SHAROF RASHIDOV NOMIDAGI SAMARQAND DAVLAT UNIVERSITETI HUZURIDAGI ILMIY DARAJALAR BERUVCHI DSc.03/30.07.2019.FM.02.01 RAQAMLI ILMIY KENGASH

SHAROF RASHIDOV NOMIDAGI SAMARQAND DAVLAT UNIVERSITETI V.I.ROMANOVSKIY NOMIDAGI MATEMATIKA INSTITUTI

SAFAROV AKBAR RAXMANOVICH

UMUMLASHGAN TEBRANUVCHAN INTEGRALLAR VA ULARNING TATBIQLARI

01.01.01 –Matematik analiz

FIZIKA–MATEMATIKA FANLARI DOKTORI (DSc) DISSERTATSIYASI AVTOREFERATI

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Dissertatsiya mavzusining dolzarbligi va zarurati. Jahonda olib borilayotgan ko‘plab ilmiy-amaliy tadqiqotlar aksariyat hollarda tebranuvchan integrallarning tekis bahosini tadqiq qilishga olib kelinadi. Bu albatta, ushbu mavzu bo‘yicha mavjud bo‘lgan keng qamrovli adabiyotlarda o‘z aksini topgan. XX asrning oltmishinchi yillarida kiritilgan trigonometrik yig‘indilar usuli hamda bu yig‘indilarning asimptotik xarakterini trigonometrik integrallar orqali ifodalanishi tebranuvchan integrallar nazariyasining rivojlanishiga asos bo‘ldi. Tebranuvchan integrallarning yana bir muhim jihatlaridan biri ularning analitik sonlar nazariyasiga tatbig‘i hisoblanadi. Bunday integrallarning xarakterini tadqiq etishning ikki jihati mavjud bo‘lib, ulardan biri trigonometrik yig‘indilar bilan, boshqasi trigonometrik integrallar bilan bog‘liq. Tebranuvchan integrallarning xarakterini tadqiq qilishga oid tadqiqotlarni rivojlantirish muhim vazifalardan biri bo‘lib qolmoqda.

Jahonda fundamental fanlarning amaliy tatbiqiga ega bo‘lgan matematik analizning dolzarb yo‘nalishlariga e’tibor kuchaytirildi. Jumladan, tebranuvchan integrallarning xarakterini o‘rganishga alohida e’tibor qaratildi. Yevklid fazosining qavariq gipersirtlarida mujassamlangan silliq Borel o‘lchovlarining Furye almashtirishlarini tadqiq qilish hamda gipersirtlarda Furye almashtirishlarining chegaralanganlik muammosi bo‘yicha salmoqli natijalarga erishildi. Algebra va matematik analiz, dinamik tizimlar nazariyasi, amaliy matematika va matematik moddellashtirish matematika fanlarining ustivor yo‘nalishlari bo‘yicha xalqaro standartlar darajasida ilmiy tadqiqotlar olib borish asosiy vazifalar va faoliyat yo‘nalishlari etib belgilanishi amaliy-nazariy jihatdan muhim ilmiy tadqiqotlardan hisoblanadi.

Respublikamizda ilmiy va amaliy ahamiyatga ega bo‘lgan fundamental fanlarga katta e’tibor berilmoqda, xususan, garmonik analiz nazariyasi masalalarini rivojlantirishga alohida e’tibor berilmoqda. “Algebra va uning tatbiqlari, differensial tenglamalar va ularning tatbiqlari, chiziqsiz tizimlar, dinamik tizimlar va ularning tatbiqlarini matematik modellashtirish, stoxastik tahlil, tibbiy-biologik informatika, hisoblash matematikasi[1]” fanlarining ustuvor yo‘nalishlari bo‘yicha xalqaro standartlar darajasida ilmiy tadqiqotlar olib borish matematika fanining asosiy vazifalari va faoliyat yo‘nalishlari etib belgilandi. Bu qaror ijrosini ta’minlashda garmonik analiz masalalarining spektral nazariyasini rivojlantirish muhim ilmiy ahamiyatga ega hisoblanadi.

O‘zbekiston Respublikasi Prezidentining 2017-yil 7-fevraldagi PF-4947-son «O‘zbekiston Respublikasini yanada rivojlantirish bo‘yicha harakatlar strategiyasi to‘g‘risida»gi Farmoni, 2019-yil 9-iyuldagi PQ-4387-son «Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qo‘llab-quvvatlash, shuningdek, O‘zbekiston Respublikasi Fanlar Akademiyasining V.I.Romanovskiy nomidagi Matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to‘g‘risida»

[1] O‘zbekiston Respublikasi Prezidentining 2019-yil 9-iyuldagi PQ-4387-son “Matematika ta’limi va fanlarini yanada rivojlantirishni davlat tomonidan qo’llab-quvvatlash, shuningdek, O’zbekiston Respublikasi Fanlar Akademiyasining V.I.Romanovskiy nomidagi Matematika instituti faoliyatini tubdan takomillashtirish chora-tadbirlari to’g’risida”gi qarori

va 2020-yil 7-maydagi PQ-4708-son «Matematika sohasidagi ta’lim sifatini oshirish va ilmiy-tadqiqotlarni rivojlantirish chora-tadbirlari to‘g‘risida»gi qarorlari hamda mazkur faoliyatga tegishli boshqa normativ-huquqiy hujjatlarda belgilangan vazifalarni amalga oshirishda ushbu dissertatsiya tadqiqoti muayyan darajada xizmat qiladi.

Tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga mosligi. Ushbu tadqiqot O‘zbekiston Respublikasi fan va texnikasining IV, “Matematika, mexanika va informatika” ustuvor yo‘nalishlariga muvofiq bajarilgan.

Dissertatsiya mavzusi bo‘yicha xorijiy ilmiy-tadqiqotlar sharhi[2]. Umumlashgan tebranuvchan integrallar va ularning tatbiqlariga oid tadqiqotlar va ilmiy izlanishlar yetakchi xorijiy davlatlarning oliy ta’lim muassalari va ilmiy-tadqiqot markazlari, jumladan, Ghent Analysis & PDE Center (Belgiya), School of Mathematical Sciences, Queen Mary University of London (Angliya), Christian-Albrechts-Universität zu Kiel (Germaniya), Ganja State University (Ozarbayjon), Moskva davlat universiteti (Rossiya) da keng qamrovli izlinishlar olib borilmoqda.

Umumlashgan tebranuvchan integrallar va ularning tatbiqlariga oid dunyoda olib borilgan tadqiqotlar natijasida qator dolzarb masalalar yechilgan, jumladan, quyidagi ilmiy natijalar olingan: bir o‘zgaruvchili umumlashgan tebranuvchan integrallarning baholari olingan (Ghent Analysis & PDE Center (Belgiya), School of Mathematical Sciences, Queen Mary University of London (Angliya)); tebranuvchan integrallarning invariant baholari topilgan (Moskva davlat universiteti (Rossiya)); Fazasi ko‘phad bo‘lgan tebranuvchan integrallar uchun jamlash ko‘rsatkichlari xususiy holda topilgan Ganja State University (Ozarbayjon)).

Dunyo miqyosida bugungi kunda umumlashgan tebranuvchan integrallar va ularning tatbiqlarini tadqiq etish bo‘yicha bir qator, jumladan: fazasi ko‘phad bo‘lgan tebranuvchan integrallarning invariant baholarini olish, fazasi maxsus ko‘rinishli ko‘p karrali tebranuvchan integrallarning tekis baholarini olish, fazasi ikki o‘zgaruvchili uchinchi darajali bir jinsli ko‘phad bo‘lgan tebranuvchan integrallarning yig‘ish ko‘rsatkichini topish, Mittag-Leffler funksiyasi bilan bog‘langan tebranuvchan integrallarning baholarini olish, Mittag-Leffler funksiyasi bilan bog‘langan tebranuvchan integrallarning asimptotikasini topish kabi ustuvor yo‘nalishlarda ilmiy tadqiqot ishlari olib borilmoqda.

Muammoning o‘rganilganlik darajasi. Tebranuvchan (trigonometrik) integrallar dastlab 1818 yil Frenelning yorug‘likning intensivligini o‘rganish bilan bog‘liq ishlarida vujudga kelgan. U bunday integrallardan foydalanib chetlardagi, ekrandagi, kichik tuynuk difraksiyalari masalalarini yechgan. 1838-yilda Eyri trigonometrik integrallardan kaustikaning atrofida yorug‘likning intensivligini o‘rganishda foydalandi. Kaustika – bu yorug‘likning intensivligi boshqa odatdagi nuqtalardagidan ancha yuqori bo‘lgan to‘plamidir. Keyinchalik, 1887-yilda Kelvin va Rimanning tadqiqotlarida tebranuvchan integrallarning xarakterini o‘rganish uchun stasionar faza usuli qo‘llanila boshlandi. 1891-1892-yillarda Puankare

[2] Dissertatsiya mavzusi bo’yicha xorijiy ilmiy-tadqiqotlar sharhi: https://www.springer/com/gp/mathematics; https://onlinelibrary.wiley.com/; www.mathnet.ru; www.scholar.google.com; www.elsevier.com/locate/camw va boshqa manbalar asosida ishlab chiqilgan.

izlanishlarida yorug‘likning intensivligini o‘rganishda tebranuvchan integrallardan keng foydalandi. Xususan, u yorug‘lik kaustikadan o‘tayotganda fazali siljishni $\frac{\pi}{2}$ ga surishini isbotladi. Oldinroq bu tasdiq Gui tomonidan tajribaviy usul bilan aniqlangan.

Keyingi tadqiqotlarda Pirsi, Lyudvig, Ursella, Konnor va boshqa olimlarning A_n maxsuslikka ega bo‘lgan tebranuvchan integrallarning matematik fizikada tadbiqlari paydo bo‘ldi. 1974-yilda J.Dyustermatning ishida umumlashgan Eyri funksiyalari, aniqrog‘i Arnoldning elliptik maxsusliklari bilan bog‘liq funksiyalar tadqiq qilindi. 1973-yilda V.I.Arnold tomonidan quyidagi gipoteza ilgari surildi. Faraz qilaylik, tebranuvchan integralning faza funksiyasi biror parametrlarga bog‘liq bo‘lsin. Albatta, faza funksiyasi parametrlar va “asosiy” o‘zgaruvchilar(integral o‘zgaruvchilari)ning silliq (hatto analitik) funksiyasi bo‘lishi mumkin. Ya’ni, biz faza funksiyalari oilasiga ega bo‘laylik. Aytaylik, bu parametrlar tayinlangan qiymatlarda nol nuqtaning biror atrofida o‘zgarsin. Shuning uchun ularni ba’zan “kichik” parametrlar deb atashadi. Bizga tebranuvchan integralning (ya’ni faza funksiyasi parametrning nol nuqtasida bo‘lgan holdagi qiymati bilan aniqlangan) tebranish ko‘rsatkichi ma’lum bo‘lsin. U holda parametrlar nol nuqtaning yetarli kichik atrofida o‘zgarganda parametrlarga bog‘liq tebranuvchan integral shu tebranish ko‘rsatkichidan ixtiyoriy kichikroq bo‘lgan katta parametrning darajasi orqali baholanadi. Boshqacha aytganda, tebranish ko‘rsatkichi fazaning kichik parametrlarining yarim uzluksiz funksiyasi bo‘ladi. Bu masala “Arnold’s Problems” nomli kitobda ham keltirilgan (bu kitobning to‘ldirilgan inglizcha nashri 2004-yilda chop qilingan).

Arnoldning gipotezasi V.N.Karpushkin tomonidan isbotlandi. Aniqrog‘i, faza funksiyasi analitik bo‘lgan holda ikki karrali tebranuvchan integrallarning tekis (faza funksiyasining yetarli kichik «siljishiga» nisbatan) baholari olindi. Boshqacha aytganda, V.N.Karpushkin V.I.Arnoldning gipotezasi analitik fazali tebranuvchan integrallar uchun o‘rinli bo‘lishini isbotladi. Keyinchalik bu natija silliq funksiyalarning ba’zi xususiy deformasiyalari uchun Xasanovning nomzodlik dissertatsiyasida isbotlandi.

Tebranuvchan integrallarni o‘rganish uchun yana bir muhim sohalardan biri analitik sonlar nazariyasiga tadbig‘i hisoblanadi. Ma’lumki, I.M.Vinogradovning tadqiqotlarida diofant tenglamalari yechimining soni bilan mos keluvchi integral (Vinogradov integrali deb ataluvchi) muhim o‘rin egallaydi. Biroq bunday integrallarning xarakterini tadqiq etishning ikki jihati mavjud bo‘lib, ulardan biri trigonometrik yig‘indilar bilan, boshqasi trigonometrik integrallar bilan bog‘liq. 1949-yilda xitoy matematigi Xua Lo Gen tomonidan trigonometrik integralning mos $L^p(R^n)$ fazoga tegishli bo‘lgan eng minimal p ni topish muammosini qo‘ydi, bunda n – fazadagi ko‘phad koeffisiyentlari fazosining o‘lchovi. Ushbu masala bir qator ishlarda qaralgan bo‘lib, Vinogradov tomonidan olingan baholar umumlashgan Van der Korput lemmasiga asoslangan. Mazkur masala bir karrali tebranuvchan integrallar uchun Arxipov G.I., V.N.Chubarikov, A.A.Karasubalarning 1979-yildagi ishida o‘z yechimini topgan va ular tomonidan ko‘p karrali trigonometrik integrallar uchun yuqori baho topilgan. Biroq bu masala umumiy holda haligacha ochiq turibdi.

Dissertatsiya tadqiqotining dissertatsiya bajarilayotgan oliy ta'lim muassasasining ilmiy-tadqiqot ishlari rejalari bilan bog'liqligi. Dissertatsiya tadqiqoti Samarqand Davlat universitetining OT-F4-69 - «Garmonik analiz, darajali geometriya va ularning matematik fizika masalalariga tatbiqlari» ilmiy-tadqiqot ishlari rejasiga muvofiq bajarilgan.

Tadqiqotning maqsadi umumlashgan tebranuvchan integrallarning baholarini olish va hamda aniq jamlash ko'rsatkichini topishdan iborat.

Tadqiqotning vazifalari:

amplituda funksiyasi uzilishga ega bo'lgan tebranuvchan integrallarning tekis baholarini ε aniqligida o'tish metodini qo'llash orqali olish;

faza funksiyasi D_∞, $D_4^\pm$ va A_r maxsusliklarga ega bo'lgan Mittag-Leffler funksiyasi bilan bog'langan umumlashgan tebranuvchan integrallarni baholash;

faza funksiyasi ko'phad bo'lgan ikki o'zgaruvchili Mittag-Leffler funksiyasi bilan bog'langan umumlashgan va klassik tebranuvchan integrallarning tekis baholarini olish;

fazasi uchinchi darajali ko'phad bo'lgan tebranuvchan integrallar uchun Yevklid tekisligidagi klassik harakatlar gruppasining invariantlari orqali baholarini olish;

faza funksiyasi ikkinchi va uchinchi darajali ko'phad bo'lgan klassik tebranuvchan integrallarning koeffisiyentlar fazosidagi aniq jamlash ko'rsatkichini topish.

Tadqiqotning ob'yekti fazasi ko'phad bo'lgan umumlashgan tebranuvchan integrallardan iborat.

Tadqiqotning predmeti faza funksiyasining koeffisiyentlari cheksizga intilgandagi tebranuvchan integrallarning holatini aniqlash, tebranuvchan integrallarning invariant klassik gruppalar orqali bahosini olishdan iborat.

Tadqiqotning usullari. Dissertatsiya ishida differensial akslantirishlarning maxsusliklar nazariyasi, analitik funksiyalar nazariyasi, analizning asimptotik usullaridan foydalanilgan.

