi bilan yechilsin.

Yechilishi. Avval, Gamilton funksiyasini topib olaylik

$$H = \frac{p^2}{2} + \frac{x^2}{2}.$$

Gamilton-Yakobi tenglamasi esa $\frac{\partial S}{\partial t} + \frac{1}{2} \left(\frac{\partial S}{\partial x} \right)^2 + \frac{x^2}{2} = 0$.

Bu yerda harakat bir o'lchovli va x - koordinata siklik emas. Lekin Gamilton funksiyasi vaqtga oshkora bog'liq emas, ya'ni $E = const$, demak, (4.26) dan

$$S = -Et + f(x).$$

Buni Gamilton-Yakobi tenglamasiga qo'ysak va $f(x)$ funksiyani topsak:

$$\frac{1}{2} \left(\frac{\partial f}{\partial x} \right)^2 + \frac{x^2}{2} = E, \quad \left(\frac{\partial f}{\partial x} \right)^2 = 2E - x^2,$$

$$f(x) = \int \sqrt{2E - x^2} dx = x \frac{\sqrt{2E - x^2}}{2} + E \arcsin \frac{x}{\sqrt{2E}}$$

Bu kattalikni S ta'sirining ifodasiga olib-borib qo'yish natijasida ushbuga ega bo'lamiz:

$$S = \frac{x\sqrt{2E - x^2}}{2} + E \arcsin \frac{x}{\sqrt{2E}} - Et.$$

Yuqoridagi ifodani yagona doimiy E bo'yicha differensiallab, uni t_0 ga tenglashtirib algebraik amallarni bajarsak, tebranma harakat ifodasini olamiz:

$$t_0 = \frac{\partial S}{\partial E} = \frac{x}{2\sqrt{2E - x^2}} + E \sqrt{E} x \frac{-E^{-3/2}}{2\sqrt{2E - x^2 + \arcsin \frac{x}{\sqrt{2E}}}} - t,$$

$$t + t_0 = \arcsin \frac{x}{\sqrt{2E}}, \quad x = \sqrt{2E} \sin(t + t_0).$$

Demak, chastotasi $\omega = 1$ ga teng garmonik tebranishlar ekan.

3-masala. Massasi $m = 1$ bo'lgan zarraning $U = \frac{1}{r}$ markaziy maydondagi

harakati integrallansin

Yechilishi. Zarraning sferik koordinat sistemasidagi Gamilton funksiyasini yozaylik

$$H = \frac{1}{2} \left[\frac{p_r^2 + p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \theta} \right] + \frac{1}{r}.$$

$$\text{Bu yerdan } \frac{\partial S}{\partial t} + \frac{1}{2} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial S}{\partial \varphi} \right)^2 \right] + \frac{1}{r} = 0$$

Bu ifodalardan ko'rinib turibdiki, faqat φ siklik koordinata. Shuning uchun, S ta'sirni quyidagi ko'rinishda qidiramiz

$$S = -Et + p_\varphi \varphi + f(r) + F(\theta), \quad (E = const, p_\varphi = const).$$

Buni Gamilton-Yakobi tenglamasiga qo'ysak ushuni olamiz:

$$2Er^2 - 2r - r^2 \left(\frac{df}{dr}\right)^2 = \frac{p_\varphi^2}{\sin^2 \theta} + \left(\frac{df}{d\theta}\right)^2.$$

Mazkur tenglamaning chap tomoni faqat r ga, o'ng tomoni esa, faqat θ ga bog'liq, demak, har ikkala tomon bitta doimiyga tengdir:

$$2Er^2 - 2r - r^2 \left(\frac{df}{dr}\right)^2 = \alpha, \quad \frac{p_\varphi^2}{\sin^2 \theta} + \left(\frac{df}{d\theta}\right)^2 = \alpha.$$

Natijada
$$f(r) = \int \sqrt{2E - \frac{2}{r} - \frac{\alpha}{r^2}} dr, \quad F(\theta) = \int \sqrt{\alpha - \frac{p_\varphi^2}{\sin^2 \theta}} d\theta.$$

Bularni integrallarini hisoblab, ularni S ta'sirning ifodasiga qo'ysak:

$$S = -Et + p_\varphi \varphi + \sqrt{2Er^2 - 2r - \alpha} - \sqrt{\alpha} \arccos \frac{\alpha - r}{r\sqrt{2E\alpha + 1}} - \frac{1}{\sqrt{2E}} \operatorname{Arch} \frac{2Er - 1}{\sqrt{2E\alpha + 1}} + \sqrt{\alpha} \arccos \frac{\sqrt{\alpha} \cos \theta}{\sqrt{\alpha - p_\varphi}} + \frac{p_\varphi^2}{2} \arcsin \frac{2 \cos^2 \theta p_\varphi^2 - (\alpha^2 + p_\varphi^2)}{\alpha^2 - p_\varphi^2}.$$

Bu ifodalarni E, p_φ, α lar bo'yicha differensiallab va ularni doimiylarga tenglashtirib, harakat trayektoriyasini olishimiz mumkin.

4.5. – Misollar.

1. Quyidagi Lagranj funksiyalariga mos Gamilton-Yakobi tenglamasi yozilsin:

a) $L = \frac{1}{2} [(1-v^2)\dot{U}^2 + (1-U^2)\dot{v}^2].$ b) $L = -mc \sqrt{1 - \frac{v^2}{c^2}}.$

v) $L = \frac{1}{2} (U^2 + v^2)(\dot{U}^2 + \dot{v}^2) - (U+v)^2.$ g) $L = \frac{m(\dot{r}^2 + r^2 \sin^2 \theta \dot{\phi}^2 + r^2 \dot{\theta}^2)}{2} - \frac{1}{r}.$

d) $L = \frac{\dot{x}^2}{x} + x\dot{y}^2 + x.$

2. Agarda, Gamilton-Yakobi tenglamalari quyidagi ko'rinishda bo'lsa, Lagranj tenglamasi tuzilsin

a) $\frac{\partial S}{\partial t} + \frac{1}{2} \left[\frac{1}{v} \left(\frac{\partial S}{\partial U} \right)^2 + \frac{1}{U} \left(\frac{\partial S}{\partial v} \right)^2 \right] = 0.$ b) $\frac{\partial S}{\partial t} + \frac{1}{2} \left[\frac{1}{U} \left(\frac{\partial S}{\partial U} \right)^2 + \frac{1}{v} \left(\frac{\partial S}{\partial v} \right)^2 \right] = 0.$

v) $\frac{\partial S}{\partial t} + \frac{1}{2m} \left[\left(\frac{\partial S}{\partial x} + ay \right)^2 + \left(\frac{\partial S}{\partial y} - ax \right)^2 \right] = 0$ g) $\frac{\partial S}{\partial t} + \frac{1}{\sqrt{xy}} \left(2 \frac{\partial S}{\partial x} \frac{\partial S}{\partial y} - 1 \right) = 0.$

d) $\frac{\partial S}{\partial t} + \frac{1}{2} \left[\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 \right] + \frac{1}{2} \left(y \frac{\partial S}{\partial x} - x \frac{\partial S}{\partial y} \right) + \frac{1}{2} (x^2 + y^2) = 0.$

3. Massasi, birga teng zarraning quyidagi maydonlaridagi harakati Gamilton-Yakobi metodi bilan integrallansin:

a) Kuchlanganligi birga teng bir jinsli maydonda

b) $U = \frac{1}{r}$ maydonda, tekislikda.

v) Aksial simmetriyaga ega $U = \frac{1}{r}$ maydonda.

4) Massasi m ga teng zarraning erkin tushishi masalasi Gamilton-Yakobi metodi bilan yechilsin..

V. ZARRALARNING TO'QNASHUVI.

Impuls va energiyaning saqlanish qonunlarini hisobga olishning o'ziga mexanik jarayonlarning muhim xususiyatlarini ochib beradi. Bunday jarayonlar turkumiga parchalanish (emirilish), zarralarning elastik to'qnashuvi, sochilish muammolari kiradi. Nisbiylik prinsipi asoslariga tayanib olingan bu natijalar sodir bo'layotgan jarayonlar qanday kuchlar ta'sirida ro'y berayotganidan qattiy nazar, saqlanib qoladigan natijalar hisoblanadi. Bunday qattiy natijalarni *kinematik munosabatlar* deyiladi

§5.1. O'z-o'zidan parchalanish (emirilish).

O'z-o'zidan parchalanayotgan zarra (ya'ni tashqi kuch ta'sirisiz) ikkita zarrachaga parchalangan holni ko'raylik. Ana shunday jarayonlarni yoritishda ushbu ikki xil sanoq sistemasi keng qo'llaniladi: *laboratoriya sistemasi (L-sistema)* va *inersiya markazi sistemasi (S-sistema)*.

S-sistemaga tegishli kattaliklarni "0" indeksi bilan belgilaymiz. Parchalanishni S-sistemada qarash qulayroq, ya'ni parchalangunga qadar parchalanuvchi zarra tinch turibdi desak $P_2 = 0$, u holda parchalanish natijasida hosil bo'lgan zarralar qarama-qarshi tomonga sochiladilar:

$$\vec{P}_{10} = -\vec{P}_{20} = \vec{P}_0, \quad |\vec{P}_{10}| = |\vec{P}_{20}| = |\vec{P}_0| \quad (5.1)$$

Parchalanish sodir bo'lishi uchun, parchalanish energiyasi - ε noldan katta bo'lishi kerak:

$$\varepsilon = E_I - E_{1I} - E_{2I} \quad (5.2)$$

bu yerda "I" indeksi, "ichki energiyani" anglatadi. Energiyaning saqlanish qonunidan:

$$\varepsilon = \frac{P_0^2}{2} \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \frac{P_0^2}{2m} \quad (5.3)$$

bu yerda m_1, m_2 - mos ravishda birinchi va ikkinchi zarralarning massalari, m keltirilgan massa har bir zarraning tezliliklari esa

$$v_{10} = \frac{P_0}{m_1}, \quad v_{20} = \frac{P_0}{m_2} \quad (5.4)$$

Agar, zarra parchalangunga qadar \vec{V} tezlik bilan harakatlanayotgan bo'lsa, quyidagi tezliklarni qo'shish formulasi orqali L - sistemaga o'tiladi.

$$\vec{v} = \vec{V} + \vec{v}_0, \quad (5.5)$$

bu yerda \vec{v} , \vec{v}_0 yemirilgandan so'ng hosil bo'lgan zarralardan birining L- va S-sistemadagi tezliklari. (5.4.) ifodadan:

$$v^2 + V^2 - 2vV \cos \theta = v_0^2 \quad (5.6)$$

bu yerda θ - shu zarraning V tezlik yo'nalishidan og'ish burchagi. Bu munosabatlar L- va S- sistemalardagi zarralarning tezliklarini aniqlab beradi. Zarralarning sochilib chiqish manzarasi ikki xil bo'lishi mumkin: $V < v_0$ va $V > v_0$, birinchi holda uchib chiqish burchagi (og'ish) ixtiyoriy bo'lsa, ikkinchi holda asosan ilgariga qarab uchib chiqiladi. Bu og'ish burchagining maksimal qiymati

$$\sin \theta_{\max} = \frac{v_0}{V} \quad (5.7)$$

L- va S- sistemadagi og'ish burchaklari θ va θ_0 lar orasidagi bog'lanish

$$\operatorname{tg} \theta = \frac{v_0 \sin \theta_0}{v_0 \cos \theta_0 + V} \quad (5.8)$$

$$\cos \theta_0 = -\frac{V}{v} \sin^2 \theta \pm \cos \theta \sqrt{1 - \frac{V^2}{(v \sin \theta)^2}}, \quad (5.9)$$

Agarda $v_0 > V$ bo'lsa, bog'lanish bir qiymatli va faqat musbatishora hisobga olinadi, $v_0 < V$ da ikkala ishora ham hisobga olinishi zarur. Uchib chiqish taqsimoti faqatgina tezlik uchun bo'lmay, balki uchib chiqish burchagi, energiyasi bo'yicha ham bo'lishi mumkin.

Masalan: S-sistemada parchalangan zarralar burchak bo'yicha izotrop taqsimlangani uchun, hamda fazoviy burchak $dO_0 = 2\pi \sin \theta_0 d\theta_0$ ni hisobga olgan holda burchak taqsimoti quyidagichadir:

$$\frac{1}{2} \sin \theta_0 d\theta_0 \quad (5.10)$$

L-sistemadagi taqsimot esa, bu ifodani (5.9) dan foydalanib o'zgartirilishidan topiladi. Masalan: L-sistemadagi energetik taqsimot

$$\frac{dT}{2m_1 v_0 V} \quad (5.11)$$

dan iborat, bu yerda m_i - i -sortli zarraning massasi, T - kinetik energiyasi. Agarda, massasi M ga teng zarra ikkitadan ko'p zarralarga parchalanayotibdi desak, ixtiyoriy m_1 massali zarra uchun energetik taqsimotda maksimal qiymat mavjuddir (S-sistemada)

$$(T_{10})_{\max} = \frac{M - m_1}{M} \mathcal{E} \quad (5.12)$$

Shuni ta'kidlash lozimki, (5.2) ifodadan $\varepsilon > 0$ dagina parchalanish sodir bo'lishini hisobga olsak, hamda parchalanayotgan zarra tinch turibdi desak,

$$E_1 > E_{11} + E_{21} \Rightarrow M > m_1 + m_2 \quad (5.13)$$

1-masala. Ikkita zarracha parchalanish jarayonida uchib chiqqan zarralarning L -sistemada og'ish burchaklari θ_1 va θ_2 orasidagi bog'lanish topilsin.

Yechilishi. S-sistemada ikkala zarraning og'ish burchaklari orasidagi munosabat $\theta_{10} = \pi - \theta_{20}$ dir. θ_{10} ni θ_0 deb belgilab (5.8) formulani har ikkala zarra uchun qo'llab, quyidagilarni olamiz:

$$V + v_{10} \cos \theta_0 = v_1 \sin \theta_0 \operatorname{ctg} \theta_1, \quad V - v_{20} \cos \theta_0 = v_2 \sin \theta_0 \operatorname{ctg} \theta_2.$$

Bu ikki munosabatdan θ_0 ni yo'qotsak, θ_1 va θ_2 orasidagi bog'lanishni topgan bo'lamiz. Uning uchun $\cos \theta_0$ va $\sin \theta_0$ ni aniqlab, $\cos^2 \theta_0 + \sin^2 \theta_0 = 1$ tenglamaga qo'yamiz:

$$\sin^2 \theta_0 + \cos^2 \theta_0 = 1 = \frac{(v_{10} + v_{20})^2 V^2 + V^2 (v_{10} \operatorname{ctg} \theta_1 - v_{20} \operatorname{ctg} \theta_2)^2}{(\operatorname{ctg} \theta_1 + \operatorname{ctg} \theta_2)^2 v_{10}^2 v_{20}^2},$$

bu yerdan

$$\frac{v_{10}^2}{\sin^2 \theta_1} + \frac{v_{20}^2}{\sin^2 \theta_2} - 2v_{10}v_{20} \frac{\cos(\theta_1 + \theta_2)}{\sin \theta_1 \sin \theta_2} = \frac{v_{10}^2 v_{20}^2 \sin^2(\theta_1 + \theta_2)}{V^2 \sin^2 \theta_1 \sin^2 \theta_2}.$$

Yuqoridagi (5.3) va (5.4) ifodalardan

$$\frac{v_{10}}{v_{20}} = \frac{m_2}{m_1}, \quad P_0 = m_1 v_{10} = m_2 v_{20}, \quad \varepsilon = \frac{P_0^2}{2m} = \frac{P_0^2 (m_1 + m_2)}{2m_1 m_2}$$

ekanligini hisobga olsak:

$$\frac{m_2}{m_1} \sin^2 \theta_2 + \frac{m_1}{m_2} \sin^2 \theta_1 - 2 \sin \theta_1 \sin \theta_2 \cos(\theta_1 + \theta_2) = \frac{2\varepsilon}{(m_1 + m_2)V^2} \sin^2(\theta_1 + \theta_2).$$

2-masala. Parchalanish bo'laklari bo'lgan zarralarning uchib chiqish burchagi (og'ish burchagi) bo'yicha taqsimoti L-sistemada topilsin.

Yechilishi. Malumki, S-sistemada uchib chiqish burchagi bo'yicha taqsimot (5.10) ifoda orqali topiladi. S- va L- sistemasidagi og'ish burchaklari θ_0 va θ orasidagi bog'lanish esa (5.9) formula orqali ifodalanadi. L - sistemadagi burchak taqsimotini topish uchun (5.9) ifodaning ikkala tomonini differensiallash va (5.10) ifoda bilan solishtirish yetarli, lekin mazkur bog'lanishning ikki qiymatli ekanligi unutmash kerak. Xususan, $v_0 > V$

bo'lganda (5.9) da radikal oldidagi musbat, $v_0 < V$ da esa, har ikkala ishora hisobga olinadi.

a) $v_0 < V$ da, $0 \leq \theta \leq \pi$, hamda (5.9) ni ikkala tomonini differensiallasak:

$$-\sin \theta_0 d\theta_0 = -\frac{V}{v_0} 2 \sin \theta \cos \theta d\theta - \sin \theta d\theta \sqrt{1 - \frac{V^2}{v_0^2 \sin^2 \theta}} - \frac{V^2 \sin \theta \cos^2 \theta d\theta}{v_0^2 \sqrt{1 - \frac{V^2}{v_0^2 \sin^2 \theta}}}$$

$$\frac{1}{2} \sin \theta_0 d\theta_0 = \frac{1}{2} \sin \theta d\theta \left(2 \frac{V}{v_0} \cos \theta + \frac{1 + \frac{V^2}{v_0^2} (\cos^2 \theta - \sin^2 \theta)}{\sqrt{1 - \frac{V^2}{v_0^2} \sin^2 \theta}} \right)$$

Demak, bu hol uchun burchak taqsimoti quyidagicha:

$$\frac{1}{2} \sin \theta_0 d\theta_0 = \frac{1}{2} \sin \theta d\theta \left(2 \frac{V}{v_0} \cos \theta + \frac{1 + \frac{V^2}{v_0^2} \cos 2\theta}{\sqrt{1 - \frac{V^2}{v_0^2} \sin^2 \theta}} \right)$$

b) $v_0 < V$ da $0 \leq \theta \leq \theta_{\max}$ (5.9) ning ikkala tomonini differensiallab va ikki qiymatlikni, hamda θ ortganda unga mos θ_0 ning bittasi ortishi va ikkinchisi kamayishini hisobga olsak (ya'ni, ikkalasining yig'indisi o'rniga ayirmasi olinadi):

$$\frac{1}{2} \sin \theta_0 d\theta_0 = \frac{1}{2} \sin \theta d\theta \left(2 \frac{V}{v_0} \cos \theta \pm \frac{1 + \frac{V^2}{v_0^2} \cos 2\theta}{\sqrt{1 - \frac{V^2}{v_0^2} \sin^2 \theta}} \right) \Rightarrow \sin \theta d\theta \left(\frac{1 + \frac{V^2}{v_0^2} \cos 2\theta}{\sqrt{1 - \frac{V^2}{v_0^2} \sin^2 \theta}} \right)$$

3-masala. Antiprotonlar \bar{p} deyteriy d bilan to'qnashib yutiladi va quyidagi reaksiya (parchalanish) sodir bo'ladi $\bar{p} + d = n + \pi^0$ deb π^0 - mezonning to'la energiyasi hisoblansin. Buyerdagi zarralarning massalari quyidagicha ($c=1$ deb olingan): $m_n = 939,5 \text{ GeV}$, $m_{\bar{p}} = m_p = 938,2 \text{ GeV}$, $m_d = 1875,5 \text{ GeV}$, $m_{\pi^0} = 135,0 \text{ GeV}$.

Yechilishi. To'rt vektorlarning quyidagi signaturasini nazarda tutamiz (to'rt-impuls P esa, energiya E va impuls \vec{p} ni o'z ichiga oladi):

$$A = (A_0, \vec{A}), \quad A^2 = A_0^2 - \vec{A}^2, \quad AB = A_0 B_0 - \vec{A} \vec{B}$$

$$p^2 = (p_0, \vec{p}) = (E, \vec{p}) \quad p^2 = p_0^2 - \vec{p}^2 = E^2 - \vec{p}^2 \Rightarrow p_0^2 = m^2$$

Ham energiyasi, ham impuls saqlanishini hisobga oladigan to'rt impuls saqlanish qonuni quyidagicha yoziladi:

$$P_{\bar{p}} + P_d = P_n + P_{\pi}$$

Parchalanish sodir bo'lgunga qadar ($\bar{p}d$) sistemaning to'rt impulsini $P = P_{\bar{p}} + p_d$ deb belgilab olsak va parchalanish S-sistemada sodir bo'layotibdi ($\vec{P} = 0$) desak

$$P = (P_0, \vec{P}) = (E, \vec{P}) \Rightarrow (m_{\bar{p}} + m_d, \vec{0}) = (M, \vec{0}), \quad M = m_{\bar{p}} + m_d,$$

chunki, tinch turgan ($\bar{p}d$) sistemaning energiyasi tashkil etuvchilarning massalari yig'indisidan iboratdir. To'rt-impuls saqlanishidan neytronning impulsini aniqlab olib, uni kvadratga oshirsak:

$$p_n = p_{\bar{p}} + p_d - p_{\pi} = P - p_{\pi}$$

$$m_n = p_n^2 = (P - p_{\pi})^2 = P^2 - 2Pp_{\pi} + p_{\pi}^2 = P^2 - 2(P_0 p_{\pi}^0 - \vec{P} \vec{p}_{\pi}) + p_{\pi}^2 = \\ = P^2 - 2P^0 p_{\pi}^0 + p_{\pi}^2 = M^2 - 2ME_{\pi} + m_{\pi}^2$$

$$\text{Demak,} \quad E_{\pi} = \frac{M^2 + m_{\pi}^2 - m_n^2}{2M} = 1,24 \text{ GeV}$$

4-masala. Energiya $\varepsilon = 1,13 \text{ GeV}$ bo'lgandagina ${}^4\text{He} + {}^{14}\text{N} = {}^{17}\text{O} + {}^1\text{H}$ reaksiya sodir bo'la boshlaydi. L -sistemada to'qnashuvga qadar azotning yadrosi tinch turibdi. Ushbu reaksiya sodir bo'lishi uchun α - zarraning minimal energiyasi topilsin.

Yechilishi. Shartga ko'ra α -zarra va azot yadrolarining nisbiy xarakati energiyasi ε_n dan katta bo'lishi kerak:

$$\frac{\mu V^2}{2} \geq \varepsilon_0, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

m_1, m_2 α - zarra va azot yadrosining massalari, \vec{v}_1, \vec{v}_2 - shu zarralarning nisbiy tezligi. Izlanayotgan energiya E quyidagicha bo'lishi kerak.

$$E_1 = \frac{m_1 \vec{v}_1^2}{2} \geq \frac{m_1}{\mu} \varepsilon_0 = \frac{m_1 + m_2}{m_2} \varepsilon_0 \approx 1,45 \text{ GeV}$$

Bu energiyani qiymatini inersiya markazi sistemasida yozib olsak,

$$E_1 = \frac{\mu \vec{R}^2}{2} + \frac{\mu V^2}{2} = \frac{m_1}{M} E_1 + \frac{m_2}{M} E_1, \quad M = m_1 + m_2.$$

Shu energiyani bir qismi $\frac{m_1 E_1}{M}$ reaksiyada ishtirok etmay, α - zarra va azot yadrosidan tashkil topgan sistemaning yalpi harakati energiyasiga aylanishini yaqqol ko'rish mumkin. Bu energiya parchalangan bo'laklarning kinetik energiyasiga aylanadi.

5.1.-Misollar.