Tadqiqotning ilmiy yangiligi quyidagilardan iborat:

amplituda funksiyasi uzilishga ega bo'lgan tebranuvchan integrallarning tekis baholarini ε aniqligida o'tish metodini qo'llash orqali olingan;

faza funksiyasi D_∞, $D_4^\pm$ va A_r maxsusliklarga ega bo'lgan Mittag-Leffler funksiyasi bilan bog'langan umumlashgan tebranuvchan integrallar baholangan;

faza funksiyasi ko'phad bo'lgan ikki o'zgaruvchili Mittag-Leffler funksiyasi bilan bog'langan umumlashgan va klassik tebranuvchan integrallarning tekis baholari olingan;

fazasi uchinchi darajali ko'phad bo'lgan tebranuvchan integrallar uchun Yevklid tekisligidagi klassik harakatlar gruppasining invariantlari orqali baholari olingan;

faza funksiyasi ikkinchi va uchinchi darajali ko'phad bo'lgan klassik tebranuvchan integrallarning koeffisiyentlar fazosidagi aniq jamlash ko'rsatkichi topilgan.

Tadqiqotning amaliy natijalari quyidagilardan iborat:

fazasi ikki o‘zgaruvchili uchinchi darajali ko‘phad bo‘lgan tebranuvchan integrallarning aniq yig‘ilish ko‘rsatkichini topishga imkon beradi;

fazasi ikki va undan ko‘p o‘zgaruvchili ko‘phad bo‘lgan umumlashgan tebranuvchan integrallar uchun Van der Korput lemmasining isbotini olishda foydalanish mumkin;

umumlashgan tebranuvchan integrallarning baholari kasr tartibli xususiy hosilali differensial tenglamalarning yechimini baholashda qo‘llanilishi mumkin.

Tadqiqot natijalarining ishonchliligi matematik analiz, funksional analiz, matematik fizika, differensial tenglamalar, garmonik analiz nazariyasi usullaridan foydalanganligi hamda qat’iy matematik mulohazalarni qo‘llash orqali, matematik isbotlashlar bilan asoslangan.

Tadqiqot natijalarining ilmiy va amaliy ahamiyati. Tadqiqot natijalarining ilmiy ahamiyati fazasi uchinchi darajali ko‘phad bo‘lgan umumlashgan tebranuvchan integrallar invariant baholarining mavjudligini ko‘rsatilganligi bilan izohlanadi.

Tadqiqot natijalarining amaliy ahamiyati, giperbolik tenglamalarning fundamental yechimlarining xarakterini tadqiq qilishda asos sifatida xizmat qilishi bilan belgilanadi.

Tadqiqot natijalarining joriy qilinishi. Umumlashgan tebranuvchan integrallarning baholari va ularning tatbiqlariga oid natijalar asosida:

fazasi ko‘phad bo‘lgan tebranuvchan integrallarning invariant baholariga oid ilmiy natijalardan yetakchi xorijiy jurnallarda (Nonlinear Analysis, Volume 207, June 2021, 112292, Analysis and Mathematical Physics volume 12, Article number: 130 (2022), Journal of Siberian Federal university. mathematics and physics 15.4 (2022): 459-466.) ba’zi nochiziqli differensial tenglamalarga mos masalalarni tadqiq qilishda foydalanilgan. Ilmiy natijalarning qo‘llanilishi qaralayotgan masalalarning spektrlari diskret bo‘lishini isbotlash imkonini bergan;

umumlashgan tebranuvchan integrallarning baholaridan Belgiya respublikasi Gent universitetining maxsus tadqiqot fondining Analiz va xususiy differensial tenglamalar hamda Medusalam programmasining № 01M01021 sonli tadqiqot loyihasida (Gent universitetining 2023-yil 18-apreldagi ma’lumotnomasi) Mittag-Leffler funksiyasi bilan bog‘langan umumlashgan tebranuvchan integrallarning xususiy hosilali differensial tenglamalarning yechimini baholashda foydalanilgan. Ilmiy natijalarning qo‘llanilishi kasr hosilali differensial tenglamalarning yechimining aprior baholarini olish imkonini bergan;

fazasi ko‘phad bo‘lgan tebranuvchan integrallarning bahosiga oid natijalar Qozog‘iston respublikasi Matematika instituti va matematik modellashtirish ilmiy-tadqiqot institutining №AR08052046 sonli “Ba’zi lokal bo‘lmagan nochiziqli xususiy hosilali differensial tenglamalar” mavzusidagi loyihada (Qozog‘iston respublikasi Matematika instituti va matematik modellashtirish ilmiy-tadqiqot institutining 2023 yil 17 apreldagi 01-06/060 son ma’lumotnomasi) Mittag-Leffler funksiyasi bilan bog‘langan umumlashgan tebranuvchan integrallarning baholarini tadqiq qilishda foydalanilgan. Ilmiy natijalarning qo‘llanilishi kasr hosilali differensial tenglamalar uchun Koshi masalasi yechimining dastlabki baholarini olish imkonini bergan.

Tadqiqot natijalarining aprobasiyasi. Tadqiqotning asosiy natijalari 14 ilmiy va ilmiy-amaliy anjumanlarda, shulardan 8 ta xalqaro va 6 ta respublika ilmiy va ilmiy-amaliy anjumanlarida muhokamadan o‘tkazilgan.

Tadqiqot natijalarining e’lon qilinganligi. Dissertatsiya mavzusi bo‘yicha jami 28 ta ilmiy ish chop etilgan bo‘lib, shulardan, O‘zbekiston Respublikasi Oliy attestasiya komissiyasining doktorlik dissertatsiyasilari asosiy ilmiy natijalarini chop etish tavsiya etilgan ilmiy nashrlarda 14 ta maqola, jumladan, 6 tasi xorijiy va 8 tasi respublika jurnallarida nashr etilgan.

Dissertatsiyaning tuzilishi va hajmi. Dissertatsiya kirish, to‘rtta bob, xulosa va foydalanilgan adabiyotlar ro‘yxatidan iborat. Dissertatsiyaning hajmi 167 betni tashkil etgan.

DISSERTATSIYANING ASOSIY MAZMUNI

Kirish qismida dissertatsiya mavzusining dolzarbligi va zarurati asoslangan, tadqiqotning respublika fan va texnologiyalari rivojlanishining ustuvor yo‘nalishlariga mosligi ko‘rsatilgan, mavzu bo‘yicha xorijiy ilmiy-tadqiqotlar sharhi, muammoning o‘rganilganlik darajasi keltirilgan, tadqiqot maqsadi, vazifalari, ob’yekti va predmeti tavsiflangan, tadqiqotning ilmiy yangiligi va amaliy natijalari bayon qilingan, olingan natijalarning nazariy va amaliy ahamiyati ochib berilgan, tadqiqot natijalarining joriy qilinishi, nashr etilgan ishlar va dissertatsiya tuzilishi bo‘yicha ma’lumotlar keltirilgan.

Dissertatsiya ishining **“Yordamchi mulohazalar”** deb nomlangan birinchi bobida biz kerakli ta’riflar, belgilar va dissertatsiya davomida foydalaniladigan ba’zi natijalarni belgilab beramiz. Shuningdek, ba’zi dastlabki tushunchalar haqidagi ma’lumotlarni ham taqdim etamiz. Xususan, koordinatalarni chiziqli almashtirish orqali oddiy maxsusliklarga ega bo‘lgan funksiyalarning normal shakliga keltiramiz. Bundan tashqari, fazasi analitik funksiya va amplitudasi uzilish nuqtalari to‘plamiga ega bo‘lgan tebranish integrallarini ko‘rib chiqamiz. Bunda musbat $\varepsilon > 0$ aniqligida bunday integrallar uchun aniq baholarni olamiz.

Ikki o‘lchovli holni qaraylik, ya’ni $n = 2$. Silliq amplitudali kompakt tashuvchiga ega bo‘lgan tebranuvchan integralni qaraylik:

$$J_0(\lambda, s) = \int_{R^2} a(x, s) e^{i\lambda\Phi(x,s)} dx.$$

Asosiy shart:

Faraz qilaylik, $s \in K$ va (η_1, η_2) uchun quyidagi yangi faza funksiyani qarab chiqaylik

$$\Phi_1(x, s, \eta) = \Phi(x, s) + x_1\eta_1 + x_2\eta_2.$$

Faraz qilaylik, $\Phi(x, s)$ analitik funksiya bo‘lib, $\Phi_1(x, s^0, \eta^0)$ funksiyaning ixtiyoriy kritik nuqtasi uchun bunda $(s^0, \eta^0) \in K \times R^2$ va uning tebranish ko‘rsatkichi $\beta \le \beta_0$ bo‘lsin. Quyidagi tasdiq o‘rinli.

1-Teorema. *Agar faza funksiya uchun asosiy shart o‘rinli bo‘lsa, u holda ixtiyoriy* $\varepsilon > 0$ *uchun quyidagi baho o‘rinli*

$$\left|J(\lambda, s)\right| \le \frac{c_\varepsilon \left\|a(\bullet, s)\right\|_{C^3}}{\lambda^{\beta_0 - \varepsilon}}.$$

Quyidagi tebranuvchan integralni qaraylik

$$J(\lambda, s) = \int_{R^2} \chi_{D(s)}(x) a(x,s) e^{i\lambda\Phi(x,s)} dx, \qquad (1)$$

bunda $D(s)-$ R^2 dagi sohalar oilasi va $\{F_1(x,s) > 0,\ F_2(x,s) > 0\} \subset D(s)$, bundan tashqari $\gamma_1, \gamma_2 \subset \partial D(0)$ bo‘lsin, bunda $\gamma_j = \{x : F_j(x,0) = 0\},\ j = 1,2.$

Quyidagi teorema o‘rinli.

2-Teorema. *$\Phi(x,0)$ funksiyaning $x = 0$ kritik nuqtadagi tebranish ko‘rsatkichi β bo‘lsin, u holda nol nuqtaning shunday $U \times V \subset R^2 \times R^m$ atrofi mavjud bo‘lib, ixtiyoriy tayinlangan musbat $\varepsilon > 0$ va ixtiyoriy $a \in C_0^\infty(U \times V)$ amplituda funksiya uchun, quyidagi baho o‘rinli:*

$$\left|J(\lambda, s)\right| \le \frac{A_\varepsilon \left\|a\right\|_{C^3}}{\lambda^{\beta-\varepsilon}}.$$

“Mittag-Leffler funksiyasi bilan bog‘langan tebranuvchan integrallar” deb nomlanuvchi dissertatsiyaning ikkinchi bobida, fazasi f va amplitudasi ψ bo‘lgan quyidagi shakldagi integralni qaraymiz:

$$I_{\alpha,\beta} = \int_U E_{\alpha,\beta}(i\lambda f(x))\psi(x)dx\ , \qquad (2)$$

bunda $0 < \alpha < 1,\ \beta > 0$, U koordinalar boshining yetarlicha kichik atrofi va $\lambda > 0$.

“Silliq fazali sodda maxsuslikga ega bo‘lgan Mittag-Leffler funksiyalarning baholari” nomli paragrafda faza funksiyasi silliq bo‘lib, D_∞, $D_4^\pm$ va A_r maxsusliklarga ega bo‘lgan Mittag-Leffler funksiyalarning baholarini qaraymiz. Tebranuvchan integrallarning eksponensial funksiyalari o‘rniga Mittag-Leffler funksiyasini qo‘yish orqali umumlashtiramiz.

Bir jinsli uchinchi darajali ikki o‘zgaruvchili ko‘phadni qaraylik. Ushbu paragrafning asosiy natijasi quyidagidan iborat.

3-Teorema. *Faraz qilaylik $-\infty < a < b < \infty$ va faza funksiya ikki o‘zgaruvchili bir jinsli ko‘phad bo‘lib hamda $\psi \in L^p[a,b]^2$, $1 < p \le \infty$ bo‘lsin. Agar $0 < \alpha < 1$, $\beta > 0$ va $\lambda > 0$ bo‘lsa, u holda quyidagi baho o‘rinli*

$$\left|\int_{[a,b]^2} E_{\alpha,\beta}(i\lambda x_1^2 x_2)\psi(x)dx\right| \le \frac{C|\psi|_{L^p}}{\lambda^{\frac{1}{2}-\frac{1}{2p}}}, \qquad (3)$$

$$\left|\int_{[a,b]^2} E_{\alpha,\beta}(i\lambda(x_1^2 x_2 \pm x_2^3))\psi(x)dx\right| \le \frac{C|\psi|_{L^p}}{\lambda^{\frac{2}{3}-\frac{1}{3p}}}, \qquad (4)$$

$$\left|\int_{[a,b]^2} E_{\alpha,\beta}(i\lambda x_1^3)\psi(x)dx\right| \le \frac{C|\psi|_{L^p}}{\lambda^{\frac{1}{3}-\frac{1}{3p}}}, \qquad (5)$$

bunda C konstanta faqatgina p ga bog‘liq.

“Fazasi ko‘phad bo‘lgan umumlashgan tebranuvchan integrallarning baholari” nomli paragrafda, fazasi bir jinsli ko‘phad bo‘lgan Mittag-Leffler funksiyasi bilan bog‘langan umumlashgan tebranuvchan integrallarning tekis baholarini qaraymiz.

Richchi-Steyn lemmaning analogini va quyidagi integralning invariant bahosini olamiz.

Quyidagi integralni qaraylik

$$I_{\alpha,\beta}(a) = \int_{Q^n} E_{\alpha,\beta}(iP(a,x))\psi(x)dx, \qquad (6)$$

bunda $0 < \alpha < 1$, $\beta > 0$, $P(a,x)$ faza funksiya, $\psi \in C^\infty(R^n)$ amplituda va $Q^n := [0,1]^n$ n o‘lchovli kubdan iborat.

4-Teorema. *Faraz qilaylik* $0 < \alpha < 1$, $\beta > 0$ *bo‘lsin. Shunday musbat* C_d *topilib, fazasi* $P(a,x)$ *bo‘lgan (6) integral uchun quyidagi baho o‘rinli*

$$\left|I_{\alpha,\beta}\right| \le \frac{C_d}{\|a\|^{\frac{1}{d}}}, \tag{7}$$

bunda $\|a\| = \sum_{|\lambda| \le d} |a_\lambda|$.

1-Eslatma. *4-teorema Richchi-Steyn lemmasining analogidir.*

1-Tasdiq. Faraz qilaylik $P: R^n \to R$ *darajasi* $\le d$ *bo‘lgan ko‘phad va* $|a| = \max\{|a_\lambda|, |\lambda| \le d\}$ *bo‘lsin. Shunday musbat* $C_{d,n}$ *o‘zgarmas topilib, agar* $a^0 \in S^N$ *(bu yerda* $N+2$ *eng yuqori darajasi* d *bo‘lgan ko‘phadlar fazosidagi o‘lchovi) tayinlangan nuqta va* $|D^\alpha P(a^0,x)| \ge \delta > 0$, $\forall x \in [0,1]^n$ *(bunda* $\alpha \in Z_+^n$ *hamda* $|\alpha| \ge 2$*), bo‘lsa u holda quyidagi baho o‘rinli*

$$\left|I_{\alpha,\beta}(\mu a^0)\right| \le \frac{C_{d,n}}{|\mu|^{\frac{1}{|\alpha|}}},$$

bu yerda $S^N = \{|a| = 1\}$ *birlik sfera va* $\mu > 0$.

“Fazasi bir jinsli ko‘phad bo‘lgan umumlashgan tebranuvchan integrallarning invariant baholari” nomli paragrafda, biz fazasi quyidagi ko‘rinishdagi umumlashgan tebranuvchan integrallarning baholarini qaraymiz

$$P_3(a,x,y) = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3, \tag{8}$$

bu yerda $a := (a_0, a_1, a_2, a_3) \in R^4$ koeffisiyentlar. Quyidagi integralni qaraylik

$$I_{\alpha,\beta}(a) = \int_{R^2} E_{\alpha,\beta}(iP_3(a,x,y))\psi(x,y)dxdy, \tag{9}$$

bunda ψ kompakt tashuvchili silliq funksiya.

Biz (9) integralni koeffisiyentlari cheksizga intilgandagi xarakterini o‘rganamiz. Evklid tekisligining harakatlar guruhining invariantlari bo‘yicha (9) integral uchun baholarini olamiz. Bu ko‘phadning diskriminanti quyidagi formula bilan aniqlanadi:

$$D = 3a_1^2a_2^2 + 6a_0a_1a_2a_3 - 4a_0a_2^3 - 4a_1^3a_3 - a_0^2a_3^2.$$

Bu $SL(2,\mathbb{C})$ guruhda invariant.