1. Tezligi V ga teng zarracha ikkita bir xil zarraga parchalanadi. Bu ikki zarra parchalanishi natijasida hosil bo'lgan zarralarning harakat yo'nalishlari orasidagi burchak θ ga teng deb, shu burchak bo'yicha taqsimot topilsin. Inersiya markazi sistemasida parchalanish izotrop deb hisoblansin va hosil bo'lgan zarrachalarning tezligi v_0 ga teng deb olinsin.

2. Parchalanayotgan K -mezonnin massasi $0,5 \text{ GeV}$ hosil bo'lgan π - mezonnik esa $0,14 \text{ GeV}$ desak, quyidagi parchalanish reaksiyalari to'g'risida nima deyish mumkin

a) $K \rightarrow 3\pi$, b) $K \rightarrow 4\pi$.

3 Massasi M ga teng zarracha ikki m_1 va m_2 massali zarrachaga parchalanadi. hosil bo'lgan zarralarning energiyasi S-sistemada topilsin ($c=1$ deb olinsin).

4. Massasi $m = 0,14 \text{ GeV}$ ga teng π - mezon $m_\mu = 0,106 \text{ GeV}$ lik μ - mezon va tinch holatdagi massasi $m_\nu = 0$ ga teng neytrinoga parchalanadi: $\pi \rightarrow \mu + \nu$. Bu ikki zarralarning kinetik energiyasi topilsin.

§5.2. Zarralarning elastik to'qnashuvi.

To'qnashuv protsesslarini yoritishda ham, ikki xil sanoq sistemasi ko'p qo'llaniladi. Ikki zarraning to'qnashish masalasi S-sistemada soddaroq hal qilinadi. Bu sistemada ikki to'qnashuvchi zarralarning inersiya markazi tinch turibdi deb hisoblanadi. Impuls saqlanish qonuniga binoan bu ikki zarraning impulslarining yo'nalishlari qarama-qarshi bulib son qiymatlari bir xildir: $\vec{v} = \vec{v}_1 - \vec{v}_2$ - nisbiy tezlik,

$$\vec{v}'_{10} = \frac{m_2}{m_1 + m_2} \vec{v}, \quad \vec{v}'_{20} = -\frac{m_1}{m_1 + m_2} \vec{v}, \quad \vec{p}'_{10} = -\vec{p}'_{20} = \vec{p}'_0,$$

$$\vec{P} = \vec{p}'_{10} + \vec{p}'_{20} = 0 = \vec{p}'_{10} + \vec{p}'_{20}, \quad |\vec{p}'_{10}| = |\vec{p}'_{20}| = p_0$$

bu yerda \vec{v}_1, \vec{v}_2 lar to'qnashgunga qadar zarralarning tezliklari, shtrix belgisi esa, to'qnashuvdan so'nggi kattaliklarni anglatadi. S-sistemada to'qnashish har ikkala zarra tezliklari yo'nalishining burilishidagina iboratdir. Agarda m zarraning to'qnashgandan keyingi yo'nalishini \vec{n}_0 bilan belgilasak:

$$\vec{v}'_{10} = \frac{m_2}{m_1 + m_2} v \vec{n}_0, \quad \vec{v}'_{20} = -\frac{m_1}{m_1 + m_2} v \vec{n}_0 \quad (5.14)$$

L-sistemada esa, inersiya markazi tezligi \vec{V} ni qo'shib qo'ysak yetarli

$$\vec{v}'_1 = \frac{m_2}{m_1 + m_2} v \vec{n}_0 + \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2},$$

$$\vec{v}'_2 = -\frac{m_1}{m_1 + m_2} v \vec{n}_0 + \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} \quad (5.15)$$

S-sistemadagi birinchi zarraning og'ish burchagi - (burilish, sochilish) χ bilan, L-sistemadagi har bir zarraning birinchi zarra yo'nalishidan og'ish burchaklari θ_1 va θ_2 orasidagi bog'lanish quyidagicha (burchak taqsimoti):

$$\operatorname{tg}\theta_1 = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi}, \quad \theta_2 = \frac{\pi - \chi}{2} \quad (5.16)$$

$\theta = \theta_1 + \theta_2$ zarralarning to'qnashishdan keyingi bir-biriga nisbatan og'ish burchagi. Agarda L-sistemada m_1 zarra m ga elastik urilayapti desak

$$m_1 > m_2 \text{ da } \theta = \theta_1 + \theta_2 < \frac{\pi}{2}$$

$$m_1 < m_2 \text{ da } \theta = \theta_1 + \theta_2 > \frac{\pi}{2} \quad (5.17)$$

Endi har bir zarraning to'qnashuvdan keyingi tezliklarining qiymatini χ burchak orqali yozsak

$$v_1' = \frac{\sqrt{m_1^2 + m_2^2 + 2m_1m_2 \cos \chi}}{m_1 + m_2}, \quad v_2' = \frac{2m_1v}{m_1 + m_2} \sin\left(\frac{\chi}{2}\right) \quad (5.18)$$

Uriluvchi zarra m_1 va L-sistemada tinch turgan zarraning massalariga qarab m zarraning to'qnashuvdan keyingi yo'nalishi har xil bo'lishi mumkin:

a) istalgan yo'nalish, agarda $m_1 < m_2$ bo'lsa,

b) $\sin \theta_{1\max} = \frac{m_1}{m_2}$ ni qanoatlantirgan $\theta_{1\max}$ dan katta burchakka og'a

olmaydi, agarda $m_1 > m_2$ bo'lsa:

Agarda $m_1 = m_2$ bo'lsa, $\theta = \frac{\chi}{2}, \theta = \frac{\pi - \chi}{2},$

$$v_1 = \cos\left(\frac{\chi}{2}\right), \quad v_2 = \sin\left(\frac{\chi}{2}\right) \quad (5.19)$$

«Pesh zarba» ($\chi = \pi$) holida

$$\vec{v}_1' = \frac{m_1 - m_2}{m_1 + m_2} \vec{v}, \quad \vec{v}_2' = \frac{2m_1}{m_1 + m_2} \vec{v} \quad (5.20)$$

Bu holdagi \vec{v}_2' ning qiymati - erishish mumkin bo'lgan eng katta qiymatdir va

$$E'_{2\max} = \frac{m_2 v_{2\max}'^2}{2} = \frac{4m_1 m_2}{(m_1 + m_2)^2} E_1,$$

(5.21)

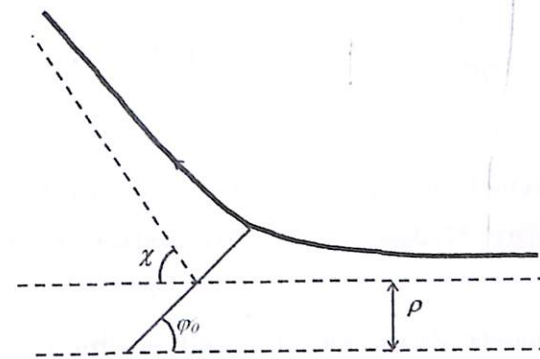
$E = \frac{m_1 v_1^2}{2}$ - urilayotgan m_1 zarraning boshlang'ich energiyasi. Odatda,

ikki zarra to'qnashuvi masalasining to'la yoritishlari (χ burchakni topish) zarralar o'zaro ta'sirining konkret qonunini hisobga olgan holda harakat tenglamalarini yechishni talab etadi.

Lekin, ikki zarra to'qnashuvi masalasi bir zarraning $U(r)$ maydondagi og'ish masalasiga kelitirilishi mumkin (5.1.-rasm)

$$\chi = |\pi - 2\phi| \quad (5.22)$$

$$\phi_0 = \int_{r_{\min}}^{\infty} \frac{\frac{M}{r^2} dr}{\sqrt{2m[E - U(r)] - \frac{M^2}{r^2}}} \quad (5.23)$$



5.1.-rasm.

(Bu integralni aniqlanish 2.1.-punktida batafsil yoritilgan).

(5.23)da energiya va impuls momentini cheksizdagi tezlik - v_x va mo'ljali masofasi - ρ orqali yozib olsak

$$E = \frac{mv_x^2}{2}, \quad M = m\rho v_x \quad (5.24)$$

$$\varphi_0 = \int_{r_{\min}}^{\infty} \frac{\left(\frac{\rho}{r}\right)^2 dr}{\sqrt{1 - \frac{\rho^2}{r^2} - \frac{2u}{mv_x^2}}} \quad (5.25)$$

Zarralar dastasining sochilishini xarakterlaydigan effektiv sochilish kesimi quyidagicha aniqlanadi:

$$d\sigma = \frac{dN}{n} \quad (5.26)$$

n -dasta ko'ndalang kesimining birlik yuzasidan vaqt birligi ichida o'tadigan zarralar soni, $dN - \chi + d\chi$ burchak oralig'ida sochiladigan zarralar soni. Bu kattaliklarning ρ va χ lar orqali ifodasi quyidagicha:

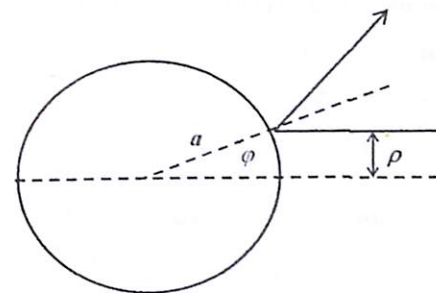
$$d\sigma = 2\pi\rho d\rho, \quad d\sigma = 2\pi\rho(\chi) \frac{|d\rho(\chi)|}{|d\chi|} d\chi \quad (5.27)$$

Effektiv kesimni L - sistemada sochilish burchagi θ ga bog'liqligini topish uchun (5.27) dagi χ ni (5.16) formulalar bo'yicha θ orqali ifodalash lozim.

1-masala. Zarralarning a radiusli absolyut qattiq sharchada sochilish effektiv kesimi topilsin (5.2- rasm)

$$U(r) = \begin{cases} \infty, & r < a \\ 0, & r > a \end{cases}$$

Yechilishi. Sharchadan tashqarida zarra erkin harakatlangani va sharchaning ichiga hech qachon kira olmasligi tufayli, zarraning harakati to'qnashish nuqtasiga



5.2.-rasm.

tortilgan, sharchaning radiusiga nisbatan simmetrik bo'lgan troyektoriya bo'yicha sodir bo'ladi. Suratdan kurinib turibdiki:

$$\rho = a \sin \varphi_0 = a \sin \frac{\pi - \chi}{2} = a \cos \left(\frac{\chi}{2} \right)$$

bu yerda, χ - S-sistemadagi og'ish burchagi, ρ - mo'ljali masofasi. Effektiv kesimni (5.27) formuladan aniqlasa bo'ladi:

$$d\sigma = 2\pi\rho(\chi) \frac{d\rho(\chi)}{d\chi} d\chi \Rightarrow \pi a \cos \left(\frac{\chi}{2} \right) \left| -\sin \left(\frac{\chi}{2} \right) \cdot a \right| d\chi = \frac{\pi a^2 \sin \chi d\chi}{2} = a^2 \frac{dO}{4},$$

$$dO = 2\pi \sin \chi d\chi$$

Bundan ko'rinib turibdiki, S-sistemada sochilish - izotrop.

Effektiv kesim $d\sigma$ ni burchak bo'yicha integrallab to'la kesimni topamiz:

$$d\sigma = a^2 \frac{4\pi}{4} = \pi a^2.$$

Demak, zarra umuman to'qnashishi mumkin bo'lgan yuz doiraning yuzasidan iborat ekan. L-sistemada effektiv kesimni sochilish burchagi θ_1 orqali ifodalash uchun (5.16) formuladan χ ni θ_1 orqali ifodalasak yetarlidir (sochiluvchi zarraning massasi m_1 "tinch" tinch turgan sharchanikini esa, m_2 deylik). χ ni θ_1 orqali aniqlash xuddi 5.1. punktdagi 2-masaladagidek bo'lgani

uchun, oxirgi javobni yozsak yetarli, chunki (5.16) va (5.8) formulalar bir biriga juda ham uxshash.

a) Agarda $m_1 < m_2$ bo'lsa,

$$d\sigma_1 = \frac{a^2}{4} \left(2 \frac{m_1}{m_2} \cos \theta_1 + \frac{1 + \frac{m_1}{m_2} \cos 2\theta_1}{\sqrt{1 - \frac{m_1^2}{m_2^2} \sin^2 \theta_1}} \right) dO_1,$$

b) Agarda $m_1 > m_2$ bo'lsa,

$$d\sigma_1 = \frac{a^2}{2} \frac{1 + \frac{m_1}{m_2} \cos 2\theta_1}{\sqrt{1 - \frac{m_1^2}{m_2^2} \sin^2 \theta_1}} dO_1,$$

v) $m_1 = m_2$ bo'lsa, yuqoridagi formulalardan $d\sigma = a^2 |\cos \theta_1| dO_1$ ni olamiz

Bu yerda modul belgisini to'g'ri ishorani olish uchun ataylab qoldirdik

((5.27) ga qarang). Shu natijani $d\sigma = \frac{a^2 dO}{4}$ ifodada $\chi = 2\theta_1$ ekanini hisobga

olib ham topish mumkin ((5.19) qarang):

$$d\sigma_1 = \frac{a^2 2\pi \sin \chi d\chi}{4} \Rightarrow \pi a^2 \sin 2\theta_1 d\theta_1 = a^2 \pi 2 \sin \theta_1 \cos \theta_1 d\theta_1 = a^2 |\cos \theta_1| d\theta_1.$$

To'qnashuvga qadar tinch turgan sharchalar uchun $\chi = \pi - 2\theta_2$ bog'lanishdan foydalansak :

$$dG_1 = \frac{a^2 dO}{4} \Rightarrow a^2 2\pi \sin(\pi - 2\theta_2) \frac{d(\pi - 2\theta_2)}{4} = a^2 \cos 2\theta_2 dO_2, \quad dO_2 = 2\pi \sin \theta_2 d\theta_2$$

2-masala. Massasi m ga teng zarraning $U = \frac{1}{2r^2}$ markaziy maydondagi

og'ish burchagi va effektiv kesimi topilsin.

Yechilishi. Ma'lumki, (5.23) ifoda xarakatlanuvchi zarraning qutb burchagining cheksizlikdan markazga eng yaqin masofaga yaqinlashguncha o'zgarishini aniqlaydi. Zarraning o'z yo'nalishidan og'ish burchagi esa (5.22) ga asosan $\chi = \pi - 2\phi_0$. Dastlab (5.23) dan ϕ_0 ni topaylik ($m=1$):

$E = \frac{M^2}{2r_0^2} + \frac{1}{2r_0^2}$ dan r_{\min} ning qiymatini topamiz.

$$r_0 = \sqrt{\frac{1+M^2}{2E}}, \quad \phi_0 = \int_{r_{\min}}^{\infty} \frac{M dr}{r^2 \sqrt{2E - \frac{M^2}{r^2} - \frac{1}{r^2}}} = \frac{M}{\sqrt{1+M^2}} \arccos \left(\frac{1}{r^2} \sqrt{\frac{1+M^2}{2E}} \right) \Big|_{r_{\min}}^{\infty} =$$

$$= \frac{M}{\sqrt{1+M^2}} [\arccos 0 - \arccos 1] = \frac{\pi M}{\sqrt{1+M^2}}.$$

Demak,

$$\chi = \pi - 2\phi_0 = \pi - \frac{\pi M}{\sqrt{1+M^2}} = \pi \frac{\sqrt{1+M^2} - M}{\sqrt{1+M^2}} = \pi \frac{\sqrt{1+m^2 \rho^2 v_x^2} - m\rho v_x}{\sqrt{1+m^2 \rho^2 v_x^2}}.$$

Bu ifoda og'ish burchagi χ ni mo'ljal masofasi ρ bilan bog'laydi.

$$\frac{\chi}{\pi} = 1 - \frac{m\rho v_x}{\sqrt{1+m^2 \rho^2 v_x^2}} \Rightarrow \rho^2 = \frac{1}{m^2 v_x^2} \frac{(\chi - \pi)^2}{\pi^2 - (\chi - \pi)^2}.$$

Bu erdan

$$d\sigma = 2\pi \rho d\rho = \frac{\pi}{m^2 v_x^2} \left\{ \frac{2(\chi - \pi) [\pi^2 - (\chi - \pi)^2] - [2(\chi - \pi)(\chi - \pi)^2]}{[\pi^2 - (\chi - \pi)^2]^2} \right\} d\chi =$$

$$= \frac{2\pi}{m^2 v_x^2} \frac{\pi^2 (\chi - \pi) d\chi}{[\pi^2 - (\chi - \pi)^2]^2},$$

yoki

$$d\sigma = \frac{\pi^2 (\chi - \pi) dO}{m^2 v_x^2 \chi^2 (2\pi - \chi)^2 \sin \chi}, \quad dO = 2\pi \sin \chi d\chi.$$

Bu S- sistemadagi effektiv kesimining ifodasidir.

3-masala. Ikkita massalari m ga teng bir xil zarra $U = \frac{1}{2r^2}$ qonuniyat bilan o'zaro ta'sirlashayotibdi deb, zarralarning biri tinch turgan sistemada effektiv kesim topilsin.

Yechilishi. Yuqoridagi 2-masalaning yechilishidan foydalanamiz. Bu holda ikki zarraning o'zaro munosabati keltirilgan massasi $\frac{m}{2}$ teng zarraning $U = \frac{1}{2r^2}$ maydondagi harakati masalasiga keltirib olinadi. Ya'ni, oldingi masalaning natijasida $m = \frac{m}{2}$ deb olishimiz kerak.

$$d\sigma = \frac{8\pi^3(\chi - \pi)d\chi}{m^2 v_x^2 \chi^2 (2\pi - \chi)^2}.$$

L-sistemada harakatda bo'lgan zarra (uriluvchi) uchun $\theta_1 = \frac{\chi}{2}$ ((5.19 ga qarang) deb

$$d\sigma_1 = \frac{\pi^3(2\theta_1 - \pi)d\theta_1}{m^2 v_x^2 \theta_1^2 (\pi - \theta_1)^2} = \frac{\pi^2(2\theta_1 - \pi)d\theta_1}{2m^2 v_x^2 \theta_1^2 (\pi - \theta_1)^2 \sin \theta_1}.$$

To'qnashishdan oldin "tinch" turgan zarra uchun $\theta_2 = \frac{\pi - \chi}{2}$,

$$d\sigma_2 = \frac{32\pi^3 \theta_2 d\theta_2}{m^2 v_x^2 (\pi^2 - 4\theta_2^2)^2} = \frac{16\pi^2 \theta_2^2 d\theta_2}{m^2 v_x^2 (\pi^2 - 4\theta_2^2)^2 \sin \theta_2}.$$

4-masala. Zarralardan biri to'qnashuvigacha tinch turgan holatda edi. Nishon tinch turgan sistemada zarralarning kinetik energiyasi va sochilish burchaklari S-sistemadagi sochilish burchagi χ orqali topilsin.

Yechilishi. Faraz qilaylikki $\vec{v}_2 = 0$ bo'lsin,

u holda, $\vec{v} = \vec{v}_1 - \vec{v}_2 = \vec{v}_1$, $|\vec{v}| = |\vec{v}_1|$ va

$$\vec{p}'_1 = m_1 \vec{v}' + \mu v \vec{n} \Rightarrow m_1 \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2} + \mu v \vec{n} = \frac{m_1^2}{m_1 + m_2} \vec{v}_1 + \mu v_1 \vec{n},$$

$$\vec{p}'_2 = m_2 \vec{v}' - \mu v \vec{n} \Rightarrow \mu \vec{v}_1 - \mu v_1 \vec{n}.$$

Bu ifodalardan har bir zarraning to'qnashuvdan keyingi kinetik energiyasini topish mumkin:

$$T'_1 = \frac{p_1'^2}{2m_1} = \frac{m_1^4 v_1^2 + 2m_1^3 m_2 v_1^2 \cos \chi + m_1^2 m_2^2 v_1^2}{(m_1 + m_2)^2 2m_1} = T_1 \frac{m_1^2 + 2m_1 m_2 \cos \chi + m_2^2}{(m_1 + m_2)^2},$$

$$ET'_2 = \frac{p_2'^2}{2m_2} = \frac{\mu^2 v_1^2 (2 - 2 \cos \chi)}{2m_2} = \frac{4\mu T_1 \sin^2 \frac{\chi}{2}}{m_1 + m_2}$$

$$T_1 = \frac{m_1 v_1^2}{2} \text{ - birinchi zarraning kinetik energiyasi}$$

Sochilish burchaklari θ_1 va θ_2 lar (5.16) formula orqali yozilishi mumkin

$$\operatorname{tg} \theta_1 = \frac{m_2 \sin \chi}{m_1 + m_2 \cos \chi}, \quad \theta_2 = \frac{\pi - \chi}{2}, \text{ yoki}$$

$$\operatorname{tg} \theta_2 = \operatorname{tg} \frac{\pi - \chi}{2} = \frac{\sin(\pi - \chi)}{1 + \cos(\pi - \chi)} = \frac{\sin \chi}{1 - \cos \chi}.$$

Oxirgi ifodalardan:

$$\operatorname{tg}(\theta_1 + \theta_2) = \operatorname{tg} \theta_{12} = \frac{\operatorname{tg} \theta_1 + \operatorname{tg} \theta_2}{1 - \operatorname{tg} \theta_1 \operatorname{tg} \theta_2} = \frac{(m_1 + m_2) \sin \chi}{(m_1 + m_2) \cos \chi (1 - \cos \chi) - m_2 \sin^2 \chi} =$$

$$= \frac{(m_1 + m_2) \sin \chi}{(m_1 - m_2)(1 - \cos \chi)} = \frac{m_1 + m_2}{m_1 - m_2} \operatorname{ctg} \frac{\chi}{2}.$$

Oxirgi natijadan ko'rinib turibdiki, $m_1 = m_2$ da $\theta_{12} = \frac{\pi}{2}$.

Bu ifodalar to'qnashayotgan zarralarning bir-biridan og'ish burchagi - θ_{12} ning S-sistemadagi sochilish burchagi χ bilan bog'lanishini beradi.

5- masala. Ushbu kattalikning Galiley almashtirishlariga nisbatan invariant

ekanligi ko'rsatilsin va uning S-sistemadagi qiymati topilsin:

$$S = \frac{(T_1 + T_2) - (\vec{p}_1 + \vec{p}_2)^2}{2(m_1 + m_2)}.$$

Bu erda, $T_i = \frac{m_i v_i^2}{2}$ - kinetik energiya.

Yechilishi. Galiley almashtirishlarining tezliklar uchun ifodasidan foydalansak:

$$S = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} - \frac{(m_1 \vec{v}_1 + m_2 \vec{v}_2)^2}{2(m_1 + m_2)} = \frac{m_1 (\vec{v}'_1 + \vec{v})^2}{2} + \frac{m_2 (\vec{v}'_2 + \vec{v})^2}{2} - \frac{(m_1 (\vec{v}'_1 + \vec{v}) + m_2 (\vec{v}'_2 + \vec{v}))^2}{2(m_1 + m_2)} =$$

$$= \frac{m_1 v_1'^2}{2} + \frac{m_2 v_2'^2}{2} - \frac{(m_1 v_1' + m_2 v_2')^2}{2(m_1 + m_2)} = (T_1' + T_2') - \frac{(\vec{p}_1' + \vec{p}_2')^2}{2(m_1 + m_2)}.$$

Demak, bu kattalik saqlanuvchi ekan.

S-sistemada (5.13) larga asosan:

$$S = \left(\frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} \right) - \frac{(\vec{p}_1 + \vec{p}_2)^2}{2(m_1 + m_2)} = \frac{m_1}{2} \left(\frac{m_1}{m_1 + m_2} \vec{v} \right)^2 + \frac{m_2}{2} \left(-\frac{m_1}{m_1 + m_2} \vec{v} \right)^2 - 0 = \mu V^2.$$

5.2.-Misollar.

1. S-sistemada zarralarning to'qnashuvigacha bo'lgan tezligi \vec{v}_2 va sochilish burchagi χ ma'lum. Bu zarralarning to'qnashuvdan keyingi tezlik va impulslari topilsin. Inersiya markazining tezligi \vec{V} , nisbiy tezlik $\vec{v} = \vec{v}_1 + \vec{v}_2$ ga teng.

2. To'qnashish protsessida ikkala zarraning bir-biridan og'ish burchagi $\theta_2 = \theta_1 + \theta_2$ ning qabul qilishi mumkin bo'lgan qiymatlarining intervali aniqlansin. Ko'rsatma: Dastlab burchaklarning tangenslarini topib olib, ekstremumga tekshirish qulay.

3. Ushbu kattalikning $t = -2m(T_2' - T_2) + (\vec{p}_2' - \vec{p}_2)$ Galiley almash-tirishidagiga nisbatan invariant qolishini ko'rsating va uning S-sistemadagi qiymatini toping.

4. Massasi m bo'lgan zarraning quyidagi markaziy maydonlardan og'ish burchagi topilsin:

$$\text{a) } U = \frac{\alpha}{r}, \quad \text{b) } U = \frac{1}{2} \left(\frac{1}{r^2} - \frac{1}{r} \right).$$

5. Quyidagi maydonlar uchun zarraning markazga tushish effektiv kesimi topilsin:

$$\text{a) } U = -\frac{\alpha}{r^2}, \quad \text{b) } U = -\frac{\alpha}{r^n}, \quad (n > 2), \quad \text{v) } U = \frac{\alpha}{r} - \frac{\beta}{r^2}, \quad \text{g) } U = \frac{\beta}{r^2} - \frac{\gamma}{r^4}$$

6. Zarraning absolyut qattiq sharchadan sochilishidagi effektiv kesimini zarraning yo'qotayotgan energiyasi ε orqali yozing.

§5.3. Relyativistik mexanika elementlari

Eynshteynning nisbiylik prinsipiga asoslangan mexanika, *relyativistik mexanika* deyiladi. Agar, K' sanoq sistemasini K sanoq sistemasiga nisbatan V tezlik bilan X o'qi bo'yicha harakatlanayapti, desak quyidagi Lorens almashtirishlari o'rinlidir.

$$x = \frac{x' + Vt'}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad y = y', \quad z = z', \quad t = \frac{t' + \frac{V}{c^2} x'}{\sqrt{1 - \frac{V^2}{c^2}}} \quad (5.28)$$

Odatda $\frac{V}{c} = \beta$, $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$ belgilashlar kiritiladi. Teskari almashtirishlar

(5.28) dan $V \rightarrow -V$ ga o'zgartirib topiladi:

$$x' = \gamma(x - Vt), \quad y' = y, \quad t' = \gamma \left(t - \frac{\beta}{c} x \right) \quad (5.29)$$

Xususiy uzunlik, vaqt, hajmlar quyidagichadir:

$$l' = \frac{l}{\gamma}, \quad t' = \frac{t}{\gamma}, \quad g' = \frac{g}{\gamma} \quad (5.30)$$

Relyativistik tezliklarni qo'shish formulalari:

$$v_x = \frac{v'_x + V}{1 + \frac{v'_x V}{c^2}}, \quad v_y = \frac{\gamma^{-1} v'_y}{1 + \frac{v'_x V}{c^2}}, \quad v_z = \frac{\gamma^{-1} v'_z}{1 + \frac{v'_x V}{c^2}} \quad (5.31)$$

$$v = \frac{v' + V}{1 + \frac{v'_x V}{c^2}}, \quad (v'_x = v') \quad (5.32)$$

kelib chiqadi. Tezlik yo'nalishining o'zgarishi quyidagicha

$$\operatorname{tg} \theta = \frac{v' \sin \theta'}{\gamma(v' \cos \theta' + V)} \quad (5.33)$$

Mexanikaning asosiy kattaliklarining relyativistik ifodalarini keltiraylik

$$\vec{P} = m\vec{v}\gamma, \quad E = mc^2\gamma, \quad L = -mc^2\sqrt{1 - \frac{v^2}{c^2}}, \quad H = c\sqrt{p^2 + m^2c^2} \quad (5.34)$$

lar mos ravishda erkin zarraning impuls, energiya, Lagranj va Gamilton funksiyalari. Energiya va impuls orasidagi bog'lanish

$$\frac{E^2}{c^2} = p^2 + m^2c^2 \quad (5.35)$$

ni 4 o'lchovli tushuncha orqali yozib olish qulay:

$$p^\mu = \left(\frac{E}{c}, \vec{p} \right) = (p^0, \vec{p}), \quad p^\mu p_\mu = p_0^2 - \vec{p}^2 = \frac{E^2}{c^2} - \vec{p}^2 \equiv m^2c^2, \quad (5.36)$$

Bu kattalik uchun Lorens almashtirishlari quyidagicha:

$$p_x = \left(p'_x + \frac{V}{c^2} E' \right) \gamma, \quad p_y = p'_y, \quad p_z = p'_z, \quad E = \gamma(E' + Vp'_x) \quad (5.37)$$

Relyativistik zarra uchun qulay munosabatlar (7.7) ifodalardan kelib chiqadi:

$$\vec{p} = \frac{E\vec{v}}{c^2} \quad (5.38)$$

Relyativistik mexanikaning asosiy bo'laklaridan biri relyativistik zarralarning kinematikasidir. Ma'lumki, energiya va impuls saqlanish qonunlari fazo va vaqtning eng umumiy xususiyatlari bilan bog'liqdir. Bu xususiyatlar *maxsus nisbiylik prinsipida* ham o'rinalidir.

Kinematika-shu xususiyatlardan kelib chiqadigan qonuniyatlar asosidagi o'zaro ta'sirni o'rganadi, lekin o'zaro ta'sirning xarakteri, kelib chiqishi bilan qiziqmaydi.

Masalan, bir grupp zarralar o'zaro ta'sirlashdi va yana qandaydir zarralar gruppasi hosil bo'ldi, boshlang'ich va oxirgi vaziyatlarda 4-impuls saqlanishi kerak:

$$a + b + \dots \rightarrow c + d + \dots, \quad p_a^\mu + p_b^\mu + \dots \rightarrow p_c^\mu + p_d^\mu + \dots, \quad \mu = 0, 1, 2, 3 \quad (5.39)$$

Bu tenglik har qanday inersial sanoq sistemasida o'rinalidir, undan tashqari 4-impulsning kvadrati invariantdir.

$$p^\mu p_\mu = p^2 = m^2c^2 \equiv inv \quad (5.40)$$

1-masala. Noturg'un zarrachalar har. xil yo'l bilan parchalanadilar.

Masalan: K^+ -mezonlar (kaonlar) quyidagi parchalanish «kanallariga» ega:

$K^+ \rightarrow \mu^+ + \nu_\mu$ - 63,6% holda, $K^+ \rightarrow \pi^+ + \pi^0$ - 21,2% holda, ya'ni, birinchi reaksiya taxminan uch marta ko'proq uchraydi. Bu parchalanish reaksiyalarining ehtimolligi «birlamchi zarracha» K -mezonning tezligiga qanday bog'liq?

Yechilishi. Nisbiylik prinsipi har qanday jarayonlar uchun o'rinalidir, shu jumladan parchalanish jarayoni uchun ham. Agar, tinch turgan kaon qandaydir proporsiya bilan parchalanar ekan, uchib borayotgan kaon o'zi bilan bog'liq bo'lgan sanoq sistemasida (o'zining «tinch» holat sanoq sistemasida) o'shanday proporsiyada, o'sha kanallarga parchalanishi kerak. Shuning uchun, parchalanish kanallari proporsiyalari tezlikga bog'liq emas. Aks holda, nisbiylik prinsipiga zid ravishda, voqiyalikning sodir bo'lishi inersial sanoq sistemasining tanlanishiga bog'liq bo'lib qolgan bo'lardi.

Birlamchi zarrachaning tezligi, ikkilamchi zarracha (π, μ, ν) larning tezliklarigagina ta'sir qiladi, xolos.

2-masala. Musbat zaryadli pion- π^+ ko'proq musbat zaryadli myuon- μ^+ va neytrinoga parchalanadi: $\pi^+ \rightarrow \mu^+ + \nu_\mu$. Tinch holatdagi π^+ -mezonlarning yarim yemirilish davri $\tau_0 = 1,8 \cdot 10^{-8}$ sekundga teng. Tajribada tezligi $v = (1 - 5 \cdot 10^{-5}) \cdot c$ ga teng bo'lgan (c -yorug'lik tezligi) π^+ -mezonlar dastasi hosil qilindi. O'lchashlar natijasida bu dastadagi pionlarning yarim yemirilish davri qancha? Shu vaqt davomida, ya'ni, pionlarning yarmi parchalanguncha, ular qancha masofa o'tadilar?

Yechilishi. Nisbiylik nazariyasining maftunkor dalillaridan biri vaqt o'tishining sekinlashish effektidir. Ayniqsa, harakatlanuvchi va tinch turgan ob'ektlar bilan

sodir bo'ladigan jarayonlarning vaqtiy xarakteristikalarini solishtirganda bu effekt yaqqol namoyonlanadi. Laboratoriya o'lchashlari shuni ko'rsatdiki har qanday harakatlanuvchi ob'ektlar bilan sodir bo'layotgan protsesslar tinch turgan ob'ektlarda sodir bo'ladigan protsesslarga ko'ra γ marta «sekinroq» o'tadi.

Yuqoridagi (5.29) formuladan

$$\gamma = (1 - \beta^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{1}{\sqrt{(1 - \frac{V}{c})(1 + \frac{V}{c})}} = \frac{1}{\sqrt{(1 - \frac{(1 - 5 \cdot 10^{-5}) \cdot c}{c})(1 + \frac{(1 - 5 \cdot 10^{-5}) \cdot c}{c})}} = \frac{1}{\sqrt{5 \cdot 10^{-5}(1 + 1 - 5 \cdot 10^{-5})}} \approx 100$$

yani γ -faktor taxminan $\gamma \approx 100$ ga teng. Demak, $\tau = \gamma \cdot \tau_0 \approx 1,8 \cdot 10^{-6}$ sekund ekan. Bu vaqt ichida pionlar $l = v \cdot \tau = 540m$ masofa o'tadilar. Demak 540 m masofa o'tilganda dastadagi pionlarning soni ikki marta kamayadi. Shunday qilib, yorug'lik tezligiga yaqin tezliklar bilan harakatlanuvchi noturg'un zarrachalarning «umri» (yashash davri) γ marotaba uzayishini aniq fakt desa bo'ladi. Keling, vaqtni sekin o'tish effekti (5.30) noo'rin deb, faraz qilaylik. U holda, yarim yemirilish davri τ_0 ga mos ravishda har 5,4 metrda dastaning yarmi parchalanib ketar edi, va 540 metr uzoqlikdagi hisoblagichga boshlang'ich dastaning atigi $\left[\frac{1}{2}\right]^{100}$ qismigina borib tushgan bo'lardi. Lekin,

$$2^{100} = (2^{10})^{10} = (1024)^{10} > (10^3)^{10} = 10^{30}$$

Demak, hisoblagichga loaqal bir dona pion borib tushish uchun dastada, kamida 10^{30} dona pion bo'lishi kerak. Bu shunday katta sonki, dunyodagi hamma tezatgichlardagi pionlar sonini yiqqanda ham shunga teng bo'lmaydi. Vaholanki, bu tajribada, yuzlab metrda joylashgan hisoblagichlar ham zarrachalarni «tutgan»lar.

3-masala. Siferblat tekisligiga perpendikulyar ravishda doimiy V tezlik bilan harakatlanayotgan soatning strelkasi siferblat tekisligida u tezlik bilan aylanyapti. Strelkaning uchi bizga nisbatan qanday tezlikda harakatlanyapti? Bu tezlikning siferblat tekisligidagi komponentasi nimaga teng?

Yechilishi. Ko'zg'almay turgan soatning strelkasi radiusi R bo'lgan siferblatni $T_0 = \frac{2\pi R}{u}$ davrda aylanib chiqadi. Harakatlanuvchi soat uchun davrning

qiymati qo'zg'almas kuzatuvchi sistemasida (5.30) ga asosan $T = \gamma T_0$. Ikkinchi tomondan, ko'ndalang o'lchamlar Lorens qisqartirishlari (7.3) ga uchramaganligi sababli, radiusning qiymati o'zgarmaydi va $T = \frac{2\pi R}{v_{\perp}}$

Bu yerda biz strelka uchining tezligi \bar{v} ni bo'ylama v_{\parallel} va ko'ndalang v_{\perp} komponentalarga ajratdik. Oxirgi ikki ifodani solishtirib

$$v_{\perp} = \frac{u}{\gamma} = u \sqrt{1 - \frac{V^2}{c^2}},$$

Bu yerda, v_{\perp} - siferblat tekisligida o'tadi. Tabiiyki, soat strelkasi tezligining bo'ylama komponentasi v_{\parallel} soat siferblatining tezligi V ga teng. Tezlikning moduli Pifagor teoremasiga binoan:

$$v^2 = v_{\parallel}^2 + v_{\perp}^2 = V^2 + u^2 \left(1 - \frac{V^2}{c^2}\right)$$

Bu natija tekislik normasi bo'yincha inersial ko'chishga duchor bo'lgan har qanday tekislikdagi harakat uchun o'rinlidir. Xususan, tekislikda yorug'likning tarqalishiga bu ifodani qo'llasak, $c = u$ va $c_{\parallel} = V$, $c_{\perp} = \sqrt{c^2 - V^2}$ yorug'lik tezligining invariantligi yaqqol kurinadi:

$$|c| = (c^2 + c_{\perp}^2)^{\frac{1}{2}} = c.$$

4-masala. Massasi M ga teng zarracha massalari m_1 va m_2 ga teng zarrachalarga parchalanayotibdi. Birlamchi zarracha tinch turgan sistemada hosil bo'lgan ikkilamchi zarrachalarning energiyasi topilsin.

Yechilishi. To'rt o'lchovli impuls saqlanish qonunini tuzaylik.

$$P'_0 = p'_{10} + p'_{20} \quad i=0,1,2,3.$$

$$\text{Bu yerda } P'_0 = (Mc^2, \vec{0}), \quad P'_{10} = \left(\frac{E_{10}}{c}, p'_{10} \right), \quad P'_{20} = \left(\frac{E_{20}}{c}, p'_{20} \right).$$

Alohida energiya va impuls uchun ham saqlanish qonuniyatini yozish mumkin:

Demak,

$$Mc^2 = E_{10} + E_{20}, \quad \vec{p}_{10} + \vec{p}_{20} = 0, \quad |\vec{p}_{10}| = |\vec{p}_{20}| = p$$

$$E_{10}^2 - (cp)^2 = m_1^2 c^4, \quad E_{20}^2 - (cp)^2 = m_2^2 c^4.$$

Bu ifodadan p ni yo'qotsak:

$$E_{10}^2 - E_{20}^2 = (m_1^2 - m_2^2) c^4, \quad E_{10} - E_{20} = \frac{(m_1^2 - m_2^2)}{M} c^2.$$

Bu yerdan

$$E_{10,20} = \frac{M^2 + m_{1,2}^2 - m_{2,1}^2}{2M} c^2. \quad (5.41)$$

Demak, zarrachalarning energiyalari massalar orqali bir qiymatli ravishda aniqlanadi.

5-masala. Tinch holatda turgan neytral kaon ikkita pionga parchalanadi.

Pionlarning massalari bir-biriga teng deb, ularning tezliklari topilsin.

Yechilishi. Massalari bir xil bo'lgani uchun, pionlarning tezliklari moduli bir xil bo'lib, qarama-qarshi tomonga yo'nalgan bo'ladi, chunki kaon tinch turgan sistemada.

$$p'_K = (M_K c^2, \vec{p}_K) = (M_K c^2, \vec{0})$$

va $|p'_\pi| = m_\pi v$; v - pionlarning tezligi.

Energiyaning saqlanishiga va (5.30) formulaga ko'ra (pionga «bog'langan» sanoq sistemasining tezligi ham v ga teng):

$$M_K c^2 = 2E_\pi \Rightarrow 2m_\pi \gamma c^2.$$

$$\text{Bu yerdan } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{M_K}{2m_\pi}, \quad v = c \sqrt{1 - \frac{1}{\gamma^2}} = c \sqrt{1 - \frac{4m_\pi^2}{M_K^2}}$$

$m_\pi = 139,6 \text{ MeV} / c^2$ $M_K = 497,7 \text{ MeV} / c^2$ ekanini hisobga olsak:

$$v = 0,83 \cdot c.$$

Bu natijaga 4-masalaning javobidan ham kelish mumkin. U yerdagi energiyaning ifodasi (5.41) da $m_1 = m_2 = m_\pi$, $M = M_K$ desak:

$$E_\pi = \frac{M_K^2}{2M_K} c^2.$$

Energiya uchun Lorens almashtirishlari (5.34) dan foydalansak

$$\frac{M_K^2}{2M_K} c^2 = m_\pi c^2 \gamma = E_\pi \Rightarrow 2m_\pi \gamma = M_K.$$

Demak, pionlarning tezligi yorug'lik tezligiga yaqin ekan.

Bu masalaning yechilishidan foydalanib, shunday simmetrik parchalanish sodir bo'lganda hosil bo'lgan zarralarning kinetik energiyasining «massa defekti» bilan qanday bog'langanligini ko'rsataylik. Mazkur parchalanishda pionlarning tezligi yorug'lik tezligiga yaqin edi. «Norelyativistik» parchalanish natijasida hosil bo'lgan ikkilamchi zarrachalarning tezligi yorug'lik tezligidan ancha kichik $v \ll c$ bo'ladi. Parchalanayotgan zarraning massasini M_0 , ikkita hosil bo'lganlarinikini esa m_0 deb, $\Delta m = M_0 - 2m_0$ - «massa defekti» ni

yuqoridagi ifodalardan ($\gamma = \frac{M_K}{2m_\pi}$) foydalanib yozib olaylik:

$$\gamma = \frac{M_0}{2m_0} = 1 + \frac{M_0 - 2m_0}{2m_0} = 1 + \frac{\Delta m}{2m_0}.$$

Kichik tezliklarda esa

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{1 - \frac{v^2}{c^2}} \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = 1 + \frac{v^2}{2c^2}$$

Bu ifodalardan

$$\square m = m_0 \frac{v^2}{c^2}.$$

Kichik tezliklarda esa energiya uchun klassik mexanika ifodalardan foydalanish mumkin. Shunday qilib, simmetrik parchalanishda, har ikki zarralarning kinetik energiyasi bir xil bo'lganligi uchun, «massa defekti» zarralarning kinetik energiyalarining yig'indisini yorug'lik tezligining kvadratiga nisbatiga tengdir

$$\square m = \frac{2T}{c^2}.$$

Harakatlanayotgan ikkilamchi zarrachaning «massa»si m ni kinetik energiyasi bilan bog'laylik: M_0 massali zarra ikkita bir xil zarra parchalanib ketganligi uchun

$$m = \frac{M_0}{2} = m_0 + \frac{\square m}{2} = m_0 + \frac{m_0 v^2}{2c^2} = m_0 + \frac{T}{c^2}$$

Shunday qilib, harakat davomida zarra massaning «ortib», borishini, norelyativistik yaqinlashishda zarraning kinetik energiyasi bilan bog'ladik.

Xuddi shu natijani «relyativistik massa»ning ifodasidan $\frac{v^2}{c^2} \ll 1$ yaqinlashish

holida ham keltirib chiqarish mumkin

$$m = m_0 \gamma \approx m_0 \left(1 + \frac{v^2}{2c^2}\right) = m_0 + \frac{T}{c^2}.$$

6-masala. Impulsi \vec{p}_k bo'lgan K^0 -mezon uchib borayotib (laboratoriya sistemasi) ikkita π mezonga parchalandi: $K^0 \rightarrow 2\pi$. K -mezon tinch turgan sistemada π -mezonning impulsi \vec{p}_π^* va u K -mezonning yo'nalishiga nisbatan

θ_π^* burchak ostida otilib chiqqan deylik. Shu π -mezonning laboratoriya sistemasidagi energiyasi va og'ish burchaklari topilsin.

Yechilishi: Avvalombor, kaonning tinch turgan sistemasidan ($\vec{p}_k = 0$) laboratoriya sistemasiga (ya'ni \vec{p}_k impulsiga ega kaonning parchalanayotgan sistemasiga, $\vec{p}_k = 0$ sanoq sistemasi emas!) o'tgandagi energiya va impulsni topib olaylik. (5.34) formuladan:

$$E_k = m_k c^2 \gamma, \quad \vec{p}_k = m_k \vec{v}_k \gamma$$

Bu yerdan

$$\gamma = \frac{E_k}{m_k c^2}, \quad \vec{v}_k \gamma = \frac{\vec{p}_k}{m_k} \quad (5.42)$$

Yuqoridagi (5.37) formulani quyidagi ko'rinishda yozib olish mumkin:

$$E = \gamma(E' + V \vec{p}'_{\parallel}), \quad \vec{p}_{\parallel} = \gamma\left(\vec{p}'_{\parallel} + \frac{V E'}{c^2}\right), \quad \vec{p}_{\perp} = \vec{p}'_{\perp} \quad (5.43)$$

Biz bu yerda impulsni bo'ylama va ko'ndalang komponentalarga yoydik:

$$\vec{p} = \vec{p}_{\parallel} + \vec{p}_{\perp}$$

Pionning energiyasi va impulsi kaon tinch turgan sistemada berilgan, ya'ni \vec{v}_k tezlik bilan harakatlanib ketayotgan sanoq sistemasida ((5.43) formuladagi (') -shtrixli) sistemaga mos keladi).

Demak (5.43) dan foydalansak:

$$E_\pi^* = \gamma(E_\pi + \vec{v}_k \vec{p}_{\pi\parallel}), \quad p_{\pi\parallel}^* = \gamma\left(\vec{p}_{\pi\parallel} + \vec{v}_k \frac{E_\pi}{c^2}\right) \quad (5.44)$$

(5.44) ga (5.42)ni olib kelib qo'ysak

$$E_\pi^* = \frac{1}{m_k c^2} (E_k E_\pi + \vec{p}_k \vec{p}_{\pi\parallel} c^2),$$

$$p_{\pi\parallel}^* = \frac{1}{m_k c^2} (\vec{p}_\pi E_k + E_\pi \vec{p}_k), \quad \vec{p}_{\pi\perp}^* = \vec{p}_{\pi\perp} \quad (5.45)$$

Agarda

$$p_{\pi\parallel} = p_\pi \cos \theta_\pi, \quad p_{\pi\perp} = p_\pi \sin \theta_\pi, \quad E_\pi = c \sqrt{m_\pi^2 c^2 + p_\pi^2}$$

belgilashlar kiritsak, (5.45)ni quyidagicha yozib olish mumkin:

$$E_x^* = \frac{1}{m_k c^2} (E_k E_x + c^2 p_x p_k \cos \theta_x)$$

$$p_{x1}^* = p_x \cos \theta_x^* = \frac{1}{m_k c^2} (E_k p_x \cos \theta_x + p_k E_x)$$

$$p_{x\perp}^* = p_x^* \sin \theta_x^* = p_x \sin \theta_x \quad (5.46)$$

Bular izlanayotgan E_x^* va p_x^* ning ifodalarini beradi. Bu ifoda, harakatlanib turib parchalanayotgan sistema uchun umumiy ifodadir, uni ikkita ketma-ket Lorens almashtirishlari orqali topdik. (5.46) dan parchalanish burchaklari (ya'ni yo'nalishning o'zgarishi) ni ham topsa bo'ladi:

$$\operatorname{tg} \theta_x^* = \frac{m_k c^2 p_x \sin \theta_x}{E_k p_x \cos \theta_x + p_k E_x} \quad (5.47)$$

7-masala. Serpuxovdagi (Rossiya) tezlatgichda protonlarning $E = 76 \text{ GeV}$ energiyali dastasini hosil qilindi. Qo'zg'almas nishonga urilgan protonlar dastasi har xil reaksiyalar sodir qiladi, yangi zarrachalar «tug'iladi», shu jumladan protonlar va antiprotonlar. Agar protonlar va antiprotonlar massasi $E = 0,94 \text{ GeV} / c^2$ bo'lsa, bir to'qnashish davomida ulardan qanchasi hosil bo'ladi?