5-Teorema. *(8) fazali (9) integral uchun quyidagi baho o‘rinli*

$$\left|I_{\alpha,\beta}(a)\right| \le \frac{c\|\psi\|_{L^\infty}}{|D|^{\frac{1}{6}}}.$$

Bunda c *o‘zgarmas* a *ga bog‘liq va* $D \ne 0$ $P_3(a,x,y)$ *ko‘phadning diskriminanti.*

Endi bir o‘lchovli integralni qaraymiz. Quyidagi baho yordamchi natija bo‘lib, keyingi paragrafda invariant bahoni olish uchun muhim hisoblanadi.

$0 < \delta < 1$ bo‘lsin. Quyidagi integralni qaraymiz

$$J_{\alpha,\beta}(p,q) = \int_R E_{\alpha,\beta}(i(x^3 + px + q))\,dx, \tag{10}$$

bunda $0 < \alpha < 1$, $\beta > 0$.

6-Teorema. *1) Faraz qilaylik* $\frac{1}{3} < \delta < \frac{1}{2}$. *bo'lsin. U holda ixtiyoriy* $(p,q) \neq (0,0)$ *uchun*

$$\left|J_{\alpha,\beta}(p,q)\right| \leq \frac{c_\delta}{\left(\frac{|p|^3}{27}+\frac{q^2}{4}\right)^{\frac{3\delta-1}{6}}}. \tag{11}$$

2) Faraz qilaylik $\frac{1}{2} < \delta < 1$. *U holda*

$$\left|J_{\alpha,\beta}(p,q)\right| \leq \frac{c_\delta}{|D|^{\delta-\frac{1}{2}}\left(\frac{|p|^3}{27}+\frac{q^2}{4}\right)^{\frac{2-3\delta}{6}}}, \tag{12}$$

bunda $D = \frac{p^3}{27} + \frac{q^2}{4} \neq 0$ $x^3 + px + q$ *ko'phadning diskriminanti va* $c_\delta > 0$ *faqatgina* δ *ga bog'liq bo'lgan musbat son.*

"Ikki o'zgaruvchili Mittag-Leffler funksiyasining baholari" paragrafida biz fazasi ikki o'zgaruvchili silliq funksiya bo'lgan umumlashgan tebranuvchan integrallarning baholarini qaraymiz. Fazasi eksponensial funksiya o'rniga ikki o'zgaruvchili Mittag-Leffler bo'lgan funksiyani qo'yib umumlashtiramiz.

Ushbu paragrafning asosiy natijasi quyidagi hisoblanadi.

7-Teorema. *Faraz qilaylik* f *koordinatalar boshining yetarlicha kichik atrofida aniqlangan ikki o'zgaruvchili silliq funksiya va* $\psi \in C_0^\infty(U)$ *bo'lsin.*

Agar $0 < \alpha < 1$, $\beta > 0$, $h > 1$ *va* $\lambda > 2$ *bo'lsa, u holda quyidagi baho o'rinli*

$$\left|\int_U E_{\alpha,\beta}\left(i\lambda f(x_1,x_2)\right)\psi(x)dx\right| \leq \frac{C|\ln\lambda|^m\|\psi\|_{L^\infty(\overline{U})}}{\lambda^{\frac{1}{h}}}, \tag{13}$$

bunda h *funksiyaning balandligi va* $m = 0,1$ *Nyuton ko'pyoqligining karraligi.*

Agar $0 < \alpha < 1$, $\beta > 0$, $h = 1$ *va* $\lambda > 2$ *bo'lsa, u holda quyidagi baho o'rinli*

$$\left|\int_U E_{\alpha,\beta}(i\lambda(x_1^2+x_2^2))\psi(x)dx\right| \leq \frac{C|\ln\lambda|\|\psi\|_{L^\infty(\overline{U})}}{\lambda} \tag{14}$$

va

$$\left|\int_U E_{\alpha,\beta}(i\lambda(x_1^2-x_2^2))\psi(x)dx\right| \leq \frac{C|\ln\lambda|^2\|\psi\|_{L^\infty(\overline{U})}}{\lambda}, \tag{15}$$

bunda C *faza va amplitudaga bog'liq bo'lmagan konstanta.*

2-Tasdiq. *Faraz qilaylik* $f:\Omega\to\mathbb{R}$ *o'lchovli funksiya bo'lib, barcha noldan farqli haqiqiy* $\lambda > 2$ *lar va barcha musbat* $\delta \neq 1$ *uchun, quyidagi o'rinli bo'lsa*

$$\left|\int_\Omega e^{i\lambda f(x)}dx\right| \leq C\,|\lambda|^{-\delta}|\ln\lambda|^m, \tag{16}$$

bunda $m \geq 0$. *U holda ushbu baho o'rinli*

$$\left|x\in\Omega: |f(x)| \leq \varepsilon\right| \leq C_\delta\varepsilon^\delta\left|\ln\varepsilon\right|^m, \quad \delta < 1 \text{ uchun}$$

$$va\ \ \delta > 1, \text{учун } |x\in\Omega: |f(x)| \leq \varepsilon| \leq C_\delta\varepsilon|,$$

bu yerda C_δ *faqatgina* δ *ga bog'liq.*

1-Natija. *Faraz qilaylik* $a(x) = \begin{cases} 1, & \text{when } |x| \le \sigma, \\ 0, & \text{when } |x| \ge 2\sigma \end{cases}$ *va* $a(x) \ge 0$ *hamda* $a \in C_0^\infty(R^2)$ *bo'lsin. U holda barcha noldan farqli* $\lambda > 2$ *va musbat* h *uchun, quyidagi baho o'rinli*

$$\left| \int_{R^2} e^{i\lambda f(x)} a(x) dx \right| \le C |\lambda|^{-\frac{1}{h}} |\ln \lambda|^m, \tag{17}$$

bunda $m \ge 0$, f *koordinatalar boshidagi silliq finit funksiya,* h *esa* f *funksiyaning balandligi va* m *Nyuton ko'pyoqligining karraligi.*

2-Natija. Faraz qilaylik $f(x_1, x_2)$ *silliq funksiya va* $\overline{\Omega}$ *koordinatalar boshining yetarlicha kichik atrofidagi kompakt to'plami bo'lsa u holda noldan farqli ixtiyoriy haqiqiy* $\varepsilon > 2$ *son uchun quyidagi baho o'rinli*

$$|x \in \Omega : |f(x)| \le \varepsilon| \le C\varepsilon^{\frac{1}{h}} |\ln \varepsilon|^m,$$

bunda h f *funksiyaning balandligi va* m *karraligi.*

2-Tasdiq. *Ixtiyoriy silliq, kompakt tashuvchiga ega* $a(x, y)$ *va* $h > 1$ *uchun quyidagi tengsizlik o'rinli*

$$\int_{\{|f(x,y)| \ge \frac{M}{\lambda}\}} \frac{a(x,y)}{1 + \lambda |f(x,y)|} dxdy \le \frac{C |\ln \lambda|}{\lambda^{\frac{1}{h}}}. \tag{18}$$

bu yerda $f(x, y)$ *silliq funksiya.*

3-Eslatma. *Agar* $h = 1$ *va* $f(x, y)$ *silliq funksiya uchun* $\nabla f(0,0) = 0$ *bo'lib. Mors lemmasini qo'llab* $f : x^2 \pm y^2$ *ga ega bo'lamiz. U holda biz quyidagi ikkita to'plamlarni baholaymiz* $\Delta = \Delta_1 + \Delta_2$, *bunda* $\Delta_1 := \{(x, y) : |x^2 \pm y^2| \le M\lambda, |x| \le 1, |y| \le 1\}$ *va* $\Delta_2 := \{(x, y) : |x^2 \pm y^2| > M\lambda, |x| \le 1, |y| \le 1\}$.

"Fazasi ko'phad bo'lgan tebranuvchan integrallarning baholari" nomli bobda fazasi ko'phad bo'lgan tebranuvchan integrallar haqidagi jamlash masalasi qaraladi.

Faraz qilaylik $P(x, a) \in R[x]$, $x \in R^n$ haqiqiy $a \in R^m$ koeffisiyentli ko'phad bo'lsin. Quyidagi integralni qaraylik

$$J(a) = \int_Q \exp\{2\pi i P(x, a)\} dx,$$

bunda $Q = \{x \in R^n : 0 \le x_l \le 1, l = 1, \dots, n\}$ n - o'lchovli birlik kub. Bu funksiyaning ahamiyati shundaki, u quyidagi shaklning maxsus integrallarida namoyon bo'ladi

$$I_p := \int_{R^m} |J(a)|^p da \tag{19}$$

(19) Terri muammosida, diofant sistemasining bir qator butun sonli yechimlari uchun asimptotik tasvir koeffisiyenti sifatida yuzaga keladi. Shuning uchun (19) integralning yaqinlashuvchi bo'lishi uchun p parametrning minimal qiymatini topish masalasi muhim.

Muammo: *Aniq qiymatini toping* $\gamma = \inf\{p \ge 1 : J \in L^p(R^m)\}$.

γ songa trigonometrik integrallar uchun jamlash ko'rsatkichi deb ataladi.

Ushbu bobda fazasi ko‘phad bo‘lgan trigonometrik integrallar uchun γ sonni topish masalasini ko‘ramiz. Faraz qilaylik, $P(x,s)\in \mathrm{R}[x]$ $x\in \mathrm{R}^k$ dagi $s\in \mathrm{R}^N$ koeffisiyentli ko‘phad bo‘lsin. Q orqali R^k dagi kompakt to‘plamni belgilaymiz.

“Fazasi to‘la kvadratik ko‘phad bo‘lmagan trigonometrik integrallarning baholari” nomli paragrafda $k=3$ holdagi trigonometrik integrallarning jamlash masalasini qaraymiz va jamlash ko‘rsatkichini $\boldsymbol{Q=[0,1]^3}$ uchun aniq topamiz.

P ko‘phad quyidagi ko‘rinishda bo‘lsin:

$$P(x,A,b)=(Ax,x)+(\beta,x),$$

bu yerda $A=\begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_1 & 0 & \alpha_3 \\ \alpha_2 & \alpha_3 & 0 \end{pmatrix}$ haqiqiy simmetrik matrisa, $\beta:=(\beta_1,\beta_2,\beta_3)$ va $(\cdot,\cdot)$ vektorlarning skalyar ko‘paytmasi. Quyidagi trigonometrik integralni qaraylik

$$T(A,\beta)=\int_{[0,1]^3} exp(\,iP(x,A,\beta))dx. \qquad (20)$$

Quyidagi integralni qaraylik

$$\theta=\int\limits_{\mathrm{R}^6} |T(A,\beta)|^p\, d\beta d\alpha,$$

bunda $d\beta=d\beta_1 d\beta_2 d\beta_3$ va $d\alpha=d\alpha_1 d\alpha_2 d\alpha_3$.

Ushbu tasdiq o‘rinli:

8-Teorema. *θ integral $p>4$ uchun yaqinlashuvchi va $p\le 4$ uchun uzoqlashuvchi. Xususan bunda jamlash ko‘rsatkichi $p_0=4$ ga teng.*

Quyidagi mulohaza bilan tanishaylik:

$$\theta_\infty=\int\limits_{\mathrm{R}^6} |T_\infty(A,\beta)|^p\, d\beta d\alpha.$$

3-Tasdiq. *θ_∞ integral $p>4$ da yaqinlashuvchi va $p\le 4$ da uzoqlashuvchi.*

Faraz qilaylik $Q:=[0,1]^3$ R^3 dagi kub bo‘lsin va

$$h(\beta)=\int_{R^3} e^{|x|^2}\chi_Q(x)e^{-2\pi i(\beta,x)}dx,$$

bunda $\chi_Q(x)$ Q to‘plamning xarakteristik funksiyasi.

4-Tasdiq. *Ixtiyoriy musbat ε son uchun, quyidagi tasdiq o‘rinli* $h\in L_{1+\varepsilon}(\mathrm{R}^3)$

.

“Fazasi kvadratik funksiyadan iborat trigonometrik integrallarning baholari” nomli paragrafda klassik qo‘yilgan masalani fazasi P kvadratik ko‘phad va $Q=[0,1]^k$ birlik kub bo‘lganda qaraymiz.

Faraz qilaylik P ko‘phad quyidagi ko‘rinishda bo‘lsin

$$P(x,A,b)=(Ax,x)+(b,x),$$

bu yerda $A=(a_{lm})_{l,m=1}^k$ simmetrik $k\times k$ o‘lchamli haqiqiy matrisa, $b:=(b_1,b_2,...,b_k)\in \mathrm{R}^k$ va $(\cdot,\cdot)$ vektorlarning skalyar ko‘paytmasi. Quyidagi trigonometrik integralni qaraylik

$$T(A,b) = \int_{R^k} \exp(iP(x,A,b))\,\chi_K(x)dx,$$

bunda K kompakt to‘plam va $\chi_K(x)$ uning xarakteristik funksiyasi.

$$\theta = \int_{R^N} |T(A,b)|^p \, dbda,$$

integralni qaraylik, bunda $db = db_1 db_2 \ldots db_k$ and $da = \prod_{1 \le l \le m \le k} da_{lm}$.

Quyidagi tasdiq o‘rinli:

9-Teorema. Faraz qilaylik K *kompakt to‘plam bo‘lsin, u holda* θ *integral* $p > 2k+2$ *yaqinlashuvchi va agar* K *ichki* x^0 *nuqtaga ega bo‘lib va* x^0 *nuqtadan o‘tuvchi* l *to‘g‘ri chiziq mavjud bo‘lib,* $\{l \cap K\}$ *to‘plam faqatgina chekli nuqtalarni saqlasa, u holda integral* $p \le 2k+2$ *da uzoqlashadi. Shunday qilib, agar* K *kompakt to‘plam bo‘sh bo‘lmagan ichki nuqtadan iborat bo‘lsa, u holda jamlash ko‘rsatkichi* $p_0 = 2k+2$ *ga teng.*

P **ikkinchi darajali bir jinsli ko‘phad bo‘lgan hol**

$P(x,A) = (Ax,x)$ bo‘lsin.

10-Teorema. *Agar* $P(x,A) = (Ax,x)$ *va* Q *ko‘pburchak bo‘lsin, u holda* $p > 2k$ *da* θ *integral yaqinlashuvchi. Agar* $Q = [0,1]^k$, *bo‘lsa, u holda* $p \le 2k$ *da* θ *integral uzoqlashuvchi bo‘ladi.*

Ko‘pburchaklar ma’nosida biz maxsusmas simplekslar birlashmasini tushunamiz.

3-Eslatma. Bu holda biz [Jong-Guk Bak, Sanghyuk Lee, 2004, "Restriction of the Fourier transform to a quadratic surface in R^n". *Mathematische Zeitschrift* No.247, pp.409-422.] ishning natijalarini $\{x_i x_j\}_{i \le j=1}^n$ silliq sirt bo‘lmaganligi tufayli qo‘llay olmaymiz.

4-Eslatma. Q to‘plamga bog‘liq holda, p ko‘rsatkich $2k$ dan kichik bo‘lishi ham mumkin. Masalan, agar $k = 2$ va Q markazi $(1,1)$ nuqtada bo‘lgan yetarlicha kichik aylanadan iborat bo‘lsa, u holda $p > 3$ da θ integral yaqinlashuvchi bo‘ladi.

Quyidagi belgilashlarni kiritamiz:

$$\theta_\infty = \int_{R^N} |T_\infty(A,b)|^p \, dbda,$$

bunda

$$N = \frac{k(k+2)}{2}.$$

5-Tasdiq. θ_∞ *integral* $p > 2k+2$ *da yaqinlashuvchi va* $p \le 2k+2$ *da uzoqlashuvchi.*

11-Teorema. *Faraz qilaylik* D *kompakt soha* $\hat{\chi}_D \in L_q(R^2)$ *va* $T(\hat{A}) = \int_D e^{i(Ax,x)}dx$ *bo‘lsin. U holda* $p > 6 - \frac{2}{q}$ *da* $T \in L^p(R^3)$. *Bundan tashqari, agar* $\hat{\chi}_D \in L_{1+0}(R^2)$ *bo‘lsa, u holda* $p > 4$ *da* $T \in L^p(R^3)$ *o‘rinli.*

5-Eslatma. [V.V.Lebedev, 2013, "On the Fourier transform of the characteristic functions of domains with C^1 boundary". *Func. anall. and its appl.* vol. 47, no. 1. pp. 33-46.] ishning natijasidan shunday ko‘pburchakdan farqli shunday D to‘plam mavjudki, ushbu $\hat{\chi}_Q \in L_{1+0}(R^2)$ o‘rinli.