Yechilishi. Protonlar dastasi odatda nishonning protonlari bilan to'qnashadi. Energiya va impulsning saqlanish qonunini yozaylik:

$$E + mc^2 = E_1 + E_2 + \dots, \quad \vec{p} + \vec{0} = \vec{p}_1 + \vec{p}_2 + \dots$$

Bu ifodaning o'ng tomonida yangi hosil bo'lgan protonlarning energiyasi va impuls. Bir qarashda, birinchi tenglikdan, hosil bo'lgan protonlarning sonini topsa bo'ladigandek (agarda hammasi proton va ularning barchasi tinch turibdi desak, ularning energiyasi $n \cdot mc^2$ ga teng

$$n = \frac{E + mc^2}{mc^2} = 81$$

Lekin, hech qachon shuncha proton xosil bo'lmaydi, chunki bu zarrachalar ikkinchi tenglikka mos impulsiga ega bo'lishi kerak. Bunday to'qnashish masalasini, ko'pincha parchalanish masalasiga keltirib olish qulay. Tezlatilgan proton nishon protoniga urilgach qandaydir massasi M ga teng «zarra» hosil bo'ladi deylik, uning qanday hosil bo'lgani kinematikani qiziqirmaydi, yuqoridagi tengliklarning chap tomoni qanotlantirilsa bas, so'ngra u protonlarga parchalanadi. Bu «fiktiv» zarraning massasi M albatta, tenglikning chap tomonidagi kattaliklardan hosil bo'lgan chunki M zarra tinch turgan sanoq sistemasida uning to'rt impuls $p_\mu = (Mc, \vec{0})$ ya'ni, energiyasi $E + mc^2$ va impuls \vec{p} ga teng to'rt impuls orqali ifodalanadi va invariantdir:

$$M^2 c^4 = (E + mc^2)^2 - (c\vec{p})^2 = 2mc^2 (E + mc^2).$$

Biz bu yerda proton uchun o'rinli bo'lgan $E^2 = (\vec{p}c)^2 + m^2 c^4$ tenglikdan foydalandik. Demak, parchalanayotgan «fiktiv» zarraning massasi (effektiv massa) tezlatilgan protonning energiyasiga bog'liq bo'ladi:

$$M = \frac{1}{c} \sqrt{2m(E + mc^2)} \quad (5.48)$$

ya'ni, $M \cong E^{\frac{1}{2}}$. Real zarralarning tinch xolatdagi parchalanish protsessi bilan «massasi» (5.48) formula orqali aniqlangan zarrachaning parchalanishi orasida prinsipial farq bor. Birinchi protsessda hosil bo'lgan zarralar massasining yig'indisi parchalanayotgannikidan katta bo'lishi mumkin emas. Ikkinchi protsessda esa, «fiktiv» zarraning massasi tezlatilgan zarraning energiyasiga bog'liq ravishda ortib boradi. Bu fiktiv zarra uzog'i bilan nechta protonga parchalanishi mumkin ekanligini topish mumkin:

$$\frac{M}{m} = \sqrt{\frac{2(E + mc^2)}{mc^2}} = \sqrt{\frac{2(76 + 0,94)}{0,94}} \cong 12$$

ya'ni, 12 ta proton va antiprotonlar hosil bo'lishi mumkin ekan.

8-masala. Uchrashuvchi protonlar tezlatgichida $E = 70 GeV$ energiyali dasta hosil qilinsa, bu tezlatgichning effektivligi qo'zg'almas nishonli oddiy tezlatgichning qanday energiyali dastalar oqiminikiga mos keladi?

Yechilishi. Ma'lumki, uchrashuvchi tezlatgichlarda (qarama-qarshi yo'nalgan dastalar tezlatgichi) har bir dasta energiyasi asosan yangi zarrachalar hosil qilish uchun ketadi. Oddiy tezlatgichlarda esa, dasta energiyasining ko'p qismi parchalanuvchi «fiktiv» zarraning yalpi xoldagi harakatini ta'minlashga ketadi. Uchrashuvchi dastalar to'qnashganda (bir xil energiyali va bir xil massali zarrachalar dastasi) inersiya markazi sistemasida dastalarning tula impulsi nolga teng:

$$\vec{P} = \vec{p}_1 + \vec{p}_2 = 0, \quad \vec{p}_1 = -\vec{p}_2 = \vec{p}$$

Bu yerda indekslar dastalarga tegishli. Yana, 7-masalaga o'xshash «fiktiv» zarra parchalanayotibdi desak, uning impulsi nolga teng, massasi esa, dastalar energiyasining yig'indisi orqali anqlanadi:

$$\vec{P} = \vec{p}_1 + \vec{p}_2, \quad Mc^2 = E_{01} + E_{02}$$

Demak,

$$M = \frac{2E_0}{c^2} \cong 140 \frac{Gev}{c^2} \quad (5.49)$$

Yuqoridagi 7-masaladan ma'lumki, shunday «fiktiv» zarraning massasi oddiy tezlatgich energiyasi bilan quyidagicha bog'lanishga ega

$$M^2 = 2m \left(m + \frac{E}{c^2} \right)$$

Bu ifoda va (5.49) formuladan foydalansak:

$$E = \left(\frac{M^2}{2m} - m \right) c^2 = \left(\frac{1}{2m} \frac{4E_0^2}{c^4} - m \right) c^2 = \frac{2E_0^2}{mc^2} - mc^2$$

Protonlar uchun $E_0^2 \gg mc^2 \cong 0,94 GeV$ ekanligini xisobga olsak:

$$E \cong \frac{2E_0^2}{mc^2} \cong 10000 GeV$$

Shunday qilib, har bir dastaning energiyasi $E = 70 GeV$ lik uchrashuvchi tezlatgichning effektivligi, $E = 10^4 GeV$ lik oddiy tezlatgichning effektivligiga to'g'ri kelar ekan. (5.49) formula bilan berilgan «effektiv massa» yangi zarralar «tug'dirish» uchun va ularga kinetik energiya berishga sarf bo'lishi mumkin, shu jumladan, protondan yuzlab marta «og'ir» zarrachalar hosil bo'lishi mumkin. Lekin, hosil bo'lgan zarrachalarning umumiy massasi M dan «og'ir» bo'lishi mumkin emas. Agar uchrashuvchi tezlatgichda proton va antiprotonlar emas, balki elektron va pozitronlar shu energiyagacha tezlantirilsa, elektronning massasi protonnikidan $\cong 2000$ marta kichikligi sababli, ularning effektivligi yanada yuqori bo'ladi - $E \cong 2 \cdot 10^7 GeV$.

9-masala. Massasi m energiyasi E ga teng neytral π^0 -mezon ikki E_1 va E_2 energiyali fotonga parchalanadi. Laboratoriya sistemasida pionning tezligi v bo'lsa, fotonlar orasidagi burchak qanday o'zgaradi? Laboratoriya sistemasida foton pionning yo'nalishiga θ burchak ostida uchib chiqadi deb $d\Omega$ fazoviy burchakda otilib chiqish ehtimolligi $P(\theta) d\Omega$ topilsin.

Yechilishi. Avval energiya va impuls saqlanish qonunidan fotonlar orasidagi burchak uzgarishini topaylik. Ma'lumki,

$$m^2 c^4 = E^2 - (c\vec{p})^2$$

- invariant kattalikdir. Saqlanish qonunidan:

$$E = E_1 + E_2, \quad \vec{p} = \vec{p}_1 + \vec{p}_2, \quad m^2 c^4 = (E_1 - E_2)^2 - c^2 (\vec{p}_1 + \vec{p}_2)^2.$$

Lekin hosil bo'lgan fotonlarning tinch holatdagi massasi nolga teng va,

$$\vec{p}_1 = \frac{E_1}{c}, \quad \vec{p}_2 = \frac{E_2}{c}.$$

Shuning uchun

$$m^2 c^4 = E_1^2 + E_2^2 + 2E_1 E_2 - c^2 (\vec{p}_1^2 + \vec{p}_2^2 + 2|\vec{p}_1||\vec{p}_2|\cos\alpha) = E_1^2 + E_2^2 + 2E_1 E_2 -$$

$$-c^2 \left(\frac{E_1^2}{c^2} + \frac{E_2^2}{c^2} + 2 \frac{E_1 E_2}{c^2} \cos \alpha \right) = 2E_1 E_2 (1 - \cos \alpha) = 4E_1 E_2 \sin^2 \left(\frac{\alpha}{2} \right).$$

Bu yerda α -fotonlar orasidagi burchak. Oxirgi ifodadan α_{\min} ni topish mumkin. Uning uchun masalan E_2 ni pionning energiyasi E orqali yozib olaylik:

$$E_2 = E - E_1 \equiv E - x \quad \text{va} \quad 4x(E-x) \sin^2 \left(\frac{\alpha}{2} \right) = m^2 c^4.$$

Agar $y = 4x(E-x)$ deb belgilab olsak, bu ifodaning maksimum qiymati $y = E^2$ ga teng va $x = \frac{E}{2}$ da unga erishadi. Ammo $y \cdot \sin^2 \left(\frac{\alpha}{2} \right) = m^2 c^4$ -

doimiydir. Demak y_{\max} qiymatida $\sin^2 \left(\frac{\alpha}{2} \right)$ minimumga erishishi kerak.

Shuning uchun $E^2 \sin^2 \left(\frac{\alpha_{\min}}{2} \right) = m^2 c^4$.

Endi, masalaning ikkinchi qismini ko'raylik. Biror burchakka sochilish ehtimolligi saqlanuvchi kattalik:

$$p(\theta) d\Omega = p'(\theta) d\Omega' \quad \text{yoki} \quad p(\theta) = p'(\theta) \frac{d \cos \theta'}{d \cos \theta}.$$

(5.46) formuladan laboratoriya va tinch holat sistemalarida sochilish burchaklarini topsa bo'ladi (faqat kaon o'rnida pion, pion o'rnida esa, foton deb hisoblash kerak).

By erda (5.41) formuladan foydalansak:

$$\operatorname{tg} \theta = \frac{1}{\gamma} \frac{c \sin \theta'}{c \cos \theta' + v} \quad (5.50)$$

Bu ifoda massasi nolra teng zarrachalarga parchalanishdagi umumiy ifodadir. Xuddi shuningdek, ko'pincha (5.45) formulalardan kelib chiqadigan quyidagi ifoda ishlatiladi:

$$\cos \theta = \frac{c \cos \theta' + v}{c + v \cos \theta'} \quad (5.51)$$

Harakatlanayotgan sanoq sistemasida yozib olish uchun almashtirish bajarish yetarli:

$$\operatorname{tg} \theta' = \frac{1}{\gamma} \frac{c \sin \theta}{c \cos \theta - v}, \quad \cos \theta' = \frac{c \cos \theta - v}{c - v \cos \theta} \quad (5.52)$$

(5.52) ifodalardan

$$P(\theta) = P'(\theta) \frac{d \cos \theta'}{d \cos \theta} = P'(\theta) \frac{1}{\gamma^2 \left(1 - \frac{v}{c} \cos \theta \right)^2} = \frac{1}{4\pi\gamma^2} \frac{1}{\left(1 - \frac{v}{c} \cos \theta \right)^2}.$$

Chunki, pion tinch turgan sistemada parchalanish izotropdir, va ehtimollik.

$$P'(\theta) = \frac{1}{4\pi}.$$

10-masala. Elastik to'qnashayotgan ikki zarra masalasidagi kattaliklarni relyativistik invariantlar orqali yozing.

Yechilishi: (5.39) ifoda bilan berilgan ikki zarraning elastik to'qnashish masalasidagi kinematik munosabatlarni topishda har qanday sanoq sistemasida o'rinni bo'lgan to'rt impulsning saqlanish qonunidan kelib chiqamiz. Avval, shu masala doirasida $c=1$ deb hisoblashni kelishib olaylik. Bunday kelishuv ko'p kinematik kattaliklarni topishda ancha noqulayliklar yaratadigan va doimo kasr tariqasida o'ziladigan kattaliklarni sodda ko'rinishga keltiradi. Bu holda massa, energiya va impulsning o'lchamligi bir xil- GeV larda. Odatdagi o'lchamliklarga o'tish uchun kerakli yerda yorug'lik tezligini yozib qo'yish yetarlidir.

Demak $p^\mu p_\mu = p^0 p_0 + p^i p_i = E^2 - \vec{p}^2 = m^2, \quad i=1,2,3.$

Odatda quyidagi uchta relyativistik parametrlar qo'llaniladi:

Boshlang'ich holatdagi to'rt impuls kvadrati,

$$s = (p_a^\mu + p_b^\mu)(p_{a\mu} + p_{b\mu}) \quad (5.53)$$

Elastik to'qnashuvda «c» zarraga uzatilgan to'rt impuls kvadrati:

$$t = (p_a^\mu - p_c^\mu)(p_{a\mu} - p_{c\mu}) \quad (5.54)$$

- "d" zarraga berilgan to'rt impuls kvadrati:

$$u = (p_a^\mu - p_d^\mu)(p_{a\mu} - p_{d\mu}) \quad (5.55)$$

Bu kattaliklar relyativistik invariant kattaliklardir, chunki to'rt impuls kvadratlaridan iborat bo'lgan kattaliklardir va ular o'zaro quyidagi munosabat orqali bog'langandirlar:

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2 \quad (5.56)$$

Keling, S-sistemada kinematik kattaliklarni invariant parametrlar orqali yozib olaylik («0» indeks S-sistemaga tegishlilikini ko'rsatadi). Avval energiyani S orqali yozib olaylik :

$$\vec{p}_{a0} = -\vec{p}_{b0}, \quad \vec{p}_{c0} = -\vec{p}_{d0}, \quad p_{a0}^\mu + p_{b0}^\mu = (E_{a0} + E_{b0}, \vec{O}),$$

$$p_{c0}^\mu + p_{d0}^\mu = (E_{c0} + E_{d0}, \vec{O}), \quad s = (E_{a0} + E_{b0}, \vec{O})^2 = (E_{c0} + E_{d0}, \vec{O})^2.$$

Ochirgi ifodada $E = \sqrt{p^2 + m^2}$ ekanligin hisobga olib, tenglikni impulsga nisbatan yechib chiqsak:

$$\vec{p}_{a0}^2 = \vec{p}_{b0}^2 = \frac{\lambda(S, m_a^2, m_b^2)}{4S}, \quad \vec{p}_{c0}^2 = \vec{p}_{d0}^2 = \frac{\lambda(S, m_c^2, m_d^2)}{4S} \quad (5.57)$$

Bu erda λ - kinematik funksiya:

$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2xz = (x + y - z)^2 - 4xy = (x + y - z)^2 - 4xz$$

(5.58)

Energiyaning ifodasi endi qulaygina yoziladi:

$$E_{a0} = \frac{S + m_a^2 - m_b^2}{2\sqrt{S}}, \quad E_{b0} = \frac{S + m_b^2 - m_a^2}{2\sqrt{S}} \quad (5.59)$$

Simmetrik ravishda $a \rightarrow c$, $b \rightarrow d$, almashtirib E_{c0} va E_{d0} ni topish mumkin.

(5.58) ifodadan t -invariantni sochilish burchagi χ bilan bog'lash mumkin:

$$t = m_a^2 + m_b^2 - 2E_{a0}E_{b0} + 2|\vec{p}_{a0}||\vec{p}_{b0}|\cos\chi \quad (5.60)$$

5.3-Misollar.

1. Tinch turgan π -mezon myuon va neytrinoga shunday parchalanadiki, myuon tezlikka ega bo'ladi. Agar tezlalgichda tezligi $v_\pi = (1 - 5 \cdot 10^{-5}) \cdot c$ ga teng pionlar dastasi hosil qilingan bo'lsa, qo'zg'almas hisoblagich qanday tezlikli myuonlarni «utib» qoladi. Ularning γ - faktori qanday?

2. Tinch holatda turgan m_2 zarrachaga m_1 zarracha kelib tushayapti, natijada massasi $M > m_1 + m_2$ bo'lgan zarrachalar guruhi hosil bo'ldi. Shu reaksiya sodir bo'lishi uchun m_1 zarrachaning yetarli bo'lgan minimal kinetik energiyasi topilsin.

3. $K^0 \rightarrow 2\pi$ parchalanishdagi pionning impulsi va energiyasi laboratoriya sistemasida $\vec{p}_\pi^*, \omega_\pi^*$ ga, K -mezoning impulsi va energiyasi esa \vec{p}_k^*, ω_k^* ga teng. K -mezon tinch holatda bo'lgan sistemada pionning impulsi va energiyasi qanday? Ko'rsatma: 6-masaladan foydalaning.

4. Zarrachaning tinch holatdagi massasi m bo'lsa, tezligi \vec{v} ni uning a) to'la energiyasi, b) kinetik energiyasi, v) impulsi orqali ifodasini toping.

5. Tinch holatdagi π -mezon $\pi \rightarrow \mu + \nu$ reaksiya ko'rinishida parchalanadi. μ -mezonning kinetik energiyasi topilsin.

6. Tashqi maydon ta'sirisiz foton elektron va pozitron juftiga aylana olmasligi ko'rsatilsin (4 impuls saqlanish qonunidan foydalaning $p_-^2 = p_+^2 = m^2 c^2$).

7. 9-masalada, $E_1 = E_2$ deb, fotonlar orasidagi burchak topilsin.

8. Yuqori energiyali E_γ foton tinch holatda turgan M bo'lgan yadro bilan to'qnashib neytral pion π^0 hosil qiladi ("foto-paydo bo'lish"). Fotonning yo'nalishiga perpendikulyar ravishda sochilgan pionlarning energiyasini toping.

9. L-sistemada ($\vec{p}_0 = 0$) ikki zarraning elastik to'qnashuvidagi zarralar energiyasining ifodasi relyativistik invariantlar orqali ifodalansin.

10. Ikki zarraning elastik to'qnashuvida ($m_i = m_{i'}$, $m_{i'} = m_{i'}$) S-sistemasidagi sochilish burchagi χ ni:

a) invariantlarning L-sistemada hisoblangan qiymatlari ((5.60) formuladan foydalanib),

b) E_a', E_b' va E_a bilan bog'lang.

VI. QATTIQ JISM HARAKATI.

Mexanikada *qattiq jism* deb harakat davomida oralaridagi masofalar o'zgaraydigan moddiy nuqtalar sistemasi tushiniladi (dattiq jism modeli). Qattiq jismni moddiy nuqtalar sistemasi deb ko'rish ba'zi masalalarni yechishni osonlashtiradi. Qattiq jismni tutash muhit (uzluksiz) deb qarash ham mumkin. Bu ikki xil qarash bir-biriga zid bo'lmasdan, balki biridan ikkinchisiga osongina o'tish mumkin. Xususan, qattiq jismni moddiy nuqtalar sistemasi degan mulohazadan tutash muhit degan mulohazaga o'tish uchun, qattiq jismning massasini moddiy nuqtalar massalari yig'indisiga teng deb qarashdan ρ zichlikli uzluksiz moddaning dV hajm bo'yicha olingan integraliga teng deb o'tish kifoyadir. Qattiq jism harakatini ifodalash uchun odatda "qo'zg'almas"- X, Y, Z va "harakatlanuvchi" - x_1, x_2, x_3 sanoq sistemalaridan foydalanamiz (belgilanishiga e'tibor bering, bosh harflar "qo'zg'almas" uchun). Harakatdagi koordinatalar sistemasi ("harakatlanuvchi") boshini *inersiya markazida* tanlab olish hisoblashlarda ko'p qulayliklarga olib keladi. Ixtiyoriy qattiq jismning erkinlik darajasi $3N - \frac{1}{2}N(N-1)$ ga teng. Odatda qattiq jism modeli bu bir-biri bilan qattiq birikkan 3 ta zarradir $N=3$, ya'ni $S=6$. Bu olti parametrning rolini *inersiya markazi radius vektori komponentalari* va uchta *burilish burchaklari* o'ynashi mumkin.

Qattiq jism ixtiyoriy nuqtasining tezligini quyidagicha ifodalash mumkin:

$$\vec{v} = \vec{V} + [\vec{\Omega}, \vec{r}].$$

(6.1)

Bunda \vec{r} -nuqtaning «qo,zg,aluvchan» sistemadagi radius vektori,
 \vec{v} - qo,zg,almas sanoq sistemasiga nisbatan shu nuqtaning tezligi,
 \vec{V} - inersiya markazining tezligi, yoki ilgari lanma harakat tezligi,
 $\vec{\Omega}$ - qattiq jism biror nuqtasi aylanishining burchak tezligi.

Qattiq jism kinetik energiyasi: $T = \sum \frac{mv^2}{2}$

Yig'indi qattiq jism tarkibidagi hamma moddiy nuqtalar bo'yicha olinadi. Bu ifodaga tezlik ifodasini (6.1) qo'y sak ushbu ko'rinishni olamiz:

$$T = \frac{\mu V^2}{2} + \frac{1}{2} \sum m \left\{ \vec{\Omega}^2 \vec{r}^2 - (\vec{\Omega}, \vec{r})^2 \right\} = \frac{\mu V^2}{2} + \frac{1}{2} I_{ik} \Omega_i \Omega_k \quad (6.2)$$

hosil bo'ladi. Bunda μ - qattiq jism massasi, I_{ik} - inersiya momenti (inersiya tenzori).

Bu ifodadagi birinchi had qattiq jismning ilgarilanma harakat energiyasi, ikkinchisi esa aylanma harakat energiyasi bo'lib, uni quyidagicha tasavvur qilish mumkin (bu tenzor ifodalardagi yig'indi alomati qaytariluvchi indeksarga tegishli ekanini unutmang):

$$T_{ayl} = \frac{1}{2} I_{ik} \Omega_i \Omega_k, \quad I_{ik} = \sum m (x_i^2 \delta_{ik} - x_i x_k). \quad (6.3)$$

Qattiq jism Lagranj funksiyasi esa, quyidagi ko'rinishga keladi:

$$L = \frac{\mu V^2}{2} + \frac{1}{2} I_{ik} \Omega_i \Omega_k - U$$

U - potensial energiya. I_{ik} - inersiya tenzori yoki inersiya momenti deyiladi:

$$I_{ik} = \begin{bmatrix} \sum m(y^2 + z^2) & -\sum mxy & -\sum mzx \\ -\sum myx & \sum m(x^2 + z^2) & -\sum myz \\ -\sum mzx & -\sum mzy & \sum m(x^2 + y^2) \end{bmatrix} \quad (6.4)$$

Ko'rinib turibdiki, I_{ik} simmetrik tenzor. Inersiya tenzorining bu an'anaviy ko'rinishida $x_1 = x, x_2 = y, x_3 = z$ deb olingan. Agar qattiq jismni tutash muhit desak:

$$I_{ik} = \int \rho (x_i^2 \delta_{ik} - x_i x_k) dV. \quad (6.5)$$

Inersiya tenzorini diagonal ko'rinishga keltirish mumkin. Bu holga mos keluvchi o'qlar x_1, x_2, x_3 inersiya bosh o'qlari, I ning komponentalari esa, tenzorning diagonal elementlari (bosh inersiya momentlari) deyiladi, jumladan

$$T_{ami} = \frac{1}{2} (I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2).$$

Shuni ta'kidlash zarurki $I_1 + I_2 \geq I_3$

$$I_1 + I_2 = \sum m(x_1^2 + x_2^2 + 2x_3^2) \geq \sum m(x_1^2 + x_2^2) = I_3 \quad (6.6)$$

Agarda jismning inersiya momentlari $I_1 \neq I_2 \neq I_3$ bo'lsa - assimetrik pirildoq, $I_1 = I_2 \neq I_3$ bo'lsa - simmetrik pirildoq, $I_1 = I_2 = I_3$ bo'lsa - sharsimon pirildoq deb ataladi. Bordiyu, qattiq jism qandaydir simmetriyaga ega bo'lsa bosh o'qlarni topish yengillashadi, inersiya markazi vaziyati va bosh o'qlar ham shu simmetriyaga ega bo'lishi kerak. Masalan, jism tekislik simmetriyasiga ega bo'lsa, inersiya markazi shu tekislikda joylashadi, hamda 2 ta bosh o'q shu tekislikda (x_1, x_2) yotadi, uchinchi esa bu tekislikka perpendikulyardir.

1. Tekislikda joylashgan zarrachalar (moddiy nuqtalar) uchun $x_3 = 0$

$$I_1 = \sum mx_2^2; \quad I_2 = \sum mx_1^2; \quad I_3 = \sum m(x_1^2 + x_2^2), \quad I_3 = I_1 + I_2 \quad (6.7)$$

Ba'zi praktik hollarda, inersiya tenzorini inersiya markazida joylashmagan koordinat sistemalarida hisoblash qulayroq bo'lishi mumkin:

$$I'_{ik} = \sum m (x_i'^2 \delta_{ik} - x_i' x_k'), \quad x_i = x_i' + a_i.$$

U holda

$$I'_{ik} = I_{ik} + \mu (a_i^2 \delta_{ik} - a_i a_k). \quad (6.8)$$

2. To'g'ri chiziqda joylashgan moddiy nuqtalar sistemasi (sterjen), deylik x_3 o'qi bo'yicha joylashgan bo'lsin, bunday sistemani rotator deyiladi. U holda

$$x_1 = x_2 = 0, \quad I_1 = I_2 = \sum mx_3^2, \quad I_3 = 0 \quad (6.9)$$

Inersiya markazida joylashgan qo.zg.aluvchan sanoq sistemasida qattiq jismning "xususiy" momenti qoladi xolos :

$$\vec{M} = \sum [\vec{r}\vec{p}] \Rightarrow \sum m [\vec{r} [\vec{\Omega}\vec{r}]] = \sum m \{ \vec{r}^2 \vec{\Omega} - \vec{r}(\vec{r}\vec{\Omega}) \} \quad (6.10)$$

yoki

$$M_i = \sum m \{ x_c^2 \Omega_i - x_i x_k \Omega_k \} = \Omega_k \sum m \{ x_c^2 \delta_{ik} - x_i x_k \} = I_{ik} \Omega_k, \quad (6.11)$$

Agar x_1, x_2, x_3 bosh o'qlar bo'lsa bu formulani quyidagicha yozib olish mumkin:

$$M_1 = I_1 \Omega_1, \quad M_2 = I_2 \Omega_2, \quad M_3 = I_3 \Omega_3 \quad (6.12)$$

Xususan sharsimon pirildoq uchun

$$I_1 = I_2 = I_3, \quad \vec{M} = I \vec{\Omega} \quad (6.13)$$

Qattiq jism oltita erkinlik darajasiga ega bo'lganligi uchun, uning harakat tenglamalari soni ham oltiga teng bo'lishi kerak. Bular

$$\frac{d\vec{P}}{dt} = \vec{F}, \quad \frac{d\vec{M}}{dt} = \vec{K} \quad (6.14)$$

Bu yerdagi \vec{P} -qattiq jism impuls, \vec{F} -qattiq jismga ta'sir etayotgan kuch, \vec{M} - qattiq jism impuls momenti, $\vec{K} = \sum [\vec{r}\vec{f}]$ - qattiq jism qismlarining kuch momentlari yig'indisi.

Biz shu vaqtgacha inersial sanoq sistemalaridagi harakat haqida mulohaza yuritdik. Agar ilgariharakat (inersiya markazi) tezligi vaqtga bog'liq holda o'zgaruvchi bo'lsa, Lagranj funksiyasi va tenglamasi mos ravishda quyidagi ko'rinishni oladi:

$$L = \frac{1}{2} m \vec{v}^2 - m \vec{W}(t) \vec{r} - U, \quad m \frac{d\vec{v}}{dt} = - \frac{\partial U}{\partial \vec{r}} - m \vec{W}(t). \quad (6.15)$$

Bunda $\vec{W}(t) = \frac{d\vec{V}}{dt}$ - ilgariharakat tezlanishi.

Agar ilgariharakat tezligi bilan birga burchak tezlik ham vaqtga bog'liq bo'lsa, ular mos ravishda ushbu ko'rinishni oladi:

$$L = \frac{1}{2} m \vec{v}^2 + m \vec{v} [\vec{\Omega} \vec{r}] + \frac{m}{2} [\vec{\Omega} \vec{r}]^2 - m \vec{W} \vec{r} - U;$$

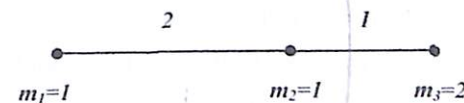
$$m \frac{d\vec{v}}{dt} = - \frac{\partial U}{\partial \vec{r}} - m \vec{W} + m [\vec{r} \vec{\Omega}] + 2m [\vec{v} \vec{\Omega}] + m [\vec{\Omega} [\vec{r} \vec{\Omega}]]. \quad (6.16)$$

Keyingi ifodadagi 4-had *Koriolis kuchi*, oxirgi had esa, *markazdan qochma kuchdir*.

§ 6.1 Inersiya tenzorini hisoblash

1-masala. Ushbu 6.1- rasmdagi sistemaning bosh inersiya momentlarini toping.

Yechilishi. Moddiy nuqtalar bir to'g'ri chiziq bo'ylab joylashgani uchun ularning shu o'q bo'yicha koordinatasini topamiz (bu o'q sifatida x o'qini olaylik, u holda $y_i = 0, z_i = 0$).



6.1.-rasm

Dastlab koordinata boshi 1-nuqtaga joylashgan deb ularning vaziyatini aniqlab olaylik : $x'_1 = 0, x'_2 = 2, x'_3 = 3$.

Inersiya markazi koordinatasini x_0 deb belgilaylik va uni shu tanlangan koordinat sistemadagi qiymatini hisoblaylik :

$$x_0 = \frac{m_1 x'_1 + m_2 x'_2 + m_3 x'_3}{m_1 + m_2 + m_3} = \frac{1 \cdot x'_1 + 1 \cdot x'_2 + 2 \cdot x'_3}{1 + 1 + 2} = 2$$

Demak, bu sistemaning inersiya markazi 2- nuqtada yotar ekan. Endi koordinata boshini inersiya markaziga ko'chirsak va simmetriyani hisobga olsak, moddiy

nuqtalarning inersiya markazida joylashgan sanoq sistemasiga nisbatan koordinatalari quyidagicha bo'ladi: $x_1 = -2$, $x_2 = 0$, $x_3 = 1$.

$$I_{11} = \sum m_a (y_a^2 + z_a^2) = 0, \quad I_{22} = \sum m_a (x_a^2 + z_a^2) = 6,$$

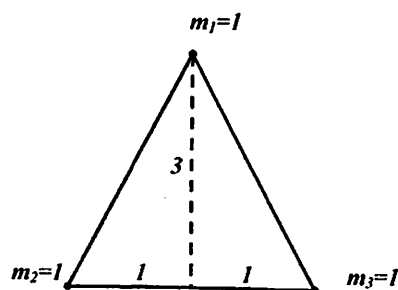
$$I_{33} = \sum m_a (y_a^2 + x_a^2) = 6.$$

(6.4) ifodaga mos ravishda hisoblangan bu kattaliklar, tabiiyki diagonal tenzorning elementlaridir (bosh inersiya momentlari).

2-masala. Ushbu 6.2- rasmdagi sistemaning bosh inersiya momentlarini toping.

Yechilishi. Bu gal nuqtalarning inersiya markazida joylashgan koordinat sistemasiga nisbatan koordinatalarini to'g'ridan-to'g'ri topaylik. Buning uchun x o'qi uchburchakning asosi, y o'qi esa uchburchak balandligi bo'ylab yo'nalgan koordinata sistemasini inersiya markazida joylashgan deyluk (z o'qi tekislikka perpendikulyar), yani koordinata boshi balandlik ustida yotadi. U holda ushbu tengliklar o'rinlidir (chunki koordinata boshi inersiya markazida joylashgan):

$$\vec{R} = \sum m_i \vec{r}_i = 0 \Rightarrow \sum m_i x_i = \sum m_i y_i = \sum m_i z_i = 0.$$



6.2.-rasm

Yuqoridagi ifodalardan va chizmadan ko'rinib turibdiki, quyidagi tengliklar o'rinli:

$$1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 = 0, \quad 1 \cdot y_1 + 1 \cdot y_2 + 1 \cdot y_3 = 0, \quad z_1 = z_2 = z_3 = 0$$

$$x_1 - x_2 = 1, \quad x_3 - x_1 = 1, \quad y_1 - y_2 = y_1 - y_3 = 3.$$

Bu tengliklarni yechib, nuqtalarning inersiya markaziga nisbatan koordinatalarini topamiz:

$$x_1 = 0, y_1 = 2, z_1 = 0, \quad x_2 = -1, y_2 = -1, z_2 = 0, \quad x_3 = 1, y_3 = -1, z_3 = 0.$$

Bulardan foydalanib bosh inersiya momentlarini hisoblab topamiz:

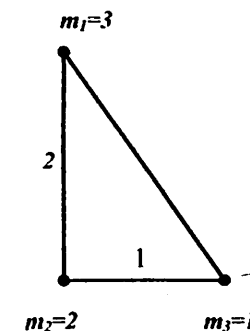
$$I_{11} = I_{xx} = 6, \quad I_{22} = I_{yy} = 2, \quad I_{33} = I_{zz} = 8.$$

Kutilganidek, bunday tanlab olingan koordinat sistemi o'qlari bosh o'qlardir, bunga nodiagonal elementlarning nolga teng ekanligini hisoblab ko'rib, ishonch hosil qilish mumkin:

$$I_{xy} = I_{xz} = I_{yz} = 0.$$

3-masala. Ushbu 6.3- rasmdagi sistemaning bosh inersiya momentlarini toping.

Yechilishi. Oldingi masaladagidek, x va y o'qlarini uchburchak katetlari bo'ylab yo'nalgan koordinata sistemasini inersiya markazida joylashgan deyluk (z o'qi tekislikka perpendikulyar).



6.3 rasm

Koordinata boshi inersiya markazida joylashganligi sababli va chizmadagi holatga ko'ra ushbu tengliklar o'rinlidir :

$$\sum m_i x_i = 0 \Rightarrow 3x_1 + 2x_2 + x_3 = 0, \quad x_3 - x_2 = 1,$$

$$\sum m_i y_i = 0 \Rightarrow 3y_1 + 2y_2 + y_3 = 0, \quad y_1 - y_2 = y_1 - y_3 = 2.$$

Natijada mazkur (яъни, инерция марказида жойлашган) koordinat sistemasida zarralarning (nuqtalarning) koordinatasini aniqlaymiz:

$$x_1 = x_2 = -\frac{1}{6}, \quad x_3 = \frac{5}{6}, \quad y_1 = 1, \quad y_2 = y_3 = -1, \quad z_1 = z_2 = z_3 = 0.$$

Inersiya tenzorining komponentalarining (6.4) ifodadagi kattaliklarini hisoblasak quidagilargina noldan farqli qiymatlarga ega bo'lishini ko'ramiz :

$$I_{xx} = 6, \quad I_{yy} = \frac{5}{6}, \quad I_{zz} = \frac{41}{6}, \quad I_{xy} = I_{yz} = 1.$$

Demak, bu sistemaning inersiya tenzori diagonal ko'rinishda emas:

$$I_{ik} = \begin{vmatrix} 6 & 1 & 0 \\ 1 & \frac{5}{6} & 0 \\ 0 & 0 & \frac{41}{6} \end{vmatrix}$$

Ammo, biz bilamizki, har qanday ikkinchi rangli T_{ik} tenzorni diagonal ko'rinishga keltirish mumkin. Buning uchun matritsa ko'rinishdagi, xususiy qiymatlar masalasini eslaylik:

$$T_{ik} A_k = \lambda A_i,$$

bu yerda λ xususiy qiymat deyiladi, \vec{A} - vektorni esa λ xususiy qiymatga mos keluvchi xususiy vektor deyishadi. Yuqoridagi tenglamaning yechimi quyidagichadir:

$$T_{ik} A_k = \lambda A_i \Rightarrow T_{ik} A_k - \lambda \delta_{ik} A_k = 0 \Rightarrow (T_{ik} - \lambda \delta_{ik}) A_k = 0.$$

Oxirgi tenglamani xususiy vektor komponentalari uchun bir jinsli tenglamalar sistemasi deb qarash mumkin va u notrivial yechimga ega bo'lishi uchun uning xarakteristik tenglamasi nolga teng bo'lishi kerak:

$$\det (T_{ik} - \lambda \delta_{ik}) = 0.$$

Uch o'lchamli fazoda xarakteristik tenglama uchinchi tartiblidir va uning uchta xususiy qiymatlariga har xil xususiy vektorlar mos keladi.

Demak, bu masaladagi I_{ik} tenzorni diagonal ko'rinishga keltirish uchun ushbu determinantni yechishimiz kerak va uning yechimlari bosh momentlar bo'ladi:

$$\begin{vmatrix} 6 - \lambda & 1 & 0 \\ 1 & \frac{5}{6} - \lambda & 0 \\ 0 & 0 & \frac{41}{6} - \lambda \end{vmatrix} = 0.$$

Determinant ochib chiqilsa:

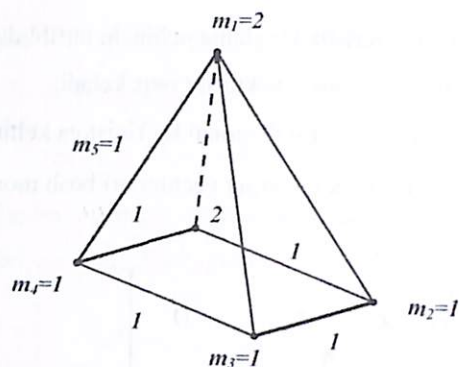
$$\left(\frac{41}{6} - \lambda \right) \left(\lambda^2 - \frac{41}{6} \lambda + 4 \right) = 0.$$

Bu yerdan I_1, I_2, I_3 bosh momentlar topiladi:

$$I_1 = \frac{41}{6}, \quad I_2 = \frac{41 + \sqrt{1105}}{12}, \quad I_3 = \frac{41 - \sqrt{1105}}{12}.$$

4-masala. Ushbu 6.4- rasmdagi sistemaning bosh inersiya momentlarini toping.

Yechilishi. Oldingi masaladagidek, x va y o'qlarini piramida asosi tomonlari bo'ylab yo'nalgan (xy tekislik asosga parallel) koordinata sistemasini inersiya markazida joylashgan deylik (z o'qi asosga perpendikulyar va balandlik bo'ylab yo'nalgan). Simmetriyaga asosan inersiya markazi shu balandlik ustida joylashadi. U holda $x_1 = y_1 = 0$ bo'lib qolgan zaralarning x va y - koordinatalarining yarmisi manfiy, yarmisi esa musbat ishoraga ega bo'ladi. Buni, chizmadagi munosabatlarni inobatga olgan holda, to'g'ridan - to'g'ri hisoblab ko'rsatish mumkin.



6.4-rasm

$$\sum m_i x_i = \sum m_i y_i = \sum m_i z_i = 0 \Rightarrow \quad , \quad 4x_1 + x_2 + x_3 + x_4 + x_5 = 0,$$

$$4y_1 + y_2 + y_3 + y_4 + y_5 = 0, \quad 4z_1 + z_2 + z_3 + z_4 + z_5 = 0$$

$$x_1 - x_2 = x_1 - x_5 = 1,$$

$$x_4 - x_1 = x_3 - x_1 = 1, \quad y_2 - y_1 = y_3 - y_1 = 1, \quad y_1 - y_4 = y_1 - y_5 = 1,$$

$$z_1 - z_2 = z_1 - z_3 = z_1 - z_4 = z_1 - z_5 = 4.$$

Bu munosabatlardan zarralarning inersiya markazida joylashgan koordinat sistemasidagi koordinatalari topiladi:

$$x_1 = y_1 = 0, \quad x_3 = x_4 = 1, \quad x_2 = x_5 = -1, \quad y_2 = y_3 = 1, \quad y_5 = y_4 = -1,$$

$$z_1 = 2, \quad z_2 = z_3 = z_4 = z_5 = -2$$

Inersiya momentlarini hisoblab topish qiyinchilik tug'dirmaydi:

$$\sum m_i x_i^2 = \sum m_i y_i^2 = 4, \quad \sum m_i z_i^2 = 32; \quad I_x = I_y = 36, \quad I_z = 8.$$

5-Masala . Sistema koordinatalari

$$-a, 3a, 5a; -3a, 8a, -5a; -a, -3a, -3a; -3a, -8a, 7a.$$

bo'lgan nuqtalarda joylashgan massalari m ga teng bo'lgan 4 ta zarrachadan iborat. Inersiyani bosh momentlarini aniqlang.

Yechilishi. Diskret zarrachalar tizimi uchun tanlab olingan koordinatalar sistemasidagi inersiya markazining o'rnini topamiz:

$$X = -\frac{1}{4}8a = -2a, \quad Y = 0, \quad Z = a.$$

Yangi koordinatalar sistemasining boshini ushbu inersiya markaziga qo'yamiz va yangi o'qlarni eski koordinata o'qlariga parallel yo'naltirilgan ravishda joylashtiramiz. Inersiya markazi aniqlanish formulasiga muvofiq yangi koordinatalar sistemasidagi barcha zarrachalarning koordinatalarini topamiz:

$$a, 3a, 4a; -a, 8a, -6a; a, -3a, -4a; -a, -8a, 6a.$$

Bu koordinatalar sistemasida inersiya tenzorining komponentlarini hisoblaymiz:

$$J_{xx} = 250ma^2, \quad J_{yy} = 0, \quad J_{zz} = 0, \quad J_{xy} = 108ma^2, \quad J_{yz} = 72ma^2, \quad J_{zx} = 150ma^2.$$

Topilgan inersiya tenzorini bosh o'qlarga nisbatan yozish uchun xarakteristik tenglamani tuzamiz:

$$\begin{vmatrix} 250ma^2 - J & 0 & 0 \\ 0 & 108ma^2 - J & 72ma^2 \\ 0 & 72ma^2 & 150ma^2 - J \end{vmatrix} = 0$$

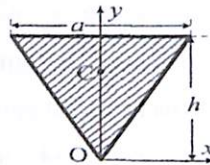
Determinantni ochib chiqib, ushuni olamiz:

$$(250ma^2 - J)[(108ma^2 - J)(150ma^2 - J) - 5178m^2a^4] = 0$$

Bu tenglamani yechib chiqib sistemaning bosh inersiya momentlarini topamiz

$$J_1 = 250ma^2, \quad J_2 = 204ma^2, \quad J_3 = 54ma^2.$$

6- Masala. Massasi m ga teng, balandligi h va asos tomoni a bo'lib teng yonli uchburchak shakliga ega bo'lgan, yupqa bir jinsli plastinkaning bosh inersiya momentlarini hisoblang.



6.5-rasm

Yechilishi. Plastinka tekis bo'lgani uchun uning inersiya markazi plastinka tekisligida joylashganligi aniq va u simmetriya sababli teng yonli uchburchakning balandligi h ustida yotadi. Demak, bosh o'qlarni balandlik bo'yicha (Y o'qi) va asosga parallel ravishda tanlanadi (X o'qi, O koordinata boshi, C esa inersiya markazi nuqtasi), bu holda C nuqtaning X koordinatasi nolga teng.

Plitaning simmetriya o'qi (teng yon tomonli uchburchak) mavjudligi sababli, inersiya markazi ushbu o'qda X koordinatasi nolga teng bo'lgan C nuqtada joylashgan va (6.6) formulaga muvofiq Y koordinatasini hisoblash.), bu erda biz plastinka yuzasi bo'ylab hisoblash uchun moddaning massa zichligini sirt va integral bilan almashtirishimiz kerak. C nuqtaning Y koordinatasini hisoblash uchun esa (6.6) formuladan foydalanamiz, faqat xajmiy zichlik o'rniga sirt zichligi $\sigma(x, y) = 2m/ah$ ni ishlatamiz:

$$Y = \frac{1}{m} \int_S \sigma(x, y) y \, dS$$

Shuni aytib o'tish lozimki, hisoblashni shu tanlab olingan koordinat sistemasida amalga oshiramiz, bosh o'qlardagi qiymatni topish uchun esa, inersiya momentlarining xossaligidan (6.8) foydalanamiz.

Integralning additivlik xossasiga ko'ra natija musbat xOy da joylashgan plastinka yuzasini hisoblab, uni 2 ko'paytirish orqali olinadi:

$$Y = \frac{4}{ah} \int_0^{a/2} \left(\int_{2hx/a}^h y \, dy \right) dx = \frac{4}{ah} \int_0^{a/2} \left[h^2 - \left(\frac{2hx}{a} \right)^2 \right] dx = \frac{2}{ah} \left[\frac{ah^2}{2} - \frac{1}{3} \cdot \frac{4h^2}{a^2} \cdot \frac{a^3}{8} \right] = \frac{2}{3}h$$

Geometriyadan yaxshi ma'lum bo'lgan natija. Plitaning simmetriyasi tufayli uning bosh inersiya o'qlaridan biri Oy o'qiga to'g'ri keladi, shuning uchun ikkinchisi Ox o'qiga parallel ravishda yo'naltiriladi. Plita tekis bo'lgani uchun uchinchi bosh inersiya o'qi uning tekisligiga perpendikulyar yo'naltirilgan bo'ladi. Yassi jism uchun (6.7-masalaga qarang), uning tekisligiga perpendikulyar bo'lgan o'qqa nisbatan inersiya momenti (bizda Oz o'qi bor) boshqa ikkita bosh o'qqa nisbatan momentlar yig'indisiga teng, shuning uchun bizga I'_x va I'_y larni hisoblash kifoya:

$$J'_y = \frac{4m}{ah} \int_0^{a/2} \left(\int_{2hx/a}^h dy \right) x^2 dx = \frac{4m}{ah} \int_0^{a/2} \left[h - \frac{2hx}{a} \right] x^2 dx = \frac{4m}{ah} \left[\frac{a^3 h}{24} - \frac{a^3 h}{32} \right] = \frac{1}{48} ma^2$$

Xuddi shuningdek I'_x

$$J'_x = \frac{4m}{ah} \int_0^{a/2} \left(\int_{2hx/a}^h y^2 dy \right) dx = \frac{4m}{3ah} \int_0^{a/2} \left[h^3 - \left(\frac{2hx}{a} \right)^3 \right] dx = \frac{4m}{3ah} \left[\frac{h^3 a}{2} - \frac{ah^3}{8} \right] = \frac{1}{2} mh^2$$

Endi inersiya markaziga nisbatan bosh inersiya momentlarini topish uchun koordinatalar sistemasini plitaning inersiya markaziga olib boramiz. Yangi bosh o'qlarni eskisi bilan bog'laydigan a vektori faqat bitta nolga teng bo'lmagan komponentaga $a_y = -2h/3$ ega bo'lganligi sababli, (6.8) ga muvofiq:

$$J_x = J'_x - \frac{4}{9}mh^2 = \frac{1}{2}mh^2 - \frac{4}{9}mh^2 = \frac{1}{18}mh^2, \quad J_y = J'_y = \frac{1}{48}ma^2.$$

Oz o'qiga mos inersiya momenti esa :

$$J_z = J_x + J_y = \frac{1}{18}mh^2 + \frac{1}{48}ma^2 = \frac{m}{144}(3a^2 + 8h^2).$$

7-masala. Qirralari a, b, c bo'lgan bir jinsli parallelepiped bosh inersiya momentlari aniqlansin

Yechilishi. Parallelepiped bir jinsli bo'lgani uchun inersiya markazi geometrik markaz bilan mos keladi va shuning uchun

$$\frac{a}{2} < x < \frac{a}{2}, \quad \frac{b}{2} < y < \frac{b}{2}, \quad -\frac{c}{2} < z < \frac{c}{2}.$$

Koordinata boshini inersiya markaziga va koordinata o'qlarini parallelepiped qirralariga parallel qilib olsak, inersiya tenzori diagonal holatga keladi:

$$I_{11} = \int \rho(y^2 + z^2) dV = \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \int_{-c/2}^{c/2} dz (y^2 + z^2) dz = \frac{\mu}{12}(b^2 + c^2),$$

$$I_{22} = \frac{\mu}{12}(a^2 + c^2), \quad I_{33} = \frac{\mu}{12}(a^2 + b^2),$$

$\mu = \int \rho dV$ - parallelepiped massasi.