3-Natija. Agar D$\subset R^2$ kompakt to‘plam va $\partial D \subset C^1$ bo‘lsa, u holda $p > 4.5$ uchun ushbu $T \in \mathsf{L}^p(\mathsf{R}^3)$ munosabat o‘rinli.

“Trigonometrik integrallar uchun L^p – chegaralanganlik” nomli paragrafida fazasi uchinchi darajali bir jinsli va chiziqli qismlar yig‘indisidan iborat ko‘phad bo‘lgan trigonometrik integrallar uchun jamlash masalasi qaraladi. Trigonometrik integrallarni baholab, jamlash ko‘rsatkichining aniq qiymati topiladi.

Biz amplituda funksiyasi ψ ham, faza funksiyasi F ham parametrga bog‘liq bo‘lgan holat uchun masalani ko‘rib chiqamiz. Avval oddiy fazali tebranish integrallari uchun jamlash masalasini hal qilamiz. Bundan tashqari, biz qutb tipidagi maxsuslikni o‘z ichiga olgan amplitudali bir o‘lchovli trigonometrik integrallarning xarakterlarini o‘rganamiz. Nihoyat, biz uzluksiz amplitudali va algebraik (lekin ko‘phad bo‘lmagan) fazali funksiyali trigonometrik integrallarning jamlash masalasini ko‘rib chiqamiz. Olingan natijalar ikki o‘lchovli trigonometrik integrallarni jamlash masalasini yechishda qo‘llaniladi.

$F(x,y,a)$ ikki o‘zgaruvchining ko‘phad funksiyasi bo‘lsin:

$$F(x,y,a) := P_3(x,y,a_3) + P_1(x,y,a_1), \tag{21}$$

bunda $P_3(x,y,a_3) = a_{30}x^3 + 3a_{31}x^2y + 3a_{32}xy^2 + a_{33}y^3$, $P_1(x,y,a_1) = a_{10}x + a_{11}y$.

$a_3 := (a_{30}, a_{31}, a_{32}, a_{33})$, $a_1 := (a_{10}, a_{11})$ lar R^4, R^2 dagi mos ravishdagi vektorlar.

Paragrafning asosiy natijalari quyidagi teoremalardir.

12-Teorema. *Faraz qilaylik* $\psi \in C_0^\infty(\mathsf{R}^2)$ *va*

$$J(a,\psi) := \int_{\mathsf{R}^2} e^{iF(x,y,a)}\psi(x,y)dxdy. \tag{22}$$

bo‘lsin. U holda ixtiyoriy $p > 8$ *uchun* $J(\cdot,\psi) \in \mathsf{L}^p(\mathsf{R}^6)$ *o‘rinli. Bundan tashqari, agar* $\psi(\cdot,\cdot)$ *koordinatalar boshining yetarli kichik atrofidagi* $\psi(0,0) \neq 0$ *bo‘lgan silliq funksiya bo‘lsa, u holda* $p \leq 8$ *uchun* $J(\cdot,\psi) \notin \mathsf{L}^p(\mathsf{R}^6)$.

Endi faza funksiyasi (21) va $\psi(x,y)\chi_Q(x,y)$ uzilishga ega amplituda funksiyali quyidagi tebranuvchan integralni ko‘rib chiqamiz

$$J(a,\psi,\chi_Q) := \int_Q e^{iF(x,y,a)}\psi(x,y)dxdy, \tag{23}$$

bu yerda $\psi \in C_0^1(\mathsf{R}^2)$ va $Q := [0,1] \times [0,1]$.

13-Teorema. *Agar* $J(\cdot,\psi,\chi_Q)$ *(23) ko‘rinishdagi integral bo‘lsa, u holda ixtiyoriy* $p > 8$ *uchun* $J(\cdot,\psi,\chi_Q) \in \mathsf{L}^p(\mathsf{R}^6)$. *Bundan tashqari, agar* $\psi(\cdot,\cdot)$ *koordinatalar boshining yetarli kichik atrofidagi* $\psi(0,0) \neq 0$ *bo‘lgan silliq funksiya bo‘lsa, u holda* $p \leq 8$ *uchun* $J(\cdot,\psi,\chi_Q) \notin \mathsf{L}^p(\mathsf{R}^6)$.

4-Natija. $J(a,\chi_Q)=\int_Q e^{iF(x,y,a)}dxdy$ integral uchun $p>8$ da $J(\cdot,\chi_Q)\in L^p(R^6)$ munosabat o‘rinli va agar $p\le 8$ bo‘lsa, u holda $J(\cdot,\chi_Q)\notin L^p(R^6)$. Boshqacha aytganda, jamlash ko‘rsatkichi $\gamma=8$.

Amplitudasi parametrga bog‘liq bir o‘zgaruvchili tebranuvchan integrallarni jamlash

$B\subset \mathsf{R}^3$ markazi koordinatalar boshida bo‘lgan yopiq kub bo‘lib, A_B orqali quyidagi shartlarni qanoatlantiruvchi $\phi\in C^\infty(\mathsf{R}^2\times B\times \mathsf{R})$ funksiyalar sinfini belgilaymiz:

• ixtiyoriy $\phi\in \mathsf{A}_B$ uchun, shunday $K_\phi\subset \mathsf{R}^2$ kompakt to‘plam mavjud bo‘lib, ixtiyoriy $(\xi,b)\in B\times R$ uchun $supp\ \phi(\cdot,\xi,b)\subset K_\phi$ bo‘lsin;

• ixtiyoriy $\phi\in \mathsf{A}_B$ va $N\ge 0$ uchun $C=C(N,\phi)>0$ mavjud bo‘lib $(\xi,b)\in B\times R$ uchun

$$\|\varphi(\cdot,\xi,b)\|_{C^N(R^2)}\le C$$

bo‘lsin.

$$R^2\times R^3\times R=\{(a_1,a_2,\eta,b):\eta\in R^3,a_1,a_2,b\in R\},$$

fazodagi R^6 da Lebeg o‘lchoviga nisbatan absolyut uzluksiz bo‘lgan, $\mu=b^3da_1da_2d\eta db$ o‘lchov bilan tanishamiz. μ-o‘lchovli $X\subset R^2\times R^3\times R$ qism to‘plamdagi p-darajasi bilan integrallanuvchi barcha μ-o‘lchovli funksiyalar sinfini $\mathsf{L}^p(X,\mu)$ orqali belgilaymiz.

$\phi\in \mathsf{A}_B$ uchun quyidagi tebranuvchan integralni qaraymiz

$$J^1(a_1,a_2,\eta,b):=\frac{1}{b^{\frac{1}{3}}}\int_R\varphi(y,b^{\frac{2}{3}}|\eta_3y^2+\eta_1|,\eta,b)e^{i(a_2y^3+a_1y)}dy, \qquad (24)$$

bunda $\eta=(\eta_1,\eta_2,\eta_3),\ b\ne 0.$

Ushbu to‘plamni qaraylik

$$X:=\{(a_1,a_2,\eta,b)\in R^2\times R^3\times(R\backslash[-1,1]):|a_1|,|a_2|\le|b|,\eta\in B\}.$$

6-Tasdiq. *Ixtiyoriy* $\phi\in \mathsf{A}_B$ *va* $p>8$ *uchun* $J^1\in \mathsf{L}^p(X,\mu)$ *o‘rinli.*

Kichik amplitudali tebranuvchan integrallarning baholari

Bunda $\frac{\psi(y,\eta,b)}{b|\eta_3y^2+\eta_1|}$, shakldagi tebranuvchan integrallarining bahosini qaraymiz, bunda $\psi(y,\eta,b)$ silliq va $\sup\psi(\cdot,\eta,b)\subset[-d,d]\cap\{b^{\frac{2}{3}}|\eta_3y^2+\eta_1|>1\}$ shartni qanoatlantiruvchi funksiya hamda $V_{-d}^d[\psi]\le$ bu yerda d musbat haqiqiy son. Tushunarliki, $\frac{\psi(y,\eta,b)}{b|\eta_3y^2+\eta_1|}$ funksiya $\boldsymbol{b^{\frac{2}{3}}|\eta_3y^2+\eta_1|>1}$ ma’nosidagi kichik amplituda funksiya.

Quyidagi tebranuvchan integralni qaraylik

$$J^2(a_1,a_2,\eta,b):=\int_R\frac{\psi(y,\eta,b)e^{i(a_2y^3+a_1y)}}{b|\eta_3y^2+\eta_1|}dy.$$

7-Tasdiq. *Ixtiyoriy* $p>8$ *uchun* $J^2\in \mathsf{L}^p(X,\mu)$ *o‘rinli.*

Uzilishga ega bo‘lgan amplituda funksiyali tebranuvchan integrallarning baholari

Ushbu tebranuvchan integralni qaraymiz

$$J^3(a_1,a_2,\eta,b) := \int_{\mathbb{R}} \frac{\psi(y,a,\eta,b)e^{iF_2(y,a,\eta,b)}}{b^{\frac{1}{2}}|\eta_3 y^2+\eta_1|^{\frac{1}{4}}}dy, \qquad (25)$$

bunda $sup\,\psi(\cdot,a,\eta,b) \in [-d,d] \cap \left\{b^{\frac{2}{3}}|\eta_3 y^2+\eta_1| > 1\right\}$, $F_2(y,a,\eta,b) = -\frac{2\sqrt{3}}{9}b|\eta_1|^{\frac{3}{2}}|\sigma y^2+1|^{\frac{3}{2}} + a_2y^3 + a_1y$ va $\sigma = \frac{\eta_3}{\eta_1}$.

Agar $\eta_3\eta_1 < 0$ bo‘lsa, u holda y yopiq nuqtani saqlovchi amplituda funksiyani saqlaydi. Bu tebranuvchan integrallarni o‘rganish jarayonidagi nozik masala hisoblanadi.

8-Tasdiq. *Ixtiyoriy* $p > 8$ *uchun* $J^3 \in \mathsf{L}^p(X,\mu)$ *o‘rinli.*

Natijaning aniqligi

$|a_{30}| = \max\{|a_{3i}|, i = \overline{0,3}\}$ va $|a_{11}| \le 1$, shartlarda $p \le 8$ uchun $J \notin L^p(\mathbb{R}^6)$ ni qanoatlantiruvchi (21) shakldagi ko‘phadni qaraylik.

(21) ko‘phadni quyidagi shaklda ifodalaymiz

$$F(x,y,a) = a_{30}\left(x_1^3 + \xi_2 x_1 y_1^2 + \xi_3 y_1^3\right) + a_{10}x + a_{11}y,$$

bunda $\xi_2 = \frac{3a_{30}a_{32}-3a_{31}^2}{a_{30}^2}$ va $\xi_3 = \frac{a_{30}^2a_{33}+2a_{31}^3-3a_{30}a_{31}a_{32}}{a_{30}^3}$ hamda $|\xi_2| \le 6$, $|\xi_3| \le 6$.

9-Tasdiq. *Koordinatalar boshining atrofida* $\psi(0,0) \neq 0$ *bo‘lib, har bir silliq* $\psi \in C_0^1(\mathbb{R}^2)$ *funksiya uchun, shunday* $\lambda_0, \varepsilon, a_1$ *sonlar topilib,* $|\xi_2 a_{30}|^{\frac{2}{3}} < \varepsilon$, $|a_{10}| \le \varepsilon a_{30}|^{\frac{1}{3}}$ *shartlar bajarilganda quyidagi baho o‘rinli*

$$|J(a)| \ge C/|a_{30}|^{1/3}.$$

“Fazasi ko‘phad bo‘lgan tebranuvchan integrallarning tekis va invariant baholari” nomli bobida fazasi D_∞ maxsuslikga ega bo‘lgan silliq funksiyali tebranuvchan integrallarning tekis baholari va fazasi ikki o‘zgaruvchili uchinchi darajali ko‘phad bo‘lgan tebranuvchan integrallarning invariant baholarini qaraymiz. [E. M. Stein, D. H. Phong, J. A. Sturm *On the growth and stability of real-analytic functions*, Amer. Journal of Math. 121 (1999), no.3, 519–554.] ishda kichik deformasiyalarda ham integral haqiqiy analitiklik masalasi qaralgan. Bir o‘zgaruvchili integrallarning baholari to‘liq qaralgan. V.N.Karpushkinning ishida faza funksiyasi ikki o‘zgaruvchili analitik funksiya bo‘lgan holda qaralgan. Uch o‘lchovli holda Varchenkoning misoli umumiy holda o‘rinli emasligini ko‘rsatdi.

“Fazasi silliq bo‘lgan tebranuvchan integrallarning baholari” nomli paragrafda fazasi D_∞ maxsuslikga ega bo‘lgan tebranuvchan integrallarning tekis bahosi qaraladi. Ushbu baho aniq va V.N.Karpushkin ishining analogi hisoblanadi.

Faraz qilaylik $U \subset V \subset \mathbb{R}^2$ koordinatalar boshining atrofi, $\overline{U}\,(\overline{V})$ esa $U\,(V)$ yopig‘i bo‘lsin. $f : \overline{V} \to \mathbb{R}$ funksiyaning ko‘rinishi quyidagicha (bunda $f \in C^N(\overline{V})$, $(N \ge 8)$) bo‘lsin

$$f(x_1, x_2) = x_1 x_2^2 + g(x_1, x_2), \tag{26}$$

bu yerda $g \in C^N(V)$, barcha $\alpha_1 + \alpha_2 \le 3$ uchun $D^\alpha g(0,0) = 0$, hamda $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$, $\alpha := (\alpha_1, \alpha_2) \in Z_+^2$ multiindeks, $Z_+ = \{0\} \cup \mathbb{N}$ nomanfiy butun sonlar.

Faraz qilaylik $\vartheta_{C^8(\overline{V})}(\varepsilon) := \{F \in C^8(\overline{V}), \|F\|_{C^8(\overline{V})} < \varepsilon\}$ bo‘lsin.

Ushbu paragrafning asosiy natijasi quyidagidan iborat.

14-Teorema. *$f \in C^8(\overline{V})$ (26) shakldan iborat bo‘lsin. U holda shunday musbat ε son va $U \subset V$ koordinatalar boshining shunday atrofi mavjud bo‘lib $a \in C_0^1(U)$ va $F \in C^8(\overline{V})$, $\|F\|_{C^8(\overline{V})} < \varepsilon$, funksiyalar uchun quyidagi baho o‘rinli:*

$$\left| \int_U e^{i\lambda(f+F)} a(x) dx \right| \frac{C \|a\|_{C^1}}{|\lambda|^{\frac{1}{2}}}.$$

“Fazasi to‘rtinchi darajali bir jinsli ko‘phad bo‘lgan tebranuvchan integrallarning tekis baholari” nomli paragrafda fazasi ko‘phad bo‘lgan tebranuvchan integrallarning tekis baholarini qaraymiz.

Faraz qilaylik, $U \subset V \subset \mathbb{R}^2$ koordinatalar boshining chegaralangan atrofi, $\overline{U}(\overline{V})$ esa $U(V)$ ning yopig‘i bo‘lsin. $f : \overline{V} \to \mathbb{R}$ quyidagi ko‘rinishda berilgan (bu yerda $f \in C^N(\overline{V})$, $(N \ge 8)$) bo‘lsin:

$$f(x_1, x_2) = f_\pi(x_1, x_2) + g(x_1, x_2), \tag{27}$$

bunda $f_\pi(x_1, x_2)$ $f_\pi(0,0)$ karraligi 2 (masalan $f_\pi(x)$ to‘rtinchi darajali ko‘phad markazi koordinalar boshida bo‘lgan birlik aylanada ikkitadan ortiq ildizga ega emas) bo‘lgan ildizga ega bo‘lgan to‘rtinchi darajali bir jinsli ko‘phad va $g \in C^N(V)$ shundayki, ixtiyoriy $\alpha_1 + \alpha_2 \le 4$ uchun $D^\alpha g(0,0) = 0$ hamda $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$, $\alpha := (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ multiindeks, $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ nomanfiy butun sonlar.