8-masala. Radiusi R , balandligi h bo'lgan silindrning bosh inersiya momentlarini aniqlang.

Yechilishi. Koordinata boshini inersiya markaziga joylashtirsak va silindrik koordinatalar sistemasidan foydalansak:

$$x = r \cos \phi, \quad y = r \sin \phi, \quad z = z, \quad 0 < r < R, \quad 0 < \phi < 2\pi, \quad -\frac{h}{2} < z < \frac{h}{2}.$$

$$I_{11} = \int \rho(y^2 + z^2) dV = \int_0^R r dr \int_0^{2\pi} d\phi \int_{-h/2}^{h/2} (r^2 \sin^2 \phi + z^2) dz = \frac{\mu}{4} \left(R^2 + \frac{h^2}{3} \right),$$

$$I_{22} = \frac{\mu}{4} \left(R^2 + \frac{h^2}{3} \right), \quad I_{33} = \frac{\mu}{2} R^2.$$

9-masala Qattiq jism kinetik energiyasi ifodasi (6.2) ni tenzor ifodalashlar orqali keltirib chiqaring

Yechilishi. Qattiq jism kinetik energiyasi ifodasini keltirib chiqarishda (T_{oyl}) biz vektorlar algebrasidan ma'lum bo'lgan ushbu tenglikdan foydalandik:

$$[\vec{\Omega}, \vec{r}]^2 = \left\{ \vec{\Omega}^2 r^2 - (\vec{\Omega} \vec{r})^2 \right\}$$

Ammo, vektor komponentalarining indeksli ifodalari va uchinchi rangli antisimmetrik tenzor (ε_{ijk} , IV bobga qarang) xossaligidan foydalangan holda birmuncha engillik bilan bularni keltirib chiqarish mumkin:

$$\vec{v} = \vec{V} + [\vec{\Omega}, \vec{r}] \Rightarrow \vec{v}_i = \vec{V}_i + [\vec{\Omega}, \vec{r}]_i, \quad [\vec{\Omega}, \vec{r}]_i = \varepsilon_{ijk} \Omega_j x_k,$$

Yuqoridagi belgilashlar orqali kinetik energiya ifodasini yozib olaylik:

$$\begin{aligned} T &= \sum \frac{mv^2}{2} = \frac{1}{2} \sum_a m_a v_i v_i = \frac{1}{2} \sum_a m_a (V_i + \varepsilon_{ijk} \Omega_j x_{ak}) (V_i + \varepsilon_{imn} \Omega_m x_{an}) = \\ &= \frac{1}{2} \sum_a m_a (V_i V_i + V_i \varepsilon_{imn} \Omega_m x_{an} + V_i \varepsilon_{ijk} \Omega_j x_{ak} + \varepsilon_{ijk} \Omega_j x_{ak} \varepsilon_{imn} \Omega_m x_{an}). \end{aligned}$$

Natijada kinetik energiya ifodasi uch qismdan iborat bo'ldi va T_{IMS} inersiya markazi sistemasida nolga aylanib ketishi yaqqol ko'rindi:

$$T_{\text{yal}} = \frac{1}{2} \sum_a m_a V_i V_i = \frac{\mu}{2} V^2$$

$$T_{IMS} = \frac{1}{2} \sum_a m_a (V_i \varepsilon_{imn} \Omega_m x_{an} + V_i \varepsilon_{ijk} \Omega_j x_{ak}) = \varepsilon_{ijk} V_i \Omega_j \sum_a m_a x_{ak} \Rightarrow 0$$

Bu ifodalarda Eynshteyn qoidasiga rioya qilingan bilan, turli zarralar bo'ycha yig'indi atomi saqlanib qolindi. T_{ayl} ning ko'rinishini ε_{ijk} ning xossalariga asosan

(ilovaga qarang) quyidagicha yozib olish mumkin:

$$\begin{aligned} T_{ayl} &= \frac{1}{2} \sum_a m_a \varepsilon_{ijk} \Omega_j x_{ak} \varepsilon_{imn} \Omega_m x_{an} = \frac{1}{2} \varepsilon_{ijk} \varepsilon_{imn} \sum_a m_a \Omega_j x_{ak} \Omega_m x_{an} = \\ &= \frac{1}{2} (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) \sum_a m_a \Omega_j \Omega_m x_{ak} x_{an} = \frac{1}{2} \sum_a m_a (\delta_{jm} \Omega_j \Omega_m x_{ak} x_{ak} - \Omega_j \Omega_m x_{aj} x_{am}) = \\ &= \frac{1}{2} \Omega_j \Omega_m \sum_a m_a (\delta_{jm} x_{ak} x_{ak} - x_{aj} x_{am}) = \frac{1}{2} I_{jm} \Omega_j \Omega_m. \end{aligned}$$

Bu yerda, I_{jm} yuqorida aniqlangan inersiya tenzoridir:

$$I_{jm} = \sum_a m_a (\delta_{jm} x_{ak} x_{ak} - x_{aj} x_{am}) = \sum_a m_a (\delta_{jm} x_{ak}^2 - x_{aj} x_{am})$$

Shunday qilib, qattiq jism kinetik energiyasi ifodasini ushbu ko'rinishda oldik:

$$T = \frac{\mu V^2}{2} + \frac{1}{2} \sum m \left\{ \bar{\Omega}^2 \bar{r}^2 - (\bar{\Omega}, \bar{r})^2 \right\} = \frac{\mu V^2}{2} + \frac{1}{2} I_{ik} \Omega_i \Omega_k$$

10-masala. Qattiq jism impuls momenti ifodasi (6.9) ni tenzor ifodalashlar orqali keltirib chiqaring

Yechilishi. Qattiq jism impuls momenti ifodasini keltirib chiqarishda biz vektorlar algebrasidan ma'lum bo'lgan qo'shloq vektor ko'paytmaning ushbu tengligidan foydalandik:

$$\left[\bar{r} \left[\bar{\Omega} \bar{r} \right] \right] = \left\{ \bar{r}^2 \bar{\Omega} - \bar{r} (\bar{r} \bar{\Omega}) \right\}$$

Impuls momenti vektorining biror komponentasi ifodasini antisimmetrik tenzor ε_{ijk} orqali yozib olish va uning xossalaridan foydalanish natijasida ushbularni olamiz (7-masala ishlanishiga qarang):

$$\begin{aligned} M_i &= \sum_a m_a (\varepsilon_{ijk} x_{aj} \varepsilon_{kmn} \Omega_m x_{an}) = \varepsilon_{ijk} \varepsilon_{kmn} \sum_a m_a (x_{aj} \Omega_m x_{an}) = \\ &= \varepsilon_{ijk} \varepsilon_{mnk} \sum_a m_a (x_{aj} \Omega_m x_{an}) = (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) \sum_a m_a x_{aj} \Omega_m x_{an} = \\ &= \sum_a m_a (x_{an} \Omega_i x_{an} - x_{aj} \Omega_j x_{ai}) = \sum_a m_a (\Omega_i x_{an}^2 - x_{ai} x_{aj} \Omega_j) = \\ &= \sum_a m_a (\Omega_j \delta_{ij} x_{an}^2 - x_{ai} x_{aj} \Omega_j) = \Omega_j \sum_a m_a (\delta_{ij} x_{an}^2 - x_{ai} x_{aj}) = I_{ij} \Omega_j \end{aligned}$$

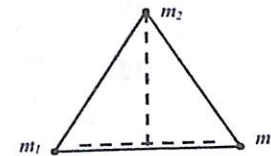
Demak,

$$M_i = I_{ik} \Omega_k.$$

Shu bilan birga yuqoridagi vektor munosabatni topish yo'li ham ko'rsatildi.

6.1-Misollar.

1. Uzunligi l bo'lgan ingichka sterjen bosh inersiya momentlari topilsin.
2. R radiusli sharning bosh inersiya momentlari topilsin.
3. Ichida r radiusli shar shaklida bo'shlig'i bo'lgan R radiusli shar bosh inersiya momentlari topilsin
4. Balandligi h va asos radiusi R bo'lgan doiraviy konus bosh inersiya momentlari topilsin.
5. Bir chiziqda joylashgan atomlardan iborat molekulaning bosh inersiya momentlari topilsin.
6. Teng yonli uchburchak shaklidagi uch atomli molekulaning bosh inersiya momentlari topilsin.



6.6 - rasm

$$b) L = m \frac{\dot{r}^2 + r^2 \sin^2 \alpha \dot{\phi}^2}{2} - \frac{a}{r}$$

$$d) L = \frac{(m_1 + m_2)}{2} \cdot a^2 \dot{\phi}^2 + m_2 \frac{b^2 \dot{\theta}^2 + 2ab\dot{\phi}\dot{\theta} \cos(\phi - \theta)}{2} + (m_1 + m_2)ag \cos \phi + m_2bg \cos \theta,$$

$$g) L = a^2 \dot{\phi}^2 \frac{m_1 \sin^2 \varphi + m_2 \cos^2 \varphi}{2} - \frac{ag}{\sqrt{2}} (m_1 \sin \varphi + m_2 \sin \theta),$$

d) $r = r_0 + vt$, $\dot{r} = v$, ikkinchi zarra uchun

$$L = \frac{m_1 a^2 \dot{\phi}^2}{2} + \frac{m_2 (\dot{\phi}^2 (r_0 + vt)^2)}{2} + m_1 ga \cos \phi + m_2 g (r_0 + vt) \cos \phi,$$

$$e) L = \frac{ml^2}{2} \cdot \dot{\varphi}^2 + mla\omega^2 \sin \omega t \cdot \cos \varphi + mg(l \cos \varphi + a \sin \omega t),$$

$$j) L = \frac{ml^2 \dot{\varphi}^2}{2} + mla\omega^2 \sin \omega t \cdot \sin \varphi + mgl \cos \varphi.$$

$$3. L = \frac{m}{2} \cdot \sum_i \dot{r}_i^2 - \frac{m^2}{2M} \left(\sum_i \dot{r}_i \right)^2 - \sum_{i,j} \frac{l_i l_j}{2r_{ij}}$$

$$4. L = \frac{1}{2} m_1 \dot{r}_1^2 + \frac{1}{2} m_2 \dot{r}_2^2 - U(\vec{r}_1, \vec{r}_2), U = -\gamma \frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|} - \gamma \frac{m_1 m_3}{|\vec{r}_1 - \vec{r}_3(t)|} - \frac{m_2 m_3}{|\vec{r}_2 - \vec{r}_3(t)|}$$

1.5-misollar

1.

$$a) E = \frac{1}{\sqrt{1-v^2}} + \phi,$$

$$b) E = \frac{m\dot{x}^2}{2} + \frac{kx^2}{2},$$

$$v) E = \dot{x}^2 + (e^x - 1)^2,$$

$$g) E = \frac{m_1 + m_2}{2} \dot{r}^2 + \frac{m_2}{2} (\dot{\phi}^2 - 2\dot{r}\dot{\phi} \sin \phi) + m_2 \sin \phi$$

2.

$$1. E = \frac{(1+q^2)\dot{q}^2}{2} + \frac{q^2}{2}$$

$$2. E = \frac{1}{\sqrt{1-\dot{q}^2}} - q$$

$$3. E = \frac{\dot{r}^2}{2} + \frac{r^2}{2} \dot{\theta}^2 - \frac{1}{r}$$

$$4. E = \frac{1}{2} \dot{U}\dot{v} - \frac{1}{\sqrt{Uv}}$$

$$5. E = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - (xy - y\dot{x})$$

$$6. E = \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + (x^2 + y^2 + xy)$$

$$7. E = \frac{1}{2} (\dot{x}^2 + \dot{x}\dot{y} + \dot{y}^2) + \frac{1}{2} (x^2 - xy + y^2)$$

$$8. E = \frac{m}{2} (\dot{\xi}^2 + \rho^2 \dot{\phi}^2) - \frac{1}{2} (\chi \xi^2 + mg\rho\phi^2)$$

9.

$$10. E = \frac{ma^2}{2} \{ \dot{\phi}^2 - 2\omega(\omega + \cos \phi) \}$$

$$E = \frac{m_1 \xi_2^2}{2} \xi_1^2 + \frac{m_2 \xi_1^2}{2} \xi_2^2 + \frac{k}{2} (\xi_2 - \xi_1)^2$$

11.

$$12. E = \frac{m_1 \xi_1^2 + m_2 \xi_2^2 - \chi(\xi_2 - \xi_1)^2}{2}$$

$$E = \frac{m}{2} (\dot{\xi}_1^2 + \dot{\xi}_2^2) + \chi(\xi_1^2 - \xi_1 \xi_2 + \xi_2^2)$$

$$13. E = \frac{e^{ax} (\dot{x}^2 + \omega^2 x^2)}{2}$$

3.

$$a) x = 2 - \frac{2}{\sqrt{3}}$$

$$b) x = 2,$$

$$v) x = \frac{\pi}{3},$$

$$g) x = \pm \frac{1}{3}$$

$$d) x = \pm \frac{1}{3}$$

$$e) x = \arctg 2$$

$$j) x = \ln 5$$

$$z) x_{1,2} = \frac{3 \pm \sqrt{5}}{2}$$

$$i) x = e$$

$$k) x = \frac{2}{5}$$

$$l) x = a \ln \left(\frac{U_0}{E} \right)$$

$$m) x = \frac{1}{k} \text{Arcch} \left(\pm \sqrt{\frac{U_0}{E_0}} \right)$$

$$n) x = 0, x = \frac{3k}{a}$$

$$p) x = 2\pi nl$$

2.1. Misollar

$$1. t = \sqrt{x^2 - 1},$$

$$3. t = \arcsin(x-1)$$

$$6. t = \frac{1}{8} \ln \frac{x-4}{x+4}$$

$$8. t = \frac{1}{\omega} \arcsin \sqrt{\frac{m}{2E}} \omega x + t_0$$

$$10. t = \frac{e^{ax} (a \sin bx - b \cos bx)}{a^2 + b^2} - \frac{be^{\frac{a}{b}x}}{a^2 + b^2}$$

$$11. t = \int \frac{dx}{\sqrt{\frac{2}{m} (E + U_0 \exp \frac{x}{a})}} = -a \sqrt{\frac{m}{2E}} \ln \frac{\sqrt{E - U_0} + \sqrt{E}}{\sqrt{E - U_0} - \sqrt{E}}$$

$$x(t) = a \ln \left[\frac{E}{U_0} \operatorname{sh}^{-2} \left(\frac{1}{a} \sqrt{\frac{E}{2m}} (T - t) \right) \right], \quad E > 0,$$

$$T = a \sqrt{\frac{m}{2E}} \ln \frac{\sqrt{E + U_0} + \sqrt{E}}{\sqrt{E + U_0} - \sqrt{E}}, \quad E = 0 \text{ da } \dot{x} = v_0 \exp \frac{x}{2a},$$

$$x(t) = -2a \ln \left(1 - \frac{v_0}{2a} t \right), \quad t_0 = a \sqrt{\frac{2m}{U_0}} \text{ da zarra cheksizga ketadi.}$$

$$13. \tau = \lim_{L \rightarrow \infty} \sqrt{\frac{m}{2}} \int_{-L}^L dx \left(\frac{1}{\sqrt{E - U_0}} - \frac{1}{\sqrt{E}} \right) = 2a \sqrt{\frac{2m}{E}} \ln \frac{2\sqrt{E}}{\sqrt{E - U_0} + \sqrt{E}}$$

2.2.- Misollar

$$2. t = \frac{2}{\sqrt{3}} \left(-\operatorname{Arctg} e^{-x} \sqrt{3} + \operatorname{Arctg} \sqrt{3} \right)$$

$$4. t = \operatorname{Arctg} x$$

$$7. t = \frac{1}{\omega} \ln \frac{x}{\omega + x}$$

$$9. t = \frac{1}{\sqrt{15}} \arccos \frac{\sqrt{15}}{4} \cos x$$

$$a) E > 2, T = 2 \arcsin \frac{2x - E}{\sqrt{E^2 - 1}} \Big|_{x_1}^{x_2} = 2\pi,$$

$$b) E < 1, T = \frac{2}{\sqrt{1-E}} \cdot \arccos \frac{e^{-x}(1-E) - 1}{\sqrt{E}} \Big|_{x_1}^{x_2} = \frac{2\pi}{\sqrt{1-E}},$$

$$d) -1 < E < 0, T = \frac{2}{\sqrt{E}} \cdot \operatorname{arcsin} \frac{xE - 1}{\sqrt{1-E}} \Big|_{x_1}^{x_2} = \frac{2\pi}{\sqrt{E}},$$

$$e) 0 < E < 1, T = \frac{2}{\sqrt{1-E}} \cdot \arcsin \sqrt{\frac{1-E}{E}} \cdot \operatorname{Sh} x \Big|_{x_1}^{x_2} = \frac{2\pi}{\sqrt{1-E}},$$

$$f) T = \frac{\pi \sqrt{2m}}{\alpha \sqrt{E + U_0}}.$$

2.3-misollar

$$1. x = t - \frac{t^3}{3},$$

$$2. x = \frac{\ln \left(1 - t \operatorname{tg} \frac{t}{2} \right)}{\left(1 + t \operatorname{tg} \frac{t}{2} \right)},$$

$$3. x = t \operatorname{tg} t,$$

$$4. x = \frac{t^2}{6},$$

$$6. x = \frac{1}{2} \sqrt{P_y^2 + E^2} \cdot \operatorname{Ch}(t - t_0) - \frac{E}{2}, \quad y = 2 \operatorname{arctg} \frac{\sqrt{P_y^2 + E^2} e^{t-t_0} - E}{P_y + y_0}.$$

$$7. t = \left(\frac{1}{2} \sqrt{2E - P_y^2} \right) \cdot \left(x \sqrt{1 - x^2} + \arcsin x \right),$$

$$y = \left(\frac{P_y}{2\sqrt{2E - P_y^2}} \right) \cdot (x\sqrt{1-x^2} + \arcsin x) - \frac{x^2}{2}.$$

$$8. x = -E \pm \sqrt{(t-t_0)^2 + (1+P_y^2)}, \quad y = P_y \operatorname{ArcCh} \frac{E+x}{\sqrt{1+P_y^2}}.$$

$$9. x = \frac{4E}{P_y^2} \cdot \cos\left(P_y \frac{t-t_0}{4}\right), \quad y = \frac{E}{P_y} \cdot (t-t_0) + \frac{2E}{P_y^2} \cdot \sin\left(P_y \frac{t-t_0}{2}\right) + Y_0.$$

$$10. t = \int \frac{d\theta}{\sqrt{2ml^2 [E - V_{\varphi\psi}(\theta)]}}, \quad V_{\varphi\psi}(\theta) = \int \frac{M_z^2}{2ml^2 \sin^2 \theta} - mgl \cos \theta,$$

Bu integrallar elliptik integrallardir. Umumlashgan koordinata φ ning aniqlash sohasi $E > V_{\varphi\psi}$ dan, chegarasi esa $E = V_{\varphi\psi}$ dan aniqlanadi. Oxirga tenglama $\cos \varphi$ ga nisbatan 3-darajali tenglama bo'lib, $(-1, +1)$ oraliqda ikkita ildizi mavjud va sferadagi ikki parallel (ham konsentrik) doira oralig'idagi trayektoriyalarni aniqlab beradi $\varphi = \frac{M_z}{l\sqrt{2m}} \int \frac{d\theta}{\sqrt{\sin^2 \theta [E - V_{\varphi\psi}(\theta)]}}$.

2.4-misollar

$$1. a) r = \frac{ma \pm \sqrt{m^2 a^2 + 2mEM^2}}{E},$$

$$b) r = \pm \sqrt{\frac{M^2 + \sqrt{M^4 + 16m^2 E}}{4mE}},$$

$$d) r = \sqrt{E \pm \sqrt{E^2 + (M^2 + 1)}},$$

$$e) r = \frac{c(1 \pm \sqrt{1-4k^2})}{2k}, \quad k = \frac{M}{c\sqrt{2mU_0}}, \quad T = \frac{\pi m^2 c^4 U_0}{M^3}, \quad 0 < M < \frac{c\sqrt{mU_0}}{2}.$$

$$2. a) \frac{1}{r} = \sqrt{\frac{2E}{1+M^2}} \cos \left[\sqrt{\frac{1+M^2}{M}} (\phi - \phi_0) \right],$$

$$b) \frac{1}{r} = \frac{1}{2\sqrt{M^2+1}} \left(1 + \sqrt{8E(M^2+1)+1} \right) \cos \left[\sqrt{\frac{M^2+1}{M^2}} (\phi - \phi_0) \right],$$

$$d) r = r_0 \exp \left[\frac{U_0 r_0^2}{2mM^2} (\phi - \phi_0)^2 + \frac{M^2}{2mr_0^2 U_0} \right].$$

3.1-misollar

$$1. a) \omega = 1, \quad b) \omega = \frac{1}{\sqrt{2}}, \quad d) \omega = \frac{1}{\sqrt{l}}, \quad e) \omega = \sqrt{\frac{k}{m}},$$

$$f) \omega^2 = \frac{1}{m} \left[l' \alpha^2 \sqrt{1 - \left(\frac{F}{l' \alpha} \right)^2} \right], \quad g) \omega = \sqrt{\frac{k}{m}}.$$

$$2. \frac{\omega'}{\omega} = \frac{m_1 m_2 (m_1' + m_2')}{m_1' m_2' (m_1 + m_2)},$$

$$3. \omega = \sqrt{\frac{k(l-a)}{lm}}, \quad l > a, \quad \omega = \sqrt{\frac{k(a^2-l^2)}{ma^2}}, \quad l < a.$$

$$4. \omega = \sqrt{\frac{F}{ml}}, \quad x=0.$$

$$5. \omega = \sqrt{\frac{F(r+l)}{lrm}}, \quad \varphi=0.$$

$$6. L = \frac{mx^2}{2} + \frac{kx^2}{2}, \quad x = C_1 \exp \left(\sqrt{\frac{k}{m}} t \right) + C_2 \exp \left(-t \sqrt{\frac{k}{m}} \right).$$

$$7. x_0 = \sqrt[3]{\frac{e^2}{8mgR^2}} < 1, \quad \frac{\sin \varphi_0}{2} = x_0,$$

$x_0 > 1$ da A nuqta barqaror muvozanat nuqtasidir, $x_0 < 1$ da esa, noturg'un.

$$8. L = \frac{m\dot{y}^2}{2} - k\left(y - l - \frac{mg}{2k}\right)^2 + Const, \quad y_0 = l + \frac{mg}{2k},$$

$$\omega^2 = \frac{2k}{m}, \quad y = y_0 + A \cos(\omega t + \varphi).$$

$$9. a) A = \sqrt{q^2 + \frac{\dot{q}^2}{\omega^2}}, \quad \text{tg} \varphi = -\frac{\dot{q}}{q\omega}, \quad b) A = \frac{1}{\omega} \sqrt{\frac{2E}{m}}, \quad \varphi = \arccos\left(q\omega \sqrt{\frac{m}{2E}}\right).$$

3.2-misollar

$$1. a) x = \frac{F_0}{m\omega^2}, \quad b) x = \frac{F_0}{m(\omega^2 - \alpha^2)} \left(\sin \alpha t - \frac{\alpha}{\omega} \sin \omega t \right),$$

$$d) x = \frac{F_0}{m\omega^2} \left(t - \frac{\sin \omega t}{\omega} \right), \quad e) x = \frac{F_0}{m(\omega^2 + \alpha^2)} \left(e^{-\alpha t} - \cos \omega t + \frac{\alpha}{\omega} \sin \omega t \right),$$

$$f) x = \frac{F_0 e^{-\alpha t}}{(\omega^2 + \alpha^2 - \beta^2)^2 + 4\alpha^2 \beta^2}$$

$$\cdot \left[(\omega^2 + \alpha^2 - \beta^2)(\cos \beta t - \cos \omega t) + 2\alpha\beta \left(\frac{\beta}{\omega} \sin \omega t - \sin \beta t \right) \right],$$

$$g) x = \frac{F_0 e^{-\alpha t}}{(\omega^2 + \alpha^2 - \beta^2)^2 + 4\alpha^2 \beta^2}$$

$$\cdot \left[2\alpha\beta(\cos \beta t - \cos \omega t) + (\omega^2 + \alpha^2 - \beta^2) \left(\sin \beta t - \frac{\beta}{\omega} \sin \omega t \right) \right].$$

$$2. a) x = \frac{at}{2m\omega} \sin \alpha t.$$

$$3. a) \frac{2F_0}{mT\omega^3} \sin \frac{\omega T}{2}, \quad b) a = \frac{F_0}{m\omega^3} \sqrt{\omega^2 T^2 - 2T\omega \sin \omega T + 2(1 - \cos \omega T)},$$

$$d) a = \frac{F_0 \pi}{m\omega^2}.$$

$$4. \text{Tebranish nodavriydir, } \lambda_{1,2} = \frac{-4 \pm \sqrt{7}}{2}.$$

$$5. F = 2k^2, \quad \lambda = 2, \quad \omega = \sqrt{5}.$$

$$6. a) C_1 = \frac{v_0 + x_0(\lambda + \sqrt{\lambda^2 - \omega_0^2})}{2\sqrt{\lambda^2 - \omega_0^2}}, \quad C_2 = \frac{-v_0 + x_0(-\lambda + \sqrt{\lambda^2 - \omega_0^2})}{2\sqrt{\lambda^2 - \omega_0^2}}.$$

$$b) C_1 = x_0, \quad C_2 = v_0 + \lambda x_0.$$

3.3-misollar

$$1. a) x_0 = 1, \quad y_0 = -1, \quad \omega_1^2 = \omega_2^2 = 6, \quad b) x_0 = y_0 = 1, \quad \omega_1^2 = 2, \quad \omega_2^2 = \frac{2}{3}.$$

$$2. a) \omega_1^2 = 6, \quad \omega_2^2 = 1, \quad x = \sqrt{\frac{1}{5}} \cdot (-2Q_1 + Q_2), \quad y = \sqrt{\frac{1}{5}} \cdot (3Q_1 + Q_2).$$

$$b) \omega_1^2 = 1, \quad \omega_2^2 = \frac{3}{4}, \quad \omega_3^2 = \frac{3}{4}.$$

$$d) \omega_1^2 = 1, \quad \omega_2^2 = \frac{9 + \sqrt{57}}{6}, \quad \omega_3^2 = \frac{9 - \sqrt{57}}{6}.$$

$$e) \omega_{1,2}^2 = \frac{1}{2} \left[(\omega_{01}^2 + \omega_{02}^2) \pm \sqrt{(\omega_{02}^2 - \omega_{01}^2)^2 + 4\alpha^2} \right],$$

$$x = Q_1 \cos \varphi - Q_2 \sin \varphi, \quad y = Q_1 \sin \varphi + Q_2 \cos \varphi.$$

$$3. \xi_1 = \rho - \rho_{uv}, \quad \xi_2 = \varphi, \quad T = \frac{m}{2} \left(\dot{\xi}_1^2 + \rho_{uv}^2 \dot{\xi}_2^2 \right),$$

$$U = \frac{1}{2}(k\xi_1^2 + mg\rho_{u,v}\xi_2^2), \quad \omega_1^2 = \frac{k}{m}, \quad \omega_2^2 = \frac{g}{\rho_{u,v}}.$$

4. a) $3n-6$, b) $3n-6=(2n-3)+(n-3)$, d) $3n-5=(n-1)+(2n-4)$.

5. $\lambda_{1,2} = \frac{-6 \pm 2\sqrt{3}}{3}$.

6.

a) $\ddot{x} + \dot{x} - \frac{3\dot{y}}{2} + 3x = 0$,

$$2\ddot{y} - 3\dot{x} + 2\dot{y} + 5y = 0.$$

b) $4\ddot{x} + \frac{3\dot{y}}{2} + \dot{x} = 0$,

$$\ddot{y} + \frac{3\dot{x}}{2} + 3\dot{y} = 0,$$

d) $x\ddot{x} + \frac{\dot{x}}{x} + 1 - \frac{1}{x^2 y} = 0$,

$$y\ddot{y} + \frac{\dot{y}}{y} + 1 - \frac{1}{xy^2} = 0$$

4.1-misollar

1.

a) $\dot{r} = (p-r^2)$, $\dot{p} = 2r(p-r^2)$

b) $\dot{x} = \frac{p_v}{tx^2 y}$, $\dot{p}_v = \frac{2p_x p_v}{tx^3 y}$, $\dot{y} = \frac{p_x}{tx^2 y}$, $\dot{p}_x = \frac{p_x p_v}{tx^2 y^2}$

v) $\dot{u} = \frac{p_u}{u^2 + v^2}$, $\dot{p}_u = \frac{u(p_u^2 + p_v^2)}{(u^2 + v^2)^2} - 1$, $\dot{v} = \frac{p_v}{u^2 + v^2}$, $\dot{p}_v = \frac{v(p_u^2 + p_v^2)}{(u^2 + v^2)^2}$.

4.2-misollar

2.

a) $H = \frac{p_y^2}{2x} + \ln p_x + 1$,

b) $H = \frac{1}{2m}(\sum \dot{p}_a^2) + \frac{1}{2M}(\sum \dot{p}_n^2) + U$,

v) $H = \frac{xp_x^2}{4} + \frac{p_x^2}{4x} - x$,

g) $H = \frac{xp_x^2}{4} + xp_x p_v$,

d) $H = 2p_u p_v - \frac{1}{\sqrt{uv}}$

e) $H = \frac{p_x^2}{2r^2} + \frac{p_x^2}{2} - \frac{1}{r}$

j) $H = \frac{p_x^2 + p_y^2}{2} + (p_x y - xp_x) + \frac{x^2 + y^2}{2}$,

z) $H = \frac{m}{2}(p_r^2 - \omega^2 r^2 \sin^2 \theta) + mgr \cos \theta$

k) $H = \frac{p^2}{2(1+2\beta x)} + \frac{\omega^2 x^2}{2} + \alpha x^3$,

l) $H = c\sqrt{m^2 c^2 + \left(\bar{p} - \frac{e\vec{A}}{c}\right)^2} + e\varphi$

m) $H = \frac{p_u^2}{2I} + \frac{p_v^2}{2I_1} + \frac{(p_\varphi - p_\varphi \cos \theta)^2}{2I \sin^2 \theta} - 3\Omega^2(I - I_1) \sin^2 \varphi \sin^2 \theta - \Omega p_\varphi$.

3.

a) $L = \ln(\dot{x}-1) + \frac{\dot{y}^2}{2} + 1$,

b) $L = -\sqrt{1-\dot{r}^2} - r$,

v) $L = \frac{\dot{r}^2}{2} + \frac{\dot{v}^2 r^2 \sin^2 v}{2} - \sin v$

g) $L = \frac{(\dot{v}^2 + \dot{u}^2)(u^2 + v^2)}{2} - (u + v^2)$.

d) $L = \dot{x}\dot{y}x^2 y - x^2 y^2$,

e) $L = \frac{\dot{r}^2}{2} + r^2 \dot{r} - \frac{1}{r}$.

j) $L = \frac{\dot{\phi}\dot{\theta}}{2} - \frac{1}{\sqrt{\phi\theta}}$,

z) $L = \frac{mr^2}{2}(\dot{r}^2 + \dot{v}^2) - \frac{1}{(r^2 + 1)}$.

k) $L = \frac{m}{2}(\dot{v} + \dot{a})^2$

4. $H = \frac{p^2}{2} \left(1 - \frac{q^2}{4}\right) + \frac{1}{2}\omega^2 q^2$

4.3-misollar

1. $-\varepsilon_{ijk} x_k$,

2. $-\left[\bar{b}\bar{a}\right]\bar{r}$,

3. $-\left[\bar{e}\bar{b}\right]\bar{M}$,

4. 0,

5. $n\bar{r}r^{(n-2)}$,

6. $2\bar{a}(\bar{a}\bar{r})$,

7. $2\left[\bar{r}\bar{M}\right]$,

8. $2\left[\bar{p}\bar{M}\right]$,

9. 0

10. $-2p_x$.

11. 0 12. $-\left[\bar{b}\bar{a}\right]\bar{p}$ 13. $n\bar{p}p^{n-2}$ 14. 0 15. 0

16. $-\frac{df(r)}{dr} \frac{r_i}{r}$ 17. $\frac{df(p)}{dp} \frac{p_i}{p}$ 18. $r \frac{df(r)}{dr}$

4.4-misolalar

6. $F_i = -e(e^Q - 1)^2 \lg p$,

8. a), b) koordinata va impuls bir-biriga almashadi,

v) aynan almashtirilgan.,

10. a) $q = \sqrt{\frac{2P}{m\omega}} \sin Q$, $p = \sqrt{2m\omega P} \cos Q$, $\dot{Q} = \omega + \dot{\omega} \frac{\sin 2Q}{2\omega}$, $\dot{p} = p\omega \frac{\cos 2Q}{\omega}$,

bu yerda P va Q ta'sir - burchak uzgaruvchilari,

b) $q = \frac{F}{m\omega^2} + \sqrt{\frac{2P}{m\omega}} \sin Q$, $p = \sqrt{2m\omega P} \sin Q$,

$\dot{Q} = +\dot{F} \sqrt{\frac{m\omega}{2P}} \cos Q$, $\dot{p} = \dot{F} \sqrt{2m\omega P} \sin Q$.

11. Gamilton funksiyasi $H' = \omega P_1$,

harakat tenglamalari $\dot{P}_1 = \dot{P}_2 = \dot{Q}_2 = 0$, $\dot{Q}_1 = \omega$.

12. Bu almashtirishlar $p = \alpha P$, $r = \frac{Q}{\alpha}$ o'xshashlik almashtirishidir.

13. a) $q = e^Q$, $p = Pe^{-Q}$, $p = \sqrt{2P}$, $q = \sqrt{2P}Q$,

b) $p = P\sqrt{2Q}$, $q = \sqrt{2Q}$,

d) $\bar{r} = \bar{r}' + \bar{V}t$, $\bar{p} = \bar{p}' + m\bar{V}$, $H' = H - \bar{V}p'$.

14. a) $q = \frac{1}{\sqrt{2m\omega}}(Q - iP)$, $p = -i\sqrt{\frac{m\omega}{2}}(Q + iP)$, $H' = -i\omega PQ$,

b) $q = \sqrt{\frac{2P}{m\omega}} \cos Q$, $\dot{p} = -\sqrt{2m\omega P} \sin Q$, $H' = \omega P$,

d) $q = \sqrt{\frac{2P}{m\omega}} \cos(\omega t + Q)$, $\dot{p} = -\sqrt{2m\omega P} \sin(\omega t + Q)$, $H' = 0$, $H = 0$.

15. b) $\dot{x} = \dot{x}' + \frac{\bar{p}'t}{m}$, $\dot{p} = \dot{p}'$, $H(x', p', t) = 0$.

4.5-misolalar

1. a) $\frac{\partial S}{\partial t} + \frac{1}{2} \left[\frac{1}{1-v^2} \left(\frac{\partial S}{\partial u} \right)^2 + \frac{1}{1-u^2} \left(\frac{\partial S}{\partial v} \right)^2 \right] = 0$,

b) $\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 - \frac{1}{c^2} \left(\frac{\partial S}{\partial t} \right)^2 + m^2 c^2 = 0$,

d) $\frac{\partial S}{\partial t} + \left\{ \frac{1}{2(u^2 + v^2)} \left[\left(\frac{\partial S}{\partial u} \right)^2 + \left(\frac{\partial S}{\partial v} \right)^2 \right] \right\} + (u+v)^2 = 0$,

e) $\frac{\partial S}{\partial t} + \left\{ \frac{1}{2m} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{r^2 \sin^2 \theta} \left(\frac{\partial S}{\partial \varphi} \right)^2 \right] \right\} + \frac{1}{r} = 0$,

f) $\frac{\partial S}{\partial t} + \left\{ \frac{1}{4x} \left[x^2 \left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 \right] \right\} - x = 0$,

2. a) $v\ddot{u} + \dot{u}\dot{v} = \frac{\dot{v}^2}{2}$, $u\ddot{v} + \dot{u}\dot{v} = \frac{\dot{u}^2}{2}$, b) $u\ddot{u} + \frac{\dot{u}^2}{2}$, $v\ddot{v} + \frac{\dot{v}^2}{2}$,

d) $m\ddot{x} = 2f\dot{y}$, $m\dot{y} = -2ax$, e) $\ddot{x}\sqrt{xy} = 1$, $m\ddot{x} = 2f\dot{y}$

f) $\ddot{x} = 2\dot{y}$, $m\dot{y} = -2\dot{x}$.

3. a) $\frac{\partial S}{\partial t} + \frac{1}{2} \left[\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 \right] - x = 0$, $x = x_0 + p_{0x}t + \frac{t^2}{2}$,

$x_0 = \frac{p_{0x}^2 + p_{0y}^2 + p_{0z}^2}{2} - E$ - boshlang'ich holat, $\frac{\partial S}{\partial E} = p_x = const.$,

$\frac{\partial S}{\partial E} = p_x = const$, $y = y_0 + p_y \sqrt{2E - p_x^2 - p_z^2 + 2x}$.

b) ko'rsatma: $\frac{\partial S}{\partial t} + \frac{1}{2} \left[\left(\frac{\partial S}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial S}{\partial \varphi} \right)^2 \right] + \frac{1}{r} = 0, \quad S = -Et + p_\varphi \varphi + f(r)$

d) ko'rsatma: $S = -Et + p_\varphi \varphi + p_z z + f(r).$

e) $H = \frac{p^2}{2m} - mgy, \quad y \frac{gt^2}{2}, \quad p = mgt.$

5.1-misollar

2. a) parchalanadi,

b) parchalana olmaydi.

3. $E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}, \quad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M}$

4. $E_\mu = 4 \text{ MeV}, \quad E_\nu = 29.8 \text{ MeV}$

5.2-misollar

1. $\vec{v}'_1 = \frac{\vec{V} + m_2 \vec{v}\vec{n}}{m_1 + m_2}, \quad \vec{v}'_2 = \frac{\vec{V} - m_1 \vec{v}\vec{n}}{m_1 + m_2}, \quad \vec{p}'_1 = m_1 \vec{V} + \mu \vec{v}\vec{n}, \quad \vec{p}'_2 = m_2 \vec{V} - \mu \vec{v}\vec{n}.$

3. $t = 4\mu^2 v^2 \sin^2 \frac{\chi}{2}.$

5. a) $\sigma = \frac{2\pi\alpha}{mv^2_\varphi},$

b) $\sigma = \pi(n-2)^{\frac{2-n}{n}} \left(\frac{\alpha}{mv^2_\varphi} \right)^n$

d) $\sigma = \begin{cases} \pi \left(\frac{\beta}{E^2} - \frac{\alpha}{4E^2} \right), & E > \frac{\alpha^2}{2\beta} \\ 0 & E < \frac{\alpha^2}{2\beta} \end{cases}$

e) $\sigma = \begin{cases} \pi \left(\frac{\beta}{E^2} - \frac{\alpha}{4E^2} \right), & E > \frac{\beta^2}{4\gamma} \\ 0 & E < \frac{\beta^2}{4\gamma} \end{cases}$

6. $d\sigma = \frac{\pi\alpha^2 dO}{\varepsilon_{\text{nnv}}}.$

6.1-misollar

1. $I_{11} = I_{22} = \frac{\mu l^2}{12}, \quad I_{33} = 0,$

2. $I_{11} = I_{22} = I_{33} = \frac{2\mu R^2}{5}.$

3. $I_{11} = I_{22} = \frac{m}{R^3 - r^3} \left[\frac{2}{5} (R^5 - r^5) - \frac{(R-r)^2 r^3 R^3}{R^3 - r^3} \right], \quad I_{33} = \frac{m}{R^3 - r^3} \frac{2}{5} (R^5 - r^5).$

4., $I_{11} = I_{22} = \frac{3}{20} \mu \left(R^2 + \frac{h^2}{4} \right), \quad I_{33} = \frac{3}{10} \mu R^2.$

5. $I_{11} = I_{22} = \frac{1}{\mu} \sum_{a \neq b} m_a m_b I_{ab}^2, \quad I_{33} = 0$

6. $I_{11} = \frac{2m_1 m_2}{\mu} h^3, \quad I_{22} = \frac{m_1}{2} a^3, \quad I_{33} = I_{11} + I_{22}, \quad a, h - \text{asos va balandlik}$

ILOVA

Ba'zi muhim matematik qo'shimchalar.

I. Egri chiziqli ortogonal koordinata sistemasida fazoviy hosilalarning ifodalari.

Nuqtaning egri chiziqli koordinatalarini q_1, q_2, q_3 deb, ularning Dekart koordinatalari orqali boshlanishlarini yozib olaylik:

$$\begin{aligned} q_1 &= q_1(x, y, z) \\ q_2 &= q_2(x, y, z) \\ q_3 &= q_3(x, y, z) \end{aligned} \quad (1.1)$$

Va aksincha

$$\vec{r} = \vec{r}(q_1, q_2, q_3) \quad (1.2)$$

Dekart va ortogonal egri chiziqli koordinata sistemalarining ortlari (birlik vektorlari)

$$\begin{aligned} \vec{i}, \vec{j}, \vec{k} &\rightarrow \vec{e}_1, \vec{e}_2, \vec{e}_3 \\ (\vec{e}_i, \vec{e}_i) &= \delta_{ii}, \quad (\vec{e}_i, [\vec{e}_2, \vec{e}_3]) = 1 \end{aligned} \quad (1.3)$$

Radius-vektor \vec{r} egri chiziqli koordinatalarning funksiyasi bo'lganligidan:

$$d\vec{r} = \sum \frac{\partial \vec{r}}{\partial q_i} dq_i = \sum \left| \frac{\partial \vec{r}}{\partial q_i} \right| dq_i \vec{e}_i \quad (1.4)$$

Quyidagi belgilashlarni kiritaylik

$$h_i = \left| \frac{d\vec{r}}{dq_i} \right| = \sqrt{\left(\frac{d\vec{r}}{dq_i} \right)^2} = \sqrt{\left(\frac{\partial x}{\partial q_i} \right)^2 + \left(\frac{\partial y}{\partial q_i} \right)^2 + \left(\frac{\partial z}{\partial q_i} \right)^2} \quad (1.5)$$

Bu h_i koeffitsiyentlar **Lame koeffitsiyentlari** deb nomlanadi.

Demak radius-vektorni egri chiziqli koordinatalardagi ifodasi quyidagicha

$$d\vec{r} = \sum h_i dq_i \vec{e}_i \quad (1.6)$$

Uning modulini $|d\vec{r}| = dS$ deb belgilasak

$$dS^2 = \sum h_i^2 dq_i^2 \quad (1.7)$$

bo'ladi. Demak radius-vektorni egri chiziqli koordinatalardagi ifodasi

quyidagicha:

$$d\vec{S}_i = dS_i \vec{e}_i = h_i dq_i \vec{e}_i$$

$$d\sigma_i = [d\vec{S}_i, d\vec{S}_k] = h_i h_k dq_i dq_k [\vec{e}_i, \vec{e}_k] \quad (1.8)$$

$$d\vec{r} = (d\vec{S}_1 [d\vec{S}_2, d\vec{S}_3]) = h_1 h_2 h_3 dq_1 dq_2 dq_3$$

Bu aniqlanishlar orqali muhim ahamiyatga ega bo'lgan ba'zi operatorlarning ifodasini keltiraylik:

$$1. \text{grad} \phi = \vec{\nabla} \phi = \sum \frac{1}{h_i} \frac{\partial \phi}{\partial q_i} \vec{e}_i \quad (1.09)$$

$$2. \text{div} \vec{a} = (\vec{\nabla} \vec{a}) = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} (a_1 h_2 h_3) + \frac{\partial}{\partial q_2} (a_2 h_1 h_3) + \frac{\partial}{\partial q_3} (a_3 h_1 h_2) \right\} \quad (1.10)$$

$$\begin{aligned} 3. \text{rot} \vec{a} = [\nabla \vec{a}] &= \frac{1}{h_3 h_2} \left\{ \frac{\partial}{\partial q_2} (a_3 h_3) - \frac{\partial}{\partial q_3} (a_2 h_2) \right\} \vec{e}_1 + \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_2} (a_3 h_3) - \frac{\partial}{\partial q_3} (a_2 h_2) \right\} \vec{e}_2 + \\ &+ \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_1} (a_2 h_2) - \frac{\partial}{\partial q_2} (a_1 h_1) \right\} \vec{e}_3. \end{aligned} \quad (1.11)$$

$$4. \Delta \phi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \phi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial q_3} \right) \right\} \quad (1.12)$$