Paragrafning asosiy natijasi quyidagi teoremadan iborat.

15-Teorema. *$f \in C^8(\overline{V})$ (27) shakldan iborat bo‘lsin. U holda shunday musbat ε son va koordinatalar boshida atrof mavjudki $U \subset V$, $a \in C_0^1(U)$ va $F \in C^8(\overline{V})$, $\|F\|_{C^8(\overline{V})} < \varepsilon$ funksiyalar uchun quyidagi baho o‘rinli:*

$$\left| \int_U e^{i\lambda(f+F)} a(x) dx \right| \le \frac{C \|a\|_{C^1} \ln(2 + |\lambda|)}{|\lambda|^{\frac{1}{2}}}.$$

“Ikki o‘zgaruvchili tebranuvchan integrallarning invariant baholari” nomli paragrafida fazasi ko‘phad bo‘lgan tebranuvchan integrallarning invariant baholarini qaraymiz. Olingan baholar Popovning trigonometrik integrallarning o‘zgarmas baholari bo‘yicha natijalarini fazasi uchinchi darajali ko‘phad bo‘lgan holda yaxshilaydi.

$W_1^n(\mathbb{R}^n)$ orqali Sobolev fazosini belgilaymiz va unda normani quyidagicha aniqlaymiz

$$\|\phi\|_{W_1^n(R^n)} = \int_{R^n}\left(\sum_{|\alpha|\le n}|D^\alpha\phi|\right)dx, \tag{28}$$

bunda $D^\alpha\varphi$ φ dan olingan umumlashgan ma'nodagi hosila.

P ko'phad quyidagicha berilgan bo'lsin

$$\mathrm{P}(x_1,x_2,a) = P_3(x_1,x_2,a_3) + P_2(x_1,x_2,a_2) + P_1(x_1,x_2,a_1),$$

bunda $P_3(x_1,x_2,a_3) := a_{30}x_1^3 + 3a_{31}x_1^2x_2 + 3a_{32}x_1x_2^2 + a_{33}x_2^3$, $P_2(x_1,x_2,a_2) := a_{20}x_1^2 + 2a_{21}x_1x_2 + a_{22}x_2^2$ va $P_1(x_1,x_2,a_1) := a_{10}x_1 + a_{11}x_2$.

Quyidagi integralni qaraymiz

$$J = \int_{R^2} e^{iP(x_1,x_2,a)}\phi(x_1,x_2)dx_1dx_2, \tag{29}$$

bu urda $\varphi \in W_1^2(\mathrm{R}^2)$. Shubhasiz, oxirgi integral odatdagi Lebeg ma'nosida yaqinlashadi.

Paragrafning asosiy natijasi quyidagidan iborat.

16-Teorema. *Shunday musbat* C *musbat soni mavjudki, ixtiyoriy* $\varphi \in W_1^2(\mathrm{R}^2)$ *uchun ushbu baho o'rinli*

$$|J| \le \frac{C\|\phi\|_{W_1^2(R^2)}}{|D(P_3)|^{\frac{1}{6}}}, \tag{30}$$

bunda $D(P_3) := 3a_{31}^2a_{32}^2 + 6a_{30}a_{31}a_{32}a_{33} - 4a_{30}a_{32}^3 - 4a_{31}^3a_{33} - a_{30}^2a_{33}^2$ $P_3(x_1,x_2,a_3)$ *ko'phadning diskriminanti.*

Ushbu 16-teoremani va [I.A. Ikromov, Invariant estimates of two-dimensional trigonometric integrals. Math. USSR Sb. Vol. 76 (1990), No. 2: 473-488.] ishning 4-teoremasini bilan birlashtirib, quyidagi natijani olamiz.

17-Teorema. *(29) integral uchun quyidagi baho o'rinli:*

$$|J| \le \frac{C\|\phi\|_{W_1^2(R^2)}}{|D(P_3)|^{\frac{1}{6}}+|H|^{\frac{1}{4}}+N^{\frac{1}{6}}}, \tag{31}$$

bunda $N := a_{30}^2 + 3a_{31}^2 + 3a_{32}^2 + a_{33}^2$, $H := a_{31}^2 + a_{32}^2 - a_{30}a_{32} - a_{31}a_{33}$.

Quyidagi mulohaza Dyustermaat teoremasining umumlashgan Eyri funksiyasining $D_4^{\pm}$ maxsusliklarga bog'liq bo'lgan chegaralanganlik haqidagi analogi hisoblanadi.

18-Teorema. *Shunday musbat* C *konstanta topilib, ixtiyoriy kompakt* T *uchburchak uchun quyidagi baho o'rinli*

$$\left|\int_T e^{i\mathrm{P}}dx_1dx_2\right| \le \frac{C}{|D(P_3)|^{\frac{1}{6}}}.$$

6-Eslatma. [I.A. Ikromov, Invariant estimates of two-dimensional trigonometric integrals. Math. USSR Sb. Vol. 76 (1990), No. 2: 473-488.] *ishning 4-teoremasi quyidagi natijani beradi*

$$|J| \le \frac{C\|\phi\|_{W_1^2(R^2)}}{\frac{|D(P_3)|^{\frac{2}{3}}}{N} + N^{\frac{1}{6}} + |H|^{\frac{1}{4}}}.$$

Shubhasiz, $\frac{|D(P_3)|^{\frac{2}{3}}}{N}$ maxrajning bosh hadi bo‘lgan vaqtda bizning (31) bahodan yomonroqligi ko‘rinadi. Olingan baho T uchburchakga bog‘liq emasligini eslatamiz.

7-Eslatma. [D. A. Popov, Remarks on uniform combined estimates of oscillatory integrals with simple singularities Izvestiya: Mathematics (2008),72(4): 793.], *da umumiyroq bo‘lgan quyidagi ko‘rinish berilgan bo‘lib*

$$P(x,a) = P_3(x,a_3) + P_2(x,a_2) + P_1(x,a_1) + R(x),$$

bunda $R(x) = O(|x|^4)$, *hamda (31) baho o‘rinli emas. Bu holda tebranuvchan integrallarning xarakterini o‘rganish murakkabroqdir.*

XULOSA

Dissertatsiya ishi tebranuvchan integrallarning tekis, klassik gruppalar invariantlari orqali ifodalangan baholari, fazasi uchinchi darajali bir jinsli ko‘phad bo‘lgan ikki karrali tebranuvchan integrallarning invariant baholari, bu integrallar uchun yig‘ish masalasi, hamda fazasi uchinchi darajali va birinchi darajali ko‘phadlarning yig‘indisi bo‘lgan ikki karrali tebranuvchan integrallarning invariant baholari masalalariga bag‘ishlangan.

Dissertatsiyada olingan ilmiy natijalar asosida quyidagi xulosalarga kelindi:

1) D_∞ maxsuslikga ega bo‘lgan Mittag-Leffler funksiyasi uchun baholar olingan.

2) Fazasi $D_4^{\pm}$ va A_∞^2 maxsusliklarga ega bo‘lgan umumlashgan tebranuvchan integrallarning baholari olingan.

3) Umumlashgan tebranuvchan integrallar uchun Richchi-Steyn lemmasining analogi isbotlangan.

4) Fazasi ikki o‘zgaruvchili bir jinsli uchinchi darajali ko‘phad bo‘lgan umumlashgan tebranuvchan integrallar uchun invariant baholar olingan.

5) Fazasi bir o‘zgaruvchili uchinchi darajali ko‘phad uchun invariant baho olingan.

6) Ikki o‘zgaruvchili Mittag-Leffler funksiyasi uchun baho oligan.

7) Fazasi to‘la kvadrat bo‘lmagan ikki o‘zgaruvchili tebranuvchan integrallarning aniq jamlash ko‘rsatkichi topilgan.

8) Fazasi to‘la kvadrat bo‘lgan ko‘p o‘zgaruvchili tebranuvchan integrallarning aniq jamlash ko‘rsatkichi topilgan.

9) Fazasi bir jinsli uchinchi darajali va chiziqli funksiyalar yig‘indisidan iborat ikki o‘zgaruvchili tebranuvchan integrallarning aniq jamlash ko‘rsatkichi topilgan.

10) Fazasi D_∞ maxsuslikga ega bo‘lgan silliq funksiya bo‘lgan tebranuvchan integrallar uchun tekis baho olingan.

11) Fazasi to‘rtinchi darajali bir jinsli ikki o‘zgaruvchili silliq funksiya bo‘lgan tebranuvchan integrallar uchun tekis baholar olingan.

12) Fazasi uchinchi darajali ikki o‘zgaruvchili ko‘phad uchun invariant baho olingan.

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GENERILIZED OSCILLATORY INTEGRALS AND THEIR APPLICATIONS

01.01.01 - Mathematical analysis

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INTRODUCTION (abstract of DSc dissertation)

Actuality and demand of the theme of dissertation. Many scientific and practical researches carried out on a global scale are mostly focused on the study of uniform estimates for oscillattory integrals. This is certainly reflected in the extensive literature available on the subject. The method of trigonometric sums introduced in the twenth of the 20th century and the expression of the asymptotic character of these sums through trigonometric integrals became the basis for the development of the theory of oscillatory integrals. Thus, important area for the study of oscillatory integrals is their application to analytic number theory. There are two aspects to the study of the nature of such integrals, one of which is related to trigonometric sums and the other to trigonometric integrals. One of the important issue remains the development of research on the behavior of oscillatory integrals.

In world science, attention was paid to the current directions of mathematical analysis with practical application of fundamental sciences. In particular, special attention was paid to the study of the character of oscillatory integrals. Remarkable results were obtained on the study of Fourier transforms of smooth Borel measures supported to hypersurfaces of Euclidean space and the Restriction problem of Fourier transforms to hypersurfaces. Algebra and mathematical analysis, theory of dynamical systems, applied mathematics and mathematical modeling are important scientific researches from a practical-theoretical point of view.

In our republic, great attention is paid to fundamental sciences of scientific and practical importance, in particular, special attention is paid to the development of the theory of harmonic analysis. Conducting scientific research at the level of international standards in the priority areas of "Algebra and its applications, differential equations and their applications, mathematical modeling of nonlinear systems, dynamic systems and their applications, stochastic analysis, medical-biological informatics, computational mathematics[3]" are the main tasks of mathematics and were defined as the directions of activity. The development of the theory of oscillatory integrals of harmonic analysis issues is considered to be of great scientific importance in ensuring the implementation of this decision.

Conducting research at the level of international standards in the priority areas of "Algebra and its applications, differential equations and their applications, linear systems, dynamic systems and mathematical modeling of their applications, stochastic analysis, medical-biological informatics, computational mathematics" is identified as one of the main tasks of the scientific researches. To ensure the implementation of the decision, it is important to develop theory of oscillatory integrals and estimates for convolution operators.

The subject and object of our dissertation are in line with tasks identified in the Decrees and Resolutions of the President of the Republic of Uzbekistan of February 7, 2017, PF-4947, "On the strategy of action for the further development of the Republic of Uzbekistan", PQ-4387 dated July 9, 2019 "On state support for the

[3] Decision of the President of the Republic of Uzbekistan No. PQ-4387 of July 9, 2019 "On state support for the further development of mathematics education and sciences, as well as measures to fundamentally improve the activities of the Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan named after V.I. Romanovsky"

further development of mathematics education and science, as well as measures to radically improve the activities of the Institute of Mathematics named after V.I. Romanovsky of the Academy of Sciences of the Republic of Uzbekistan" and PQ-4708 of May 7, 2020 "On measures to improve the quality of education and research in the field of mathematics" as well as in other regulations related to this activity.

Connection of research to priority directions of development of science and technologies of the Republic. This study was performed in accordance with the priority areas of science and technology of the Republic of Uzbekistan, IV, "Mathematics, Mechanics and Computer Science".

Review of foreign research on the topic of the dissertation.[4] The scientific research on generalized oscillatory integrals and their applications in higher education institutions and scientific research centers of leading foreign countries, including Ghent Analysis and PDE Center (Belgium), School of Mathematical Sciences, Queen Mary University of London (England), Christian- Extensive research is being conducted at Albrechts-Universitat zu Kiel (Germany), Ganja State University (Azerbaijan), Moscow State University (Russia).

As a result of research conducted in the world on generalized oscillatory integrals and their applications, a number of topical issues were solved, including the following scientific results: estimates of one-variable generalized oscillatory integrals were obtained (Ghent Analysis and PDE Center (Belgium), School of Mathematical Sciences, Queen Mary University of London (England)); found invariant estimates of oscillatory integrals (Moscow State University (Russia)); Summation exponents for polynomial phase oscillatory integrals partially were found (Ganja State University (Azerbaijan)).

Today, there are a number of researches on generalized oscillatory integrals and their applications, including: obtaining invariant estimates of oscillating integrals with phase polynomials, obtaining uniform estimates of oscillatory integrals with special polynomials phase, summation of oscillatory integrals with homogeneous third-order polynomials with phase two variables scientific research works are being carried out in the priority areas such as finding the oscillating index, obtaining estimates of the oscillatory integrals associated with the Mittag-Leffler function, and finding the asymptotics of the oscillatory integrals associated with the Mittag-Leffler function.

The degree of scrutiny of the problem. Oscillatory (trigonometric) integrals first appeared in 1818 in Fresnel's work related to the study of light intensity. Using such integrals, he solved the problems of edge, screen, and small-hole diffraction. In 1838, Airy used trigonometric integrals to study the intensity of light around caustics. A caustic is a collection of light whose intensity is much higher than at other ordinary points. Later, in 1887, in the studies of Kelvin and Riemann, the stationary phase method was used to study the character of oscillatory integrals. In 1891-1892, Poincare made extensive use of oscilatory integrals in his research on the intensity of light. In particular, he proved that when light passes through a

[4] Review of foreign research on the topic of the dissertation: https://www.springer/com/gp/mathematics; https://onlinelibrary.wiley.com/; www.mathnet.ru; www.scholar.google.com; Developed from www.elsevier.com/locate/camw and other sources.

caustic, the phase shifts to $\frac{\pi}{2}$. This assertion was previously confirmed experimentally by Gui. In the later researches, Pearce, Ludwig, Ursell, Connor and other scientists appeared in mathematical physics applications of oscillatory integrals with special phases. In 1974, in the work of J. Dustermat, generalized Airy functions, more precisely, functions related to Arnold's elliptic singularities, were studied.

In 1973, V. I. Arnold proposed the following hypothesis. Suppose that the phase function of the oscillatory integral depends on some parameters. Of course, the phase function can be a smooth (even analytic) function of parameters and "basic" variables (integral variables). That is, let us have a family of phase functions. Let these parameters vary around the zero point at the assigned values. Therefore, they are sometimes called small parameters. Let us know the oscillation index of the oscillatory integral (that is, the phase function is determined by the value of the parameter at the zero point). Then, when the parameters change around the zero point sufficiently small, the oscillatory integral depending on the parameters is evaluated by the level of a large parameter arbitrarily smaller than this oscillatory index. In other words, the oscillation index is a semi-continuous function of the "subparameters" of the phase. This issue is also covered in Arnold's Problems (an expanded English edition of this book was published in 2004).

Arnold's hypothesis was proved by V. N. Karpushkin. More specifically, smooth (with respect to a sufficiently small "shift" of the phase function) estimates of the double oscillatory integrals were obtained, with the phase function being analytic. In other words, V. N. Karpushkin proved that V. I. Arnold's hypothesis is valid for oscillatory integrals with analytic phase. This result was later proved in Hasanov's PhD thesis for some special deformations of smooth functions.

The important area for the study of oscillatory integrals is their application to analytic number theory. It is known that in the studies of I.M. Vinogradov, the integral corresponding to the number of solutions of Diophantine equations (the so-called Vinogradov integral) occupies an important place. However, there are two aspects of investigating the behavior of such integrals, one of which is related to trigonometric sums and the other to trigonometric integrals. In 1949, the Chinese mathematician Hua Lo Gen posed the problem of finding the minimum exponent of the trigonometric integral belonging to a suitable space. This issue has been considered in a number of works, and estimates obtained by Vinogradov are based on variant of classical Van der Korput lemma. This problem was solved in 1979 by G.I. Arkhipov, V.N. Chubarikov, and A.A. Karatsuba for one-dimensional oscillatory integrals, and they found an upper boun for multiple trigonometric integrals. However, this issue is still open in general.

After the works of J. Mokenhoupt, interest in this issue increased. It turns out that this problem is related to the Restriction problem of Fourier transforms of integrable functions. More specifically, it becomes clear that the lower bound for the summation exponent is the same as that obtained for the bounded operator. It should be noted that the estimate obtained by G.I. Arkhipov, V.N.Chubarikov and A.A.Karatsubas allows to obtain the sharp lower estimate for the index of the space of integrable functions with some degree. In this case, the Restriction problem on

model curves with torsion is considered. However, the boundedness problem for curves of finite order and the Restriction problem of the Fourier transform in Euclidean space are still open.

The connection of the topic of the dissertation with the research work of the higher educational institution, in which the dissertation is carried out. The dissertation was done in accordance with the planned theme of scientific research of fundamental project OT-F4-69 "Harmonic analysis, power geometry and their applications to the problems of mathematical physics" (OT-F4-69, Samarkand State University, 2017-2020).

The aim of the research is to obtain estimates of generalized oscillatory integrals and to find an exact summation index.

Research tasks:

obtain smooth estimates for oscillatory integrals with a discontinuity of the amplitude function using the transition method in accuracy ε;

estimation for generalized oscillatory integrals associated with the Mittag-Leffler function, whose phase function has singularities D_∞, $D_4^\pm$ and A_r;

obtain smooth estimates for the generalized and classical oscillatory integrals associated with the two-variable Mittag-Leffler function, the phase function of which is polynomial;

for oscillating integrals whose phase is a third-order polynomial, obtaining estimates of the group of classical actions in the Euclidean plane through invariants;

to find the exact summation exponent in the coefficient space of the classical oscillatory integrals where the phase function is a polynomial of the second and third order.

The research object. Consists of generalized oscillatory integrals whose phase is polynomial.

The research subject. State of oscillating integrals when the coefficients of the phase function tend to infinity, estimation of oscillatory integrals by means of invariants of classical groups

Research methods. The theory of singularities of differential maps, the theory of analytic functions, and asymptotic methods of analysis were used in the thesis work.

The scientific novelty of the research is as follows:

it is obtained smooth estimates for oscillatory integrals with a discontinuity of the amplitude function using the transition method in accuracy ε;

it is estimated for generalized oscillatory integrals associated with the Mittag-Leffler function, whose phase function has singularities D_∞, $D_4^\pm$ and A_r;

it is obtained smooth estimates for the generalized and classical oscillatory integrals associated with the two-variable Mittag-Leffler function, the phase function of which is polynomial;

for oscillating integrals whose phase is a third-order polynomial is obtained estimates of the group of classical actions in the Euclidean plane through invariants;

it is finded the exact summation exponent in the coefficient space of the classical oscillatory integrals where the phase function is a polynomial of the second and third order.

The practical results of the research are as follows:

it consists in the fact that the exact summation exponent of the oscillatory integrals whose phase is a third-order polynomial in two variables is found;

it can be used to obtain the proof of Van der Korput's lemma for generalized oscillatory integrals whose phase is a polynomial in two or more variables;

estimates of generalized oscillating integrals can be used to estimate solutions of partial differential equations of fractional order.

The reliability of the results is based on the use of methods of mathematical analysis, functional analysis, mathematical physics, differential equations, harmonic analysis theory, as well as mathematical proofs by applying rigorous mathematical considerations.

Scientific and practical significance of research results. The scientific significance of the research results is explained by the fact that the existence of invariant estimates of the generalized oscillatory integrals, which is a third-order polynomial phase, is shown.

The practical importance of the research results is determined by the fact that they serve as a basis for researching the nature of the fundamental solutions of hyperbolic equations.

Implementation of research results. Based on the estimations of the generalized oscillatory integrals and the results related to their applications:

scientific results on invariant estimates of oscillatory integrals with polynomial phase in leading foreign journals (Nonlinear Analysis, Volume 207, June 2021, 112292, Analysis and Mathematical Physics volume 12, Article number: 130 (2022), Journal of Siberian Federal university. mathematics and physics 15.4 (2022): 459-466.) have been used in the study of problems corresponding to some nonlinear differential equations. Application of scientific results made it possible to prove that the spectra of the considered issues are discrete;

Generalized oscillatory integrals and their applications have been used in the research project number 01M01021 by the FWO Odysseus 1 grant G.0H94.18N: Analysis and Partial Differential Equations and by the Methusalem programme of the Ghent University Special Research Fund (BOF) (reference of Ghent University dated Aprel 18, 2023). The application of scientific results made aprior solution to partial differential equations with fractional derivatives;

Results of estimates for generalized oscillatory integrals with polynomial phase in the project number AP08052046 "Some non-local analogs of non-linear partial differential equations" (reference number 01-06/060 of Institute of Mathematics and mathematical modelling dated Aprel 17, 2023) estimates for generalized oscillatory integrals with Mittag-Leffler functions. Application of scientific results made it possible to estimated solution to Koshi problem for partial differential equations with fractional derivatives.

Approval of research results. The main results of the research were discussed at 14 scientific and scientific-practical conferences, including 8 international and 6 national scientific and scientific-practical conferences.

Publication of research results. A total of 28 scientific works were published on the topic of the dissertation, of which 14 articles were published in scientific publications, including 6 foreign and 8 republic journals, in which the main scientific

results of doctoral dissertations of the Higher Attestation Commission of the Republic of Uzbekistan were recommended for publication.

The structure and scope of the dissertation. The dissertation consists of an introduction, four chapters, a conclusion and a list of references. The volume of the dissertation was 167 pages.

THE MAIN CONTENT OF THE DISSERTATION

In the introduction is given the actuality and relevance of the thesis topics are described, the appropriate research priority areas of science and technology of the Republic are determined. Moreover, we give a review of international research on the theme of the dissertation and the degree of scrutiny of the problem, formulate goals and objectives, identify the object and subject of study. The scientific novelty and practical results of the research, the theoretical and practical importance of the obtained results, information on the implementation of the research results about the published works and the structure of dissertation are also presented in this chapter.

In the first chapter of the thesis, titled **"Pleminaries",** we define necessary definitions, notations and some results which will be used through the thesis. Also, we provide proofs of some preliminary statements. In particular, we reduce normal form of functions, with respect to linear change of coordinates, having simple singularities. Further, we consider oscillatory integrals with analytic phase function and amplitude having set of discontinuity points. We obtain estimates for such integrals which are sharp up to a positive number $\varepsilon > 0$.

We will restrict ourselves to the two-dimensional case, i.e. $n = 2$. Let us consider the corresponding oscillatory integral with a smooth amplitude having a compact support:

$$J_0(\lambda, s) = \int_{R^2} a(x, s) e^{i\lambda\Phi(x,s)} dx.$$

The basic condition:

Let $s \in K$ and for (η_1, η_2) consider a new phase function

$$\Phi_1(x, s, \eta) = \Phi(x, s) + x_1\eta_1 + x_2\eta_2.$$

Main assuption: suppose that $\Phi(x, s)$ is an analytic function such that for any critical point of the function $\Phi_1(x, s^0, \eta^0)$ where $(s^0, \eta^0) \in K \times R^2$, and its oscillation exponent $\beta \le \beta_0$. We have the following statement.

Theorem 1. *If the phase function satisfies the main condition, then for any positive number* $\varepsilon > 0$ *the following estimate is valid*

$$\left|J(\lambda, s)\right| \le \frac{c_\varepsilon \left\|a(\bullet, s)\right\|_{C^3}}{\lambda^{\beta_0 - \varepsilon}}.$$

Consider the oscillatory integral

$$J(\lambda, s) = \int_{R^2} \chi_{D(s)}(x) a(x, s) e^{i\lambda\Phi(x,s)} dx, \quad (1)$$

where $D(s)$ – family of regions in R^2 and $\{F_1(x, s) > 0, F_2(x, s) > 0\} \subset D(s)$, furthermore $\gamma_1, \gamma_2 \subset \partial D(0)$, where $\gamma_j = \{x : F_j(x, 0) = 0\}, j = 1, 2.$

The following theorem is true.

Theorem 2. *If the oscillation index of the critical point* $x = 0$ *of the function* $\Phi(x,0)$ *is* β*, then there exists a neighborhood of zero* $U \times V \subset R^2 \times R^m$ *such that, for any fixed positive number* $\varepsilon > 0$*, for any amplitude function* $a \in C_0^\infty(U \times V)$ *the following estimate is valid:*

$$\left|J(\lambda, s)\right| \leq \frac{A_\varepsilon \left\|a\right\|_{C^3}}{\lambda^{\beta-\varepsilon}}.$$

In the second chapter of the thesis, titled **"Oscillatory integrals with Mittag-Leffler functions",** we consider the following integral with phase f and amplitude ψ is an integral of the form:

$$I_{\alpha,\beta} = \int_U E_{\alpha,\beta}(i\lambda f(x))\psi(x)dx \tag{2}$$

where $0 < \alpha < 1$, $\beta > 0$, U is a sufficiently small neighborhood of the origin and $\lambda > 0$.

In **"Estimates for Mittag-Leffler functions with smooth phase having simple singularities"** section we consider the problem on uniform estimates for Mittag-Leffler functions with the smooth phase functions having singularities D_∞, $D_4^\pm$ and A_r. The generalisation is that we replace the exponential function with the Mittag-Leffler-type function, to study oscillatory type integrals.

Let we consider homogenous polynomial third degree with two variables. The main result of the section is the following.

Theorem 3. *Let* $-\infty < a < b < \infty$*. Let phase function has the a homogenous polynomial third degree with two variables and let* $\psi \in L^p[a,b]^2$, $1 < p \leq \infty$*. If* $0 < \alpha < 1$, $\beta > 0$ *and* $\lambda > 0$ *then we have following estimates*

$$\left|\int_{[a,b]^2} E_{\alpha,\beta}(i\lambda x_1^2 x_2)\psi(x)dx\right| \leq \frac{C|\psi|_{L^p}}{\lambda^{\frac{1}{2}-\frac{1}{2p}}}, \tag{3}$$

$$\left|\int_{[a,b]^2} E_{\alpha,\beta}(i\lambda(x_1^2 x_2 \pm x_2^3))\psi(x)dx\right| \leq \frac{C|\psi|_{L^p}}{\lambda^{\frac{2}{3}-\frac{1}{3p}}}, \tag{4}$$

$$\left|\int_{[a,b]^2} E_{\alpha,\beta}(i\lambda x_1^3)\psi(x)dx\right| \leq \frac{C|\psi|_{L^p}}{\lambda^{\frac{1}{3}-\frac{1}{3p}}}, \tag{5}$$

where constant C is depend only p.

In **"Estimates for generalizated oscillatory integrals with polynomial phase"** section we consider the problem on uniform estimates for generalized oscillatory integrals given by Mittag- Leffler functions with the homogeneous polynomial phase.

We obtain a variant of the Ricci-Stein lemma and invariant estimates for corresponding integrals.

Let

$$I_{\alpha,\beta}(a) = \int_{Q^n} E_{\alpha,\beta}(iP(a,x))\psi(x)dx, \tag{6}$$

where $0 < \alpha < 1$, $\beta > 0$, $P(a,x)$ is a phase, $\psi \in C^\infty(R^n)$ is an amplitude and $Q^n := [0,1]^n$ is n dimensional cube.

Theorem 4. *Let* $0 < \alpha < 1,\ \beta > 0$. *There exists positive number* C_d *such that for the integral (6) with phase* $P(a,x)$ *following inequality holds*

$$\left|I_{\alpha,\beta}\right| \le \frac{C_d}{\|a\|^{\frac{1}{d}}}, \tag{7}$$

where $\|a\| = \sum_{|\lambda|\le d} |a_\lambda|$.

Remark 1. *Theorem 4 is an analog of the Ricci-Stein lemma.*

Proposition 1. *Let* $P: R^n \to R$ *be a polynomial of degree* $\le d$ *and* $|a| = \max\{|a_\lambda|, |\lambda| \le d\}$. *There exists a constant* $C_{d,n}$ *such that if* $a^0 \in S^N$ *(where* $N+2$ *is the dimension of space of polynomials of degree at most* d*) is a fixed point and* $|D^\alpha P(a^0, x)| \ge \delta > 0,\ \forall x \in [0,1]^n$ *(where* $\alpha \in Z_+^n$ *with* $|\alpha| \ge 2$*), then the following inequality holds*

$$\left|I_{\alpha,\beta}(\mu a^0)\right| \le \frac{C_{d,n}}{|\mu|^{\frac{1}{|\alpha|}}},$$

here $S^N = \{|a| = 1\}$ is the unit sphere and $\mu > 0$.

In **"Invariant estimates for homogeneous polynomial phase"** section, we consider generalized oscillatory integrals with phase function which is a homogeneous polynomial of degree three in two variables

$$P_3(a,x,y) = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3, \tag{8}$$

here $a := (a_0, a_1, a_2, a_3) \in R^4$ are coefficients. Consider

$$I_{\alpha,\beta}(a) = \int_{R^2} E_{\alpha,\beta}(iP_3(a,x,y))\psi(x,y)dxdy, \tag{9}$$

where ψ is a smooth function of compact support.

We study the behavior of integral (9) in the case when the coefficients of the polynomial tend to infinity. Let us obtain estimates for integral (9) in terms of the invariants of the group of motions of the Euclidean plane. Note that, the discriminant of a polynomial denoted by D is defined by the formula:

$$D = 3a_1^2a_2^2 + 6a_0a_1a_2a_3 - 4a_0a_2^3 - 4a_1^3a_3 - a_0^2a_3^2.$$

It is the invariant of the group $SL(2, \mathbb{C})$.

Theorem 5. *For the integral (9) with phase (8) following inequality*

$$\left|I_{\alpha,\beta}(a)\right| \le \frac{c\|\psi\|_{L^\infty}}{|D|^{\frac{1}{6}}}$$

holds. Here c *is a constant independent of* a *and* $D \ne 0$ *is discriminant of the polynomial* $P_3(a,x,y)$.

Then we consider one-dimensional integrals. The results of this section will be used in the next section to get invariant estimates. Now we consider the following example for $d = 3$.

Let $0 < \delta < 1$. We consider the integral

$$J_{\alpha,\beta}(p,q) = \int_R E_{\alpha,\beta}(i(x^3 + px + q))\,dx, \tag{10}$$

where $0 < \alpha < 1,\ \beta > 0$.

Theorem 6. *1) Let* $\frac{1}{3} < \delta < \frac{1}{2}$. *Then*

$$\left|J_{\alpha,\beta}(p,q)\right| \le \frac{c_\delta}{\left(\frac{|p|^3}{27}+\frac{q^2}{4}\right)^{\frac{3\delta-1}{6}}}, \tag{11}$$

for any $(p,q) \neq (0,0)$.

2) Let $\frac{1}{2} < \delta < 1$. *Then*

$$\left|J_{\alpha,\beta}(p,q)\right| \le \frac{c_\delta}{|D|^{\delta-\frac{1}{2}}\left(\frac{|p|^3}{27}+\frac{q^2}{4}\right)^{\frac{2-3\delta}{6}}}, \tag{12}$$

where $D = \frac{p^3}{27}+\frac{q^2}{4} \neq 0$ *is discriminant of polynomial* $x^3 + px + q$ *and* $c_\delta > 0$ *a positive number depending only on* δ.

In **"Estimates for Mittag-Leffler functions with two variables"** section we consider the problem estimates for Mittag-Leffler functions with two variables. The generalisation is that we replace the exponential function with the Mittag-Leffler-type function, to study oscillatory type integrals.