A. Sferik koordinatalar sistemasi

$$x = r \cdot \sin \theta \cos \varphi, \quad y = r \cdot \sin \theta \sin \varphi, \quad z = r \cdot \cos \theta.$$

$$h_i = \left| \frac{d\vec{r}}{dq_i} \right| = \sqrt{\left(\frac{d\vec{r}}{dq_i} \right)^2} = \sqrt{\left(\frac{\partial x}{\partial q_i} \right)^2 + \left(\frac{\partial y}{\partial q_i} \right)^2 + \left(\frac{\partial z}{\partial q_i} \right)^2}$$

$$h_1 = \left| \frac{d\vec{r}}{dr} \right| = \sqrt{\left(\frac{d\vec{r}}{dr} \right)^2} = \sqrt{\left(\frac{\partial (r \cdot \sin \theta \cos \varphi)}{\partial r} \right)^2 + \left(\frac{\partial (r \cdot \sin \theta \sin \varphi)}{\partial r} \right)^2 + \left(\frac{\partial (r \cdot \cos \theta)}{\partial r} \right)^2} = 1$$

$$h_2 = \left| \frac{d\vec{r}}{d\theta} \right| = \sqrt{\left(\frac{d\vec{r}}{d\theta} \right)^2} = \sqrt{\left(\frac{\partial (r \cdot \sin \theta \cos \varphi)}{\partial \theta} \right)^2 + \left(\frac{\partial (r \cdot \sin \theta \sin \varphi)}{\partial \theta} \right)^2 + \left(\frac{\partial (r \cdot \cos \theta)}{\partial \theta} \right)^2} = r$$

$$h_3 = \left| \frac{d\vec{r}}{d\varphi} \right| = \sqrt{\left(\frac{d\vec{r}}{d\varphi} \right)^2} = \sqrt{\left(\frac{\partial(r \cdot \sin \theta \cos \varphi)}{\partial \varphi} \right)^2 + \left(\frac{\partial(r \cdot \sin \theta \sin \varphi)}{\partial \varphi} \right)^2 + \left(\frac{\partial(r \cos \theta)}{\partial \varphi} \right)^2} = r \sin \theta.$$

$$h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_\varphi = r \sin \theta$$

$$dS^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\varphi)^2$$

$$1. \text{ grad } \phi = \nabla \phi = \sum \frac{1}{h_i} \frac{\partial \phi}{\partial q_i} \vec{e}_i = \frac{1}{h_r} \frac{\partial \phi}{\partial r} \vec{e}_r + \frac{1}{h_\theta} \frac{\partial \phi}{\partial \theta} \vec{e}_\theta + \frac{1}{h_\varphi} \frac{\partial \phi}{\partial \varphi} \vec{e}_\varphi \Rightarrow \frac{\partial \phi}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \vec{e}_\varphi.$$

$$2. \text{ div } \vec{a} = (\nabla \vec{a}) = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} (a_1 h_2 h_3) + \frac{\partial}{\partial q_2} (a_2 h_3 h_1) + \frac{\partial}{\partial q_3} (a_3 h_1 h_2) \right\} \Rightarrow$$

$$= \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} (a_1 r^2 \sin \theta) + \frac{\partial}{\partial \theta} (a_2 r \sin \theta) + \frac{\partial}{\partial \varphi} (a_3 r) \right\};$$

3).

$$\text{rot } \vec{a} = [\nabla \vec{a}] = \frac{1}{h_3 h_2} \left\{ \frac{\partial}{\partial q_2} (a_3 h_1) - \frac{\partial}{\partial q_3} (a_2 h_1) \right\} \vec{e}_1 + \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_2} (a_3 h_1) - \frac{\partial}{\partial q_3} (a_2 h_1) \right\} \vec{e}_2 +$$

$$+ \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_2} (a_2 h_1) - \frac{\partial}{\partial q_3} (a_1 h_1) \right\} \vec{e}_3 \Rightarrow \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial \theta} (a_3 r \sin \theta) - \frac{\partial}{\partial \varphi} (a_2 r) \right\} \vec{e}_r +$$

$$+ \frac{1}{r} \left\{ \frac{\partial}{\partial \theta} (a_3 r \sin \theta) - \frac{\partial}{\partial \varphi} (a_2 r) \right\} \vec{e}_\theta + \frac{1}{r} \left\{ \frac{\partial}{\partial r} (a_2 r) - \frac{\partial}{\partial \theta} (a_1) \right\} \vec{e}_\varphi;$$

$$4. \Delta \phi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial q_3} \right) \right\} \Rightarrow$$

$$= \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{r \sin \theta}{r} \frac{\partial \phi}{\partial \theta} \right) + \frac{\partial}{\partial \varphi} \left(\frac{r}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right) \right\};$$

B. Silindrik koordinata sistemasi:

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad z = z.$$

$$h_i = \left| \frac{d\vec{r}}{dq_i} \right| = \sqrt{\left(\frac{d\vec{r}}{dq_i} \right)^2} = \sqrt{\left(\frac{\partial x}{\partial q_i} \right)^2 + \left(\frac{\partial y}{\partial q_i} \right)^2 + \left(\frac{\partial z}{\partial q_i} \right)^2}$$

$$h_1 = \left| \frac{d\vec{r}}{d\rho} \right| = \sqrt{\left(\frac{d\vec{r}}{d\rho} \right)^2} = \sqrt{\left(\frac{\partial(\rho \cos \varphi)}{\partial \rho} \right)^2 + \left(\frac{\partial(\rho \sin \varphi)}{\partial \rho} \right)^2 + \left(\frac{\partial z}{\partial \rho} \right)^2} = 1;$$

$$h_2 = \left| \frac{d\vec{r}}{d\varphi} \right| = \sqrt{\left(\frac{d\vec{r}}{d\varphi} \right)^2} = \sqrt{\left(\frac{\partial(\rho \cos \varphi)}{\partial \varphi} \right)^2 + \left(\frac{\partial(\rho \sin \varphi)}{\partial \varphi} \right)^2 + \left(\frac{\partial z}{\partial \varphi} \right)^2} = \rho;$$

$$h_3 = \left| \frac{d\vec{r}}{dz} \right| = \sqrt{\left(\frac{d\vec{r}}{dz} \right)^2} = \sqrt{\left(\frac{\partial(\rho \cos \varphi)}{\partial z} \right)^2 + \left(\frac{\partial(\rho \sin \varphi)}{\partial z} \right)^2 + \left(\frac{\partial z}{\partial z} \right)^2} = 1.$$

$$h_1 = h_\rho = 1, \quad h_2 = h_\varphi = \rho, \quad h_3 = h_z = 1.$$

$$dS^2 = (d\rho)^2 + \rho^2 (d\varphi)^2 + (dz)^2.$$

$$1. \text{ grad } \phi = \nabla \phi = \sum \frac{1}{h_i} \frac{\partial \phi}{\partial q_i} \vec{e}_i = \frac{1}{h_\rho} \frac{\partial \phi}{\partial \rho} \vec{e}_\rho + \frac{1}{h_\varphi} \frac{\partial \phi}{\partial \varphi} \vec{e}_\varphi + \frac{1}{h_z} \frac{\partial \phi}{\partial z} \vec{e}_z \Rightarrow \frac{\partial \phi}{\partial \rho} \vec{e}_\rho + \frac{1}{\rho} \frac{\partial \phi}{\partial \varphi} \vec{e}_\varphi + \frac{\partial \phi}{\partial z} \vec{e}_z;$$

$$2. \text{ div } \vec{a} = (\nabla \vec{a}) = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} (a_1 h_2 h_3) + \frac{\partial}{\partial q_2} (a_2 h_3 h_1) + \frac{\partial}{\partial q_3} (a_3 h_1 h_2) \right\} \Rightarrow$$

$$= \frac{1}{\rho} \left\{ \frac{\partial}{\partial \rho} (a_1 \rho) + \frac{\partial}{\partial \varphi} (a_2) + \frac{\partial}{\partial z} (a_3 \rho) \right\};$$

3).

$$\text{rot } \vec{a} = [\nabla \vec{a}] = \frac{1}{h_3 h_2} \left\{ \frac{\partial}{\partial q_2} (a_3 h_1) - \frac{\partial}{\partial q_3} (a_2 h_1) \right\} \vec{e}_1 + \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_2} (a_3 h_1) - \frac{\partial}{\partial q_3} (a_2 h_1) \right\} \vec{e}_2 +$$

$$+ \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_2} (a_2 h_1) - \frac{\partial}{\partial q_3} (a_1 h_1) \right\} \vec{e}_3 = \frac{1}{\rho} \left\{ \frac{\partial}{\partial \varphi} (a_3) - \frac{\partial}{\partial z} (a_2 \rho) \right\} \vec{e}_\rho + \frac{1}{\rho} \left\{ \frac{\partial}{\partial \varphi} (a_3) - \frac{\partial}{\partial z} (a_2 \rho) \right\} \vec{e}_\varphi +$$

$$h_3 = \left| \frac{d\vec{r}}{d\varphi} \right| = \sqrt{\left(\frac{d\vec{r}}{d\varphi} \right)^2} = \sqrt{\left(\frac{\partial(r \cdot \sin \theta \cos \varphi)}{\partial \varphi} \right)^2 + \left(\frac{\partial(r \cdot \sin \theta \sin \varphi)}{\partial \varphi} \right)^2 + \left(\frac{\partial(r \cos \theta)}{\partial \varphi} \right)^2} = r \sin \theta.$$

$$h_1 = h_r = 1, \quad h_2 = h_\theta = r, \quad h_3 = h_\varphi = r \sin \theta$$

$$dS^2 = (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\varphi)^2$$

$$1. \text{grad} \phi = \vec{\nabla} \phi = \sum \frac{1}{h_i} \frac{\partial \phi}{\partial q_i} \vec{e}_i = \frac{1}{h_r} \frac{\partial \phi}{\partial r} \vec{e}_r + \frac{1}{h_\theta} \frac{\partial \phi}{\partial \theta} \vec{e}_\theta + \frac{1}{h_\varphi} \frac{\partial \phi}{\partial \varphi} \vec{e}_\varphi \Rightarrow \frac{\partial \phi}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \vec{e}_\varphi.$$

$$2). \text{div} \vec{a} = (\vec{\nabla} \cdot \vec{a}) = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} (a_1 h_2 h_3) + \frac{\partial}{\partial q_2} (a_2 h_3 h_1) + \frac{\partial}{\partial q_3} (a_3 h_1 h_2) \right\} \Rightarrow$$

$$= \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} (a_1 r^2 \sin \theta) + \frac{\partial}{\partial \theta} (a_2 r \sin \theta) + \frac{\partial}{\partial \varphi} (a_3 r) \right\};$$

3).

$$\text{rot} \vec{a} = [\vec{\nabla} \vec{a}] = \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_2} (a_3 h_1) - \frac{\partial}{\partial q_3} (a_2 h_1) \right\} \vec{e}_r + \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_2} (a_3 h_1) - \frac{\partial}{\partial q_3} (a_2 h_1) \right\} \vec{e}_\theta +$$

$$+ \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_2} (a_2 h_1) - \frac{\partial}{\partial q_3} (a_1 h_1) \right\} \vec{e}_\varphi \Rightarrow \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial \theta} (a_3 r \sin \theta) - \frac{\partial}{\partial \varphi} (a_2 r) \right\} \vec{e}_r +$$

$$+ \frac{1}{r} \left\{ \frac{\partial}{\partial \theta} (a_3 r \sin \theta) - \frac{\partial}{\partial \varphi} (a_2 r) \right\} \vec{e}_\theta + \frac{1}{r} \left\{ \frac{\partial}{\partial r} (a_2 r) - \frac{\partial}{\partial \theta} (a_1) \right\} \vec{e}_\varphi;$$

$$4). \Delta \phi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial q_3} \right) \right\} \Rightarrow$$

$$= \frac{1}{r^2 \sin \theta} \left\{ \frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{r \sin \theta}{r} \frac{\partial \phi}{\partial \theta} \right) + \frac{\partial}{\partial \varphi} \left(\frac{r}{r \sin \theta} \frac{\partial \phi}{\partial \varphi} \right) \right\};$$

B. Silindrik koordinata sistemasi:

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad z = z.$$

$$h_i = \left| \frac{d\vec{r}}{dq_i} \right| = \sqrt{\left(\frac{d\vec{r}}{dq_i} \right)^2} = \sqrt{\left(\frac{\partial x}{\partial q_i} \right)^2 + \left(\frac{\partial y}{\partial q_i} \right)^2 + \left(\frac{\partial z}{\partial q_i} \right)^2}$$

$$h_1 = \left| \frac{d\vec{r}}{d\rho} \right| = \sqrt{\left(\frac{d\vec{r}}{d\rho} \right)^2} = \sqrt{\left(\frac{\partial(\rho \cos \varphi)}{\partial \rho} \right)^2 + \left(\frac{\partial(\rho \sin \varphi)}{\partial \rho} \right)^2 + \left(\frac{\partial z}{\partial \rho} \right)^2} = 1;$$

$$h_2 = \left| \frac{d\vec{r}}{d\varphi} \right| = \sqrt{\left(\frac{d\vec{r}}{d\varphi} \right)^2} = \sqrt{\left(\frac{\partial(\rho \cos \varphi)}{\partial \varphi} \right)^2 + \left(\frac{\partial(\rho \sin \varphi)}{\partial \varphi} \right)^2 + \left(\frac{\partial z}{\partial \varphi} \right)^2} = \rho;$$

$$h_3 = \left| \frac{d\vec{r}}{dz} \right| = \sqrt{\left(\frac{d\vec{r}}{dz} \right)^2} = \sqrt{\left(\frac{\partial(\rho \cos \varphi)}{\partial z} \right)^2 + \left(\frac{\partial(\rho \sin \varphi)}{\partial z} \right)^2 + \left(\frac{\partial z}{\partial z} \right)^2} = 1.$$

$$h_1 = h_\rho = 1, \quad h_2 = h_\varphi = \rho, \quad h_3 = h_z = 1.$$

$$dS^2 = (d\rho)^2 + \rho^2 (d\varphi)^2 + (dz)^2.$$

$$1). \text{grad} \phi = \vec{\nabla} \phi = \sum \frac{1}{h_i} \frac{\partial \phi}{\partial q_i} \vec{e}_i = \frac{1}{h_\rho} \frac{\partial \phi}{\partial \rho} \vec{e}_\rho + \frac{1}{h_\varphi} \frac{\partial \phi}{\partial \varphi} \vec{e}_\varphi + \frac{1}{h_z} \frac{\partial \phi}{\partial z} \vec{e}_z \Rightarrow \frac{\partial \phi}{\partial \rho} \vec{e}_\rho + \frac{1}{\rho} \frac{\partial \phi}{\partial \varphi} \vec{e}_\varphi + \frac{\partial \phi}{\partial z} \vec{e}_z;$$

$$2). \text{div} \vec{a} = (\vec{\nabla} \cdot \vec{a}) = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} (a_1 h_2 h_3) + \frac{\partial}{\partial q_2} (a_2 h_3 h_1) + \frac{\partial}{\partial q_3} (a_3 h_1 h_2) \right\} \Rightarrow$$

$$= \frac{1}{\rho} \left\{ \frac{\partial}{\partial \rho} (a_1 \rho) + \frac{\partial}{\partial \varphi} (a_2) + \frac{\partial}{\partial z} (a_3 \rho) \right\};$$

3).

$$\text{rot} \vec{a} = [\vec{\nabla} \vec{a}] = \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_2} (a_3 h_1) - \frac{\partial}{\partial q_3} (a_2 h_1) \right\} \vec{e}_1 + \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_2} (a_3 h_1) - \frac{\partial}{\partial q_3} (a_2 h_1) \right\} \vec{e}_2 +$$

$$+ \frac{1}{h_1 h_2} \left\{ \frac{\partial}{\partial q_2} (a_2 h_1) - \frac{\partial}{\partial q_3} (a_1 h_1) \right\} \vec{e}_3 = \frac{1}{\rho} \left\{ \frac{\partial}{\partial \varphi} (a_3) - \frac{\partial}{\partial z} (a_2 \rho) \right\} \vec{e}_\rho + \frac{1}{\rho} \left\{ \frac{\partial}{\partial \varphi} (a_3) - \frac{\partial}{\partial z} (a_2 \rho) \right\} \vec{e}_\varphi +$$

$$+ \frac{1}{\rho} \left\{ \frac{\partial}{\partial \varphi} (a_2 \rho) - \frac{\partial}{\partial z} (a_1) \right\} \vec{e}_z;$$

$$4). \Delta \phi = \frac{1}{h_1 h_2 h_3} \left\{ \frac{\partial}{\partial q_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \phi}{\partial q_1} \right) + \frac{\partial}{\partial q_2} \left(\frac{h_1 h_3}{h_2} \frac{\partial \phi}{\partial q_2} \right) + \frac{\partial}{\partial q_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \phi}{\partial q_3} \right) \right\} \Rightarrow$$

$$= \frac{1}{\rho} \left\{ \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \phi}{\partial \rho} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{\rho} \frac{\partial \phi}{\partial \varphi} \right) + \frac{\partial}{\partial z} \left(\rho \frac{\partial \phi}{\partial z} \right) \right\};$$

2. Birlik psevdotenzorning birlik tenzor orqali ifodalanishi.

Determinantlar nazariyasidan foydalanib, birlik psevdotenzorni birlik tenzor komponentalaridan tashkil topgan determinant shaklida yozish mumkin.

$$\varepsilon_{i_1 i_2 \dots i_n} = \begin{vmatrix} \delta_{1i_1} & \delta_{1i_2} & \dots & \delta_{1i_n} \\ \delta_{2i_1} & \delta_{2i_2} & \dots & \delta_{2i_n} \\ \dots & \dots & \dots & \dots \\ \delta_{ni_1} & \delta_{ni_2} & \dots & \delta_{ni_n} \end{vmatrix}$$

yoki i_1, i_2, \dots, i_n indekslar o'rniga j_1, j_2, \dots, j_n indekslar olinsa:

$$\varepsilon_{j_1 j_2 \dots j_n} = \begin{vmatrix} \delta_{1j_1} & \delta_{1j_2} & \dots & \delta_{1j_n} \\ \delta_{2j_1} & \delta_{2j_2} & \dots & \delta_{2j_n} \\ \dots & \dots & \dots & \dots \\ \delta_{nj_1} & \delta_{nj_2} & \dots & \delta_{nj_n} \end{vmatrix}$$

boladi. Bu ikki psevdotenzorlar ko'paytmasini tuzaylik:

$$\varepsilon_{i_1 i_2 \dots i_n} \varepsilon_{j_1 j_2 \dots j_n} = \begin{vmatrix} \delta_{1i_1} & \delta_{1i_2} & \dots & \delta_{1i_n} & \left| \delta_{1j_1} & \delta_{1j_2} & \dots & \delta_{1j_n} \right. \\ \delta_{2i_1} & \delta_{2i_2} & \dots & \delta_{2i_n} & \left| \delta_{2j_1} & \delta_{2j_2} & \dots & \delta_{2j_n} \right. \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \delta_{ni_1} & \delta_{ni_2} & \dots & \delta_{ni_n} & \left| \delta_{nj_1} & \delta_{nj_2} & \dots & \delta_{nj_n} \right. \end{vmatrix}$$

Mos ustunlar elementlarini ko'paytirib qo'shish qoidasi va birlik tenzor ta'rifiga muvofiq bunday yozamiz:

$$\delta_{1i_1} \delta_{1j_1} + \delta_{2i_1} \delta_{2j_1} + \dots + \delta_{ni_1} \delta_{nj_1} = \delta_{i_1 j_1}$$

$$\delta_{1i_1} \delta_{1j_2} + \delta_{2i_1} \delta_{2j_2} + \dots + \delta_{ni_1} \delta_{nj_2} = \delta_{i_1 j_2}$$

$$\dots$$

$$\delta_{1i_n} \delta_{1j_n} + \delta_{2i_n} \delta_{2j_n} + \dots + \delta_{ni_n} \delta_{nj_n} = \delta_{i_n j_n}$$

Demak ikki psevdotenzorlar ko'paytmasining elementlari birlik tenzorlardan iborat determinant ekan:

$$\varepsilon_{i_1 i_2 \dots i_n} \varepsilon_{j_1 j_2 \dots j_n} = \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \dots & \delta_{i_1 j_n} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \dots & \delta_{i_2 j_n} \\ \dots & \dots & \dots & \dots \\ \delta_{i_n j_1} & \delta_{i_n j_2} & \dots & \delta_{i_n j_n} \end{vmatrix}$$

Jumladan, uch o'lchovli fazoda bu ifoda quyidagi ko'rinishni oladi:

$$\varepsilon_{i_1 i_2 i_3} \varepsilon_{j_1 j_2 j_3} = \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} & \delta_{i_1 j_3} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} & \delta_{i_2 j_3} \\ \delta_{i_3 j_1} & \delta_{i_3 j_2} & \delta_{i_3 j_3} \end{vmatrix}$$

Bu determinantni ochib yozadigan bo'lsak ushuni olamiz:

$$\begin{aligned} \varepsilon_{i_1 i_2 i_3} \varepsilon_{j_1 j_2 j_3} &= \delta_{i_1 j_1} \delta_{i_2 j_2} \delta_{i_3 j_3} + \delta_{i_1 j_2} \delta_{i_2 j_3} \delta_{i_3 j_1} + \delta_{i_1 j_3} \delta_{i_2 j_1} \delta_{i_3 j_2} - \\ &- \delta_{i_1 j_2} \delta_{i_2 j_1} \delta_{i_3 j_3} - \delta_{i_1 j_1} \delta_{i_2 j_3} \delta_{i_3 j_2} - \delta_{i_1 j_3} \delta_{i_2 j_2} \delta_{i_3 j_1}. \end{aligned}$$

Mazkur ifodani mos indekslar bo'yicha yig'ishirish mumkin.

a) Masalan, $i_3 = j_3$ bo'lsin, u holda $\delta_{j_3 j_3} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ bo'ladi, hamda ushbularni hisobga olsak

$$\delta_{i_2 j_3} \delta_{j_3 i_1} = \delta_{i_2 i_1}, \quad \delta_{i_1 j_3} \delta_{j_3 i_2} = \delta_{i_1 i_2},$$

$$\delta_{i_2 j_3} \delta_{j_3 i_2} = \delta_{i_2 i_2}, \quad \delta_{i_1 j_3} \delta_{j_3 i_1} = \delta_{i_1 i_1}.$$

determinant natijasini quyidagicha yozib olamiz:

$$\begin{aligned} \varepsilon_{i_1 i_2 i_3} \varepsilon_{j_1 j_2 j_3} &= 3\delta_{i_2 j_2} \delta_{i_1 j_1} + \delta_{i_2 j_2} \delta_{i_1 j_1} + \delta_{i_2 j_1} \delta_{i_1 j_2} - 3\delta_{i_2 j_2} \delta_{i_1 j_1} - \delta_{i_1 j_2} \delta_{i_2 j_1} - \delta_{i_2 j_1} \delta_{i_1 j_2} = \\ &= \delta_{i_1 j_1} \delta_{i_2 j_2} - \delta_{i_2 j_1} \delta_{i_1 j_2} = \begin{vmatrix} \delta_{i_1 j_1} & \delta_{i_1 j_2} \\ \delta_{i_2 j_1} & \delta_{i_2 j_2} \end{vmatrix} \end{aligned}$$

b) So'ngi ifodada $i_2 = j_2$ deb hissoqlab ,yana yig'ishtirish amalini ishlataylik:

$$\varepsilon_{i_1 i_2 i_3} \varepsilon_{j_1 i_2 j_3} = \delta_{i_1 j_1} \delta_{i_2 i_2} - \delta_{i_1 i_2} \delta_{i_2 j_1} = 3\delta_{i_1 j_1} - \delta_{i_1 j_1} = 2\delta_{i_1 j_1}.$$

v) Va nihoyat, bu natijani ham $i_1 = j_1$ deb hissoqlab , so'ngra yig'ishtirsak:

$$\varepsilon_{i_1 i_2 i_3} \varepsilon_{i_1 i_2 i_3} = 2\delta_{i_1 i_1} = 6$$

Yuqoridagi natijalarning amaliyotdagi roli muhimligi uchun ularni uch o'lchovli fazoda ko'p ishlatiladigan an'anaviy ko'rinishlarini qayta keltiraylik:

$$\begin{aligned} \varepsilon_{ikn} \varepsilon_{lmn} &= 3\delta_{il} \delta_{km} + \delta_{im} \delta_{kl} + \delta_{kl} \delta_{im} - 3\delta_{im} \delta_{kl} - \delta_{il} \delta_{km} - \delta_{km} \delta_{il} = \delta_{il} \delta_{km} - \delta_{im} \delta_{kl}, \\ \varepsilon_{ikn} \varepsilon_{lmn} &= \delta_{il} \delta_{km} - \delta_{im} \delta_{kl}, \quad \varepsilon_{ikn} \varepsilon_{lkn} = 2\delta_{il}, \quad \varepsilon_{ikn} \varepsilon_{ikn} = 6. \end{aligned}$$

$$\varepsilon_{ijk} = \begin{cases} 1 \\ -1 \\ 0 \end{cases}$$

ε_{123} - indekslar siklik almashtirilganda.

ε_{213} - indekslar siklik almashtirilganda.

ijk larning boshqa qiymatlarida.

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