The main result of this section is the following.

Theorem 7. *Let* f *be a smooth finite type function with two variables defined in a sufficiently small neighborhood of the origin and let* $\psi \in C_0^\infty(U)$.

If $0 < \alpha < 1$, $\beta > 0$, $h > 1$ and $\lambda > 2$ then we have following estimate

$$\left|\int_U E_{\alpha,\beta}\left(i\lambda f(x_1,x_2)\right)\psi(x)dx\right| \le \frac{C|\ln\lambda|^m \|\psi\|_{L^\infty(\overline{U})}}{\lambda^{\frac{1}{h}}}, \tag{13}$$

where h *is a height of the function also* $m = 0,1$ *multiplicity of the Newton polyhedron.*

If $0 < \alpha < 1$, $\beta > 0$, $h = 1$ *and* $\lambda > 2$ *then we have following estimate*

$$\left|\int_U E_{\alpha,\beta}(i\lambda(x_1^2+x_2^2))\psi(x)dx\right| \le \frac{C|\ln\lambda|\|\psi\|_{L^\infty(\overline{U})}}{\lambda} \tag{14}$$

and

$$\left|\int_U E_{\alpha,\beta}(i\lambda(x_1^2-x_2^2))\psi(x)dx\right| \le \frac{C|\ln\lambda|^2\|\psi\|_{L^\infty(\overline{U})}}{\lambda}, \tag{15}$$

where constant C *is independent of the phase and amplitude.*

Proposition 2. *Let* $f:\Omega \to \mathbb{R}$ *be a measurable function such that for all real* $\lambda > 2$ *and for any positive* $\delta \neq 1$, *we have*

$$\left|\int_\Omega e^{i\lambda f(x)}dx\right| \le C\,|\lambda|^{-\delta}|\ln\lambda|^m, \tag{16}$$

with $m \ge 0$. *Then, we have*

$$\left|x \in \Omega : |f(x)| \le \varepsilon| \le C_\delta \varepsilon^\delta |\ln\varepsilon|^m, \text{for } \delta < 1,\right.$$

$$\text{and for } \delta > 1, |x \in \Omega: |f(x)| \le \varepsilon| \le C_\delta \varepsilon|,$$

where C_δ *depends only on* δ.

Corollary 1. *Let* $a(x) = \begin{cases} 1, & \text{when } |x| \le \sigma, \\ 0, & \text{when } |x| \ge 2\sigma \end{cases}$ *and* $a(x) \ge 0$ *with* $a \in C_0^\infty(R^2)$. *Then for all real* $\lambda > 2$ *and for any positive* h, *we have*

$$\left| \int_{R^2} e^{i\lambda f(x)} a(x)\,dx \right| \le C\,|\lambda|^{-\frac{1}{h}} |\ln \lambda|^m, \tag{17}$$

where $m \ge 0$, f is a smooth finite type function at the origin, h is the height of the function f and m is the multiplicity of the Newton polygon.

Corollary 2. *Let* $f(x_1, x_2)$ *be a smooth function and* $\overline{\Omega}$ *is a sufficiently small compact set of the origin that for all real* $\varepsilon > 2$ *we have*

$$|\{x \in \Omega : |f(x)| \le \varepsilon\}| \le C\varepsilon^{\frac{1}{h}} |\ln \varepsilon|^m,$$

where h *is the height of* f *and* m *is a multiple.*

Proposition 2. *For any smooth function with compact support* $a(x, y)$ *and for* $h > 1$ *the following inequality holds*

$$\int_{\{|f(x,y)| \ge \frac{M}{\lambda}\}} \frac{a(x,y)}{1 + \lambda\,|f(x,y)|}\,dxdy \le \frac{C\,|\ln \lambda|}{\lambda^{\frac{1}{h}}}. \tag{18}$$

where $f(x, y)$ is smooth function.

Remark 2. *If* $h = 1$ *and as* $f(x, y)$ *is smooth function with* $\nabla f(0,0) = 0$. *Using Morse lemma we have* $f : x^2 \pm y^2$. *So in this case we estimate two sets* $\Delta = \Delta_1 + \Delta_2$, *where* $\Delta_1 := \{(x, y) : |x^2 \pm y^2| \le M\lambda, |x| \le 1, |y| \le 1\}$ *and* $\Delta_2 := \{(x, y) : |x^2 \pm y^2| > M\lambda, |x| \le 1, |y| \le 1\}$.

In **"Estimates for oscillatory integrals with polynomial phase"** chapter it is considered the summation problem for trigonometric integrals with polynomial phase.

Let $P(x, a) \in R[x]$, $x \in R^n$ be a polynomial with real coefficients $a \in R^m$. We consider the trigonometric integral given by

$$J(a) = \int_Q \exp\{2\pi i P(x, a)\}\,dx,$$

where $Q = \{x \in R^n : 0 \le x_l \le 1, l = 1, \dots, n\}$ is an n-dimensional unit cube. Importance of this function is that it appears in the special integrals of the form

$$I_p := \int_{R^m} |J(a)|^p\,da \tag{19}$$

in Tarry's problem, arising as a coefficient of an asymptotic representation for a number of integer solutions of a Diophantine system. Therefore, it is important to find a minimal value of the parameter p such that the integral (19) is convergent.

Problem: *Find the number* $\gamma = \inf\{p \ge 1 : J \in L^p(R^m)\}$.

The number γ is called to be the sharp convergence exponent for the trigonometric integrals.

In this chapter, we find the number γ for trigonometric integrals with polynomial phase function. Let $P(x,s) \in \mathbb{R}[x]$ be a polynomial of $x \in \mathbb{R}^k$ with coefficients $s \in \mathbb{R}^N$. By Q we denote the compact set in $\mathbb{R}^k$.

In **"Estimates for trigonometric integrals with incomplete quadratic phase"** section we consider the summability problem of trigonometric integrals when $k = 3$ and obtain the exact convergence index p_0 in the case when $Q = [0,1]^3$.

In the case when P is a homogeneous polynomial of degree two, we obtain the exact value p_0.

Let polynomial P be given by:

$$P(x, A, b) = (Ax, x) + (\beta, x),$$

where $A = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_1 & 0 & \alpha_3 \\ \alpha_2 & \alpha_3 & 0 \end{pmatrix}$ is a real symmetric matrix, $\beta := (\beta_1, \beta_2, \beta_3)$ and $(\cdot,\cdot)$ the inner product of vectors. The following trigonometric integral

$$T(A, \beta) = \int_{[0,1]^3} \exp(iP(x, A, \beta))\, dx, \tag{20}$$

arises in the study of analytic number theory (see [1]).

We consider the following integral

$$\theta = \int_{\mathbb{R}^6} |T(A, \beta)|^p \, d\beta d\alpha,$$

where $d\beta = d\beta_1 d\beta_2 d\beta_3$ and $d\alpha = d\alpha_1 d\alpha_2 d\alpha_3$.

The following is true:

Theorem 8. *The integral θ converges for $p > 4$ and for $p \le 4$ it diverges. In particular in this case $p_0 = 4$.*

Let us introduce the following notation:

$$\theta_\infty = \int_{\mathbb{R}^6} |T_\infty(A, \beta)|^p \, d\beta d\alpha.$$

Proposition 3. *The integral θ_∞ converges at $p > 4$ and diverges at $p \le 4$.*

Let $K := [0,1]^3$ be cube in $\mathbb{R}^3$ and

$$h(\beta) = \int_{\mathbb{R}^3} e^{|x|^2} \chi_K(x) e^{-2\pi i(\beta, x)} dx,$$

where $\chi_K(x)$ is the characteristic function of the set K.

Proposition 4. *For any positive number ε, the following inclusion holds true* $h \in L_{1+\varepsilon}(\mathbb{R}^3)$.

In **"On estimates for trigonometric integrals with quadratic phase"** section we study the problem by considering the classical setting. In other words, P is a quadratic polynomial function and $Q = [0,1]^k$ is a unit cube. It should be noted that the condition of Makenhaupt does not hold for this case. When P is a homogeneous polynomial of degree two, we obtain the exact value of p_0.

Let polynomial P be given by

$$P(x, A, b) = (Ax, x) + (b, x),$$

where $A = (a_{lm})_{l,m=1}^{k}$ is a symmetric $k \times k$ matrix with real entries, $b := (b_1, b_2, ..., b_k) \in \mathbb{R}^k$ and $(\cdot,\cdot)$ is the inner product of vectors. Consider the trigonometric integral

$$T(A, b) = \int_{\mathbb{R}^k} \exp(iP(x, A, b))\, \chi_K(x)\, dx,$$

where K is a compact set and $\chi_K(x)$ is its characteristic function.

Consider the trigonometric integral

$$\theta = \int_{\mathbb{R}^N} |T(A, b)|^p \, dbda,$$

where $db = db_1 db_2 ... db_k$ and $da = \prod_{1 \le l \le m \le k} da_{lm}$.

The following is true:

Theorem 9. *Let K be a compact set, then the integral θ converges at $p > 2k + 2$ and if K contains an interior point x^0 and there exists a line l passing through point x^0 such that the set $\{l \cap K\}$ contains only a finite number of points, then at $p \le 2k + 2$ the integral diverges. Thus, if K is a compact set with a non-empty interior, then $p_0 = 2k + 2$.*

The case when P is a homogeneous polynomial of the second degree

Now suppose that $P(x, A) = (Ax, x)$.

Theorem 10. *If $P(x, A) = (Ax, x)$ and Q is a polyhedron, then for $p > 2k$ the integral θ converges. If $Q = [0,1]^k$, then at $p \le 2k$ the integral θ diverges.*

By polyhedron we mean the finite union of nondegenerate simplexes.

Remark 3. In this case, we cannot apply the results of [Jong-Guk Bak, Sanghyuk Lee, 2004, "Restriction of the Fourier transform to a quadratic surface in $\mathbb{R}^n$". *Mathematische Zeitschrift* No.247, pp.409-422.] as the corresponding set $\{x_i x_j\}_{i \le j=1}^{n}$ is not a smooth surface.

Remark 4. Depending on the set Q, the exponent p may be smaller than $2k$. For example, if $k = 2$ and Q is a sufficiently small square centered at $(1,1)$, then it can be proved that for $p > 3$ the integral θ converges.

Let us introduce the following notation:

$$\theta_\infty = \int_{\mathbb{R}^N} |T_\infty(A, b)|^p \, dbda,$$

where

$$N = \frac{k(k+2)}{2}.$$

Proposition 5. *The integral θ_∞ converges when $p > 2k + 2$ and diverges when $p \le 2k + 2$.*

Theorem 11. *Let D be a compact domain such that $\hat{\chi}_D \in L_q(R^2)$ and $T(A) = \int_D e^{i(Ax,x)} dx$. Then $T \in L^p(\mathbb{R}^3)$ at $p > 6 - \frac{2}{q}$. Moreover, if $\hat{\chi}_D \in L_{1+0}(R^2)$, then, for any $p > 4$, the inclusion $T \in L^p(\mathbb{R}^3)$ is valid.*

Remark 5. From the results [V. V.Lebedev, 2013, "On the Fourier transform of the characteristic functions of domains with C^1 boundary". *Func. anall. and its appl.* vol. 47, no. 1. pp. 33-46.] given in it follows that there exists a set D other than a polygon such that $\hat{\chi}_Q \in L_{1+0}(R^2)$.

Corollary 3. If $\subset \mathsf{R}^2$ is a compact set such that $\partial D \subset C^1$, then for $p > 4.5$ the relation $T \in \mathsf{L}^p(\mathsf{R}^3)$ holds.

In **"On the L^p – bound for trigonometric integrals"** section we consider the summation problem for two-dimensional trigonometrical integrals with a third-order polynomial phase, which is sum of homogeneous polynomials of degree three and one. We find the sharp convergence exponent using the estimates for trigonometric integrals.

We consider the problem for the case when both the amplitude function ψ and the phase function F depend on parameter a. First we solve the summation problem for oscillatory integrals with a simple phase. Further, we investigate the behavior of the one-dimensional trigonometric integrals with an amplitude containing a pole-type singularity. Finally, we consider more difficult summation problem of the trigonometric integrals with a discontinuous amplitude function and with an algebraic (but not a polynomial) phase function. The obtained results will be applied to the solution of the summation problem for two-dimensional trigonometric integrals.

Let $F(x, y, a)$ be a polynomial function of two variables:

$$F(x, y, a) := P_3(x, y, a_3) + P_1(x, y, a_1), \tag{21}$$

where $P_3(x, y, a_3) = a_{30}x^3 + 3a_{31}x^2 y + 3a_{32}xy^2 + a_{33}y^3$, $P_1(x, y, a_1) = a_{10}x + a_{11}y$.

$a_3 := (a_{30}, a_{31}, a_{32}, a_{33})$, $a_1 := (a_{10}, a_{11})$ are vectors in R^4, R^2 respectively.

The main results of the section are the following theorems.

Theorem 12. *Let $\psi \in C_0^\infty(\mathsf{R}^2)$ and*

$$J(a, \psi) := \int_{\mathsf{R}^2} e^{iF(x,y,a)} \psi(x, y)\, dxdy. \tag{22}$$

Then $J(\cdot, \psi) \in \mathsf{L}^p(\mathsf{R}^6)$ for any $p > 8$. Moreover, if $\psi(\cdot,\cdot)$ is a smooth function supported in a sufficiently small neighborhood of the origin with $\psi(0,0) \neq 0$, then $J(\cdot, \psi) \notin \mathsf{L}^p(\mathsf{R}^6)$ for all $p \leq 8$.

Now we consider the oscillatory integral with the phase function (21) and a discontinuous amplitude function $\psi(x, y)\chi_Q(x, y)$ so that

$$J(a, \psi, \chi_Q) := \int_Q e^{iF(x,y,a)} \psi(x, y)\, dxdy, \tag{23}$$

where $\psi \in C_0^1(\mathsf{R}^2)$ and $Q := [0,1] \times [0,1]$.

Theorem 13. *If $J(\cdot, \psi, \chi_Q)$ is the trigonometrical integral given by (23), then $J(\cdot, \psi, \chi_Q) \in \mathsf{L}^p(\mathsf{R}^6)$ for all $p > 8$. Moreover, if $\psi(\cdot,\cdot)$ is a smooth function, concentrated in a neighborhood of the origin with $\psi(0,0) \neq 0$, then $J(\cdot, \psi, \chi_Q) \notin \mathsf{L}^p(\mathsf{R}^6)$ for all $p \leq 8$.*

Corollary 4. *For the integral*

$$J(a,\chi_Q)=\int_Q e^{iF(x,y,a)}dxdy,$$

the relation $J(\cdot,\chi_Q)\in L^p(R^6)$ holds for all $p>8,$ and if $p\le 8$ then $J(\cdot,\chi_Q)\notin L^p(R^6)$. In other words we have $\gamma=8.$

Summation of the one dimensional oscillatory integrals with amplitude depending on parameters

We introduce a class of amplitudes and obtain estimates for corresponding one-dimensional oscillatory integrals which will be used in the proofs of main results in general case.

Given a closed cube $B\subset \mathsf{R}^3$ centered at the origin, denote by A_B the class of functions $\phi\in C^\infty(\mathsf{R}^2\times B\times\mathsf{R})$ satisfying the following conditions:

• for any $\phi\in \mathsf{A}_B$ there exists a compact set $K_\phi\subset \mathsf{R}^2$ such that $supp\ \phi(\cdot,\xi,b)\subset K_\phi$ for any $(\xi,b)\in B\times\mathsf{R}$;

• for any $\phi\in \mathsf{A}_B$ and $N\ge 0$ there is $C=C(N,\phi)>0$ such that

$$\|\varphi(\cdot,\xi,b)\|_{C^N(R^2)}\le C$$

for any $(\xi,b)\in B\times\mathsf{R}.$

We introduce the measure $\mu=b^3da_1da_2d\eta db$ in

$$R^2\times R^3\times R=\{(a_1,a_2,\eta,b):\eta\in R^3,a_1,a_2,b\in R\},$$

which is absolutely continuous with respect to Lebesgue measure in R^6. Given a μ-measurable subset $X\subset R^2\times R^3\times R$ we denote by $\mathsf{L}^p(X,\mu)$ the class of all μ-measurable functions whose p-th power is μ-integrable. For $\phi\in \mathsf{A}_B$ consider the following oscillatory integral

$$J^1(a_1,a_2,\eta,b):=\frac{1}{b^{\frac{1}{3}}}\int_R\varphi(y,b^{\frac{2}{3}}|\eta_3y^2+\eta_1|,\eta,b)e^{i(a_2y^3+a_1y)}dy, \qquad (24)$$

where $\eta=(\eta_1,\eta_2,\eta_3),\ b\ne 0.$

Set

$$X:=\{(a_1,a_2,\eta,b)\in R^2\times R^3\times(R\backslash[-1,1]):|a_1|,|a_2|\le|b|,\eta\in B\}.$$

Proposition 6. *For any $\phi\in \mathsf{A}_B$ and $p>8$ the inclusion $J^1\in \mathsf{L}^p(X,\mu)$ holds.*

Estimates for oscillatory integrals with small amplitude function

We will estimate the oscillatory integral with amplitude of the form $\frac{\psi(y,\eta,b)}{b|\eta_3y^2+\eta_1|}$, where $\psi(y,\eta,b)$ be a smooth function satisfying conditions $\sup\psi(\cdot,\eta,b)\subset[-d,d]\cap\{b^{\frac{2}{3}}|\eta_3y^2+\eta_1|>1\}$ and also $V_{-d}^d[\psi]\le 1$, where d is a positive real number. It is clear that, $\frac{\psi(y,\eta,b)}{b|\eta_3y^2+\eta_1|}$ is of a small amplitude function in some sense for $\boldsymbol{b^{\frac{2}{3}}|\eta_3y^2+\eta_1|>1}$.

Consider the oscillatory integral given by

$$J^2(a_1,a_2,\eta,b):=\int_R\frac{\psi(y,\eta,b)e^{i(a_2y^3+a_1y)}}{b|\eta_3y^2+\eta_1|}dy.$$

Proposition 7. *For any* $p > 8$ *the inclusion* $J^2 \in L^p(X,\mu)$ *holds true.*

Estimates for oscillatory integrals with discontinuous amplitude function

We consider the oscillatory integral

$$J^3(a_1,a_2,\eta,b) := \int_R \frac{\psi(y,a,\eta,b)e^{iF_2(y,a,\eta,b)}}{b^{\frac{1}{2}}|\eta_3 y^2+\eta_1|^{\frac{1}{4}}}dy, \tag{25}$$

where $sup\,\psi(\cdot,a,\eta,b) \in [-d,d] \cap \left\{b^{\frac{2}{3}}|\eta_3 y^2+\eta_1| > 1\right\}$, $F_2(y,a,\eta,b) = -\frac{2\sqrt{3}}{9}b|\eta_1|^{\frac{3}{2}}|\sigma y^2+1|^{\frac{3}{2}} + a_2y^3 + a_1y$ and $\sigma = \frac{\eta_3}{\eta_1}$.

If $\eta_3\eta_1 < 0$ then the amplitude contains a function, which is discontinuous at some point and can be big when y close to that point. This is more subtle case in the investigations of behavior of the oscillatory integrals.

Proposition 8. *For any* $p > 8$ *the inclusion* $J^3 \in L^p(X,\mu)$ *holds.*

The sharpness of the result

Take any polynomial of the form (21) satisfying $|a_{30}| = \max\{|a_{3i}|, i = \overline{0,3}\}$, and $|a_{11}| \le 1$, such that $J \notin L^p(R^6)$ for all $p \le 8$.

We represent the polynomial (21) in the form

$$F(x,y,a) = a_{30}\left(x_1^3 + \xi_2 x_1 y_1^2 + \xi_3 y_1^3\right) + a_{10}x + a_{11}y,$$

where $\xi_2 = \frac{3a_{30}a_{32} - 3a_{31}^2}{a_{30}^2}$ and $\xi_3 = \frac{a_{30}^2 a_{33} + 2a_{31}^3 - 3a_{30}a_{31}a_{32}}{a_{30}^3}$. Notice that $|\xi_2| \le 6$, $|\xi_3| \le 6$.

Proposition 9. *For every smooth function* $\psi \in C_0^1(R^2)$ *supported in a neighborhood of* $(0,0)$ *with* $\psi(0,0) \neq 0$ *there exist positive numbers* $\lambda_0, \varepsilon, a_1$ *such that in the region* $|\xi_2 a_{30}|^{\frac{2}{3}} < \varepsilon$, $|a_{10}| \le \varepsilon a_{30}|^{\frac{1}{3}}$ *the following estimate holds for integral (25):*

$$|J(a)| \ge C/|a_{30}|^{1/3}.$$

In **"Uniform and invariant estimates for oscillatory integrals with polynomial phases"** chapter we consider the uniform estimates of oscillatory integrals with smooth phase functions in the case when the phase function has singularity of type D_∞ and in the case when the phase function is a homogeneous polynomial of degree four in two variables also we obtain invariant estimate for oscillatory integrals with polynomial phase third degree of two variables. The issue of whether integrals of real-analytic functions remain finite under small deformations was considered in [E. M. Stein, D. H. Phong, J. A. Sturm *On the growth and stability of real-analytic functions*, Amer. Journal of Math. 121 (1999), no.3, 519–554.]. An approach based on uniform estimates for certain classes of one-dimensional integrals is introduced. It is strong enough to recover the stability properties of real integrals in two dimensions which follow from the work of V. N. Karpushkin, and to obtain new results in higher dimensions. In dimension three, the new stability results in general can not be true, as shown by the well-known example of Varchenko.

In section **"Uniform estimates oscillatory integrals with smooth phase"** we

consider the problem on uniform estimates for oscillatory integrals with the smooth phase functions having singularities of type D_∞. The estimate is sharp and analogy to estimates of the work by V.N.Karpushkin.

Let $U \subset V \subset \mathbb{R}^2$ be bounded neighborhoods of the origin, $\overline{U}\,(\overline{V})$ the closure of $U(V)$, respectively. Suppose that the function $f : \overline{V} \to \mathbb{R}$ (where $f \in C^N(\overline{V}),\ (N \geq 8)$) has the form

$$f(x_1, x_2) = x_1 x_2^2 + g(x_1, x_2), \tag{26}$$

where $g \in C^N(V)$, such that $D^\alpha g(0,0) = 0$ for all $\alpha_1 + \alpha_2 \leq 3$, where D^α is $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$, $\alpha := (\alpha_1, \alpha_2) \in Z_+^2$ is a multi-index, $Z_+ = \{0\} \cup \mathbb{N}$ are the non-negative integers.

Let $\vartheta_{C^8(\overline{V})}(\varepsilon) := \{F \in C^8(\overline{V}), \|F\|_{C^8(\overline{V})} < \varepsilon\}$.

The main result of this section is the following.

Theorem 14. *Let $f \in C^8(\overline{V})$ have the form (26). Then there exists a positive number ε and a neighborhood $U \subset V$ of the origin such that for any functions $a \in C_0^1(U)$ and $F \in C^8(\overline{V})$, $\|F\|_{C^8(\overline{V})} < \varepsilon$, the following estimate holds:*

$$\left| \int_U e^{i\lambda(f+F)} a(x)dx \right| \frac{C\|a\|_{C^1}}{|\lambda|^{\frac{1}{2}}}.$$

In section **"Uniform estimates for oscillatory integrals with homogeneous polynomial phases of degree 4"** we consider the problem on uniform estimates for oscillatory integrals with the homogeneous polynomial phases of degree 4.

Let $U \subset V \subset \mathbb{R}^2$ be bounded neighborhoods of the origin, $\overline{U}\,(\overline{V})$ the closure of $U(V)$, respectively. Suppose that the function $f : \overline{V} \to \mathbb{R}$ (where $f \in C^N(\overline{V}),\ (N \geq 8)$) has the form:

$$f(x_1, x_2) = f_\pi(x_1, x_2) + g(x_1, x_2), \tag{27}$$

where $f_\pi(x_1, x_2)$ is a homogeneous polynomial of degree 4 having the root $f_\pi(0,0)$ of multiplicity at most 2 (e.g. polynomial $f_\pi(x)$ of order 4 has at most two roots of multiplicity two on the unit circle in $\mathbb{R}^2$ centered at the origin), and $g \in C^N(V)$ is such that $D^\alpha g(0,0) = 0$ for all $\alpha_1 + \alpha_2 \leq 4$, where D^α is $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$, $\alpha := (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$ is a multi-index, $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ are the non-negative integers.

The main result of this section is the following.

Theorem 15. *Let $f \in C^8(\overline{V})$ have the form (27). Then there exists a positive number ε and a neighborhood $U \subset V$ of the origin such that for any functions $a \in C_0^1(U)$ and $F \in C^8(\overline{V})$, $\|F\|_{C^8(\overline{V})} < \varepsilon$, the following estimate holds:*

$$\left| \int_U e^{i\lambda(f+F)} a(x)dx \right| \leq \frac{C\|a\|_{C^1} \ln(2 + |\lambda|)}{|\lambda|^{\frac{1}{2}}}.$$

In **"Invariant estimates of two-dimensional oscillatory integrals"** section we consider invariant estimates of oscillatory integrals with polynomial phase. The

obtained estimates improve Popov's well-known results on invariant estimates of trigonometric integrals in the case where the phase function is a third-degree polynomial.

Let $W_1^n(\mathbb{R}^n)$ denote the Sobolev space whose norm is defined by

$$\|\phi\|_{W_1^n(R^n)} = \int_{R^n}\left(\sum_{|\alpha|\le n}|D^\alpha\phi|\right)dx, \tag{28}$$

here $D^\alpha\varphi$ is the derivative of φ in the sense of distributions. It is known that the Sobolev space can be defined as the completion of the class $C_0^\infty(\mathbb{R}^n)$ of infinitely smooth compactly supported functions in the norm (28).

Let P be a polynomial of the form

$$\mathrm{P}(x_1,x_2,a) = P_3(x_1,x_2,a_3) + P_2(x_1,x_2,a_2) + P_1(x_1,x_2,a_1),$$

where $P_3(x_1,x_2,a_3) := a_{30}x_1^3 + 3a_{31}x_1^2x_2 + 3a_{32}x_1x_2^2 + a_{33}x_2^3$, $P_2(x_1,x_2,a_2) := a_{20}x_1^2 + 2a_{21}x_1x_2 + a_{22}x_2^2$ and $P_1(x_1,x_2,a_1) := a_{10}x_1 + a_{11}x_2$.

Consider the intergal

$$J = \int_{R^2} e^{iP(x_1,x_2,a)}\phi(x_1,x_2)dx_1dx_2, \tag{29}$$

where $\varphi \in W_1^2(\mathbb{R}^2)$. Obviously, the last integral converges in the usual Lebesgue sense.

The main result of this section is the following statement.

Theorem 16. *There exists a positive number C such that, for any element $\varphi \in W_1^2(\mathbb{R}^2)$*

$$|J| \le \frac{C\|\phi\|_{W_1^2(R^2)}}{|D(P_3)|^{\frac{1}{6}}}, \tag{30}$$

where $D(P_3) := 3a_{31}^2a_{32}^2 + 6a_{30}a_{31}a_{32}a_{33} - 4a_{30}a_{32}^3 - 4a_{31}^3a_{33} - a_{30}^2a_{33}^2$ is the discriminant of the polynomial $P_3(x_1,x_2,a_3)$.

Combining this Theorem 16 with Theorem 4 of [I.A. Ikromov, Invariant estimates of two-dimensional trigonometric integrals. Math. USSR Sb. Vol. 76 (1990), No. 2: 473-488.], we obtain the following result.

Theorem 17. *For the integral (29), the following estimate holds:*

$$|J| \le \frac{C\|\phi\|_{W_1^2(R^2)}}{|D(P_3)|^{\frac{1}{6}}+|H|^{\frac{1}{4}}+N^{\frac{1}{6}}}, \tag{31}$$

where $N := a_{30}^2 + 3a_{31}^2 + 3a_{32}^2 + a_{33}^2$, $H := a_{31}^2 + a_{32}^2 - a_{30}a_{32} - a_{31}a_{33}$.

The following statement is an analog of Duistermaat's theorem on the boundedness of generalized Airy functions associated with singularities of type $D_4^\pm$.

Theorem 18. *There exists a constant C such that, for an arbitrary compact triangle T*

$$\left|\int_T e^{i\mathrm{P}}dx_1dx_2\right| \le \frac{C}{|D(P_3)|^{\frac{1}{6}}}.$$

Remark 6. *Theorem 4 of* [I.A. Ikromov, Invariant estimates of two-dimensional trigonometric integrals. Math. USSR Sb. Vol. 76 (1990), No. 2: 473-488.] *gives the estimate*

$$|J| \le \frac{C\|\phi\|_{W_1^2(R^2)}}{\frac{|D(P_3)|^{\frac{2}{3}}}{N}+N^{\frac{1}{6}}+|H|^{\frac{1}{4}}}.$$

Obviously, in the case where $\frac{|D(P_3)|^{\frac{2}{3}}}{N}$ is the dominant term of the denominator, this estimate is much worse than (31). Note that, the estimate does not depend on the triangle T.

Remark 7. *As mentioned in* [D. A. Popov, Remarks on uniform combined estimates of oscillatory integrals with simple singularities Izvestiya: Mathematics (2008),72(4): 793.], *for a more general phase function of the form*

$$\mathrm{P}(x,a) = P_3(x,a_3) + P_2(x,a_2) + P_1(x,a_1) + R(x),$$

where $R(x) = O(|x|^4)$, an estimate of the form (31) does not hold. In this case, the study of the behavior of oscillatory integrals is substantially more difficult.

CONCLUSION

Generally the results obtained in the thesis allow us to speak about the achievement of the goal of the thesis work. All main results are new and together they make a certain contribution to the theory of oscillatory integrals and harmonic analysis. This dissertation is devoted to the study of estimates of oscillator integrals and invariant estimates of oscillator integral with polynomial phase, summability of oscillator integral with phase, which is a polynomial of the third degree of two variables. The main results of the study are as follows:

1) It is obtained estimates for Mittag-Leffler functions with singularities of type D_∞.

2) It is proved that estimates for generalized oscillatory integrals with singularities of type $D_4^{\pm}$ and A_∞^2.

3) It is proved analogous of Ricci-Stein Lemma for generalized oscillatory integrals.

4) It is given that invariant estimates for generalized oscillatory integrals with homogeneous phase.

5) We obtained the estimates for one-dimensional oscillatory integrals with simple phase functions.

6) It is obtained estimates for integrals with Mittag-Leffler functions with two variables.

7) We found sharp convergent exponent for trigonometric integrals with incomplete quadratic phase.

8) We found sharp convergent exponent for trigonometrical integrals with quadratic phase.

9) We proved the problem summation for oscillatory integrals with polynomial phase third degree and showed sharpness of the obtained estimates.

10) We obtain estimate for oscillatory integrals with smooth phase having singularities of type D_∞.

11) We obtain uniform estimates for oscillatory integrals with the phase function is a smooth perturbation of homogeneous polynomial of degree four in two variables.

12) We obtain invariant estimates for oscillatory integrals with the polynomial phase of degree four in two variables.

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ПО ПРИСУЖДЕНИЮ УЧЕНЫХ СТЕПЕНЕЙ ПРИ
САМАРКАНДСКОМ ГОСУДАРСТВЕННОМ УНИВЕРСИТЕТЕ
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САФАРОВ АКБАР РАХМАНОВИЧ

ОБОБЩЕННЫЕ ОСЦИЛЛЯТОРНЫЕ ИНТЕГРАЛЫ И ИХ ПРИЛОЖЕНИЯ

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АВТОРЕФЕРАТ
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Защита диссертации состоится «15» 08 2023 года в 10.00 часов на заседании Научного совета DSc.03/30.12.2019.FM.02.01 при Самаркандском государственном университете имени Шарофа Рашидова. (Адрес: 140104, г. Самарканд, Университетский бульвар, 15. Тел.: (+99866) 231-06-32, факс: (+99866) 235-19-38, e-mail: patent@samdu.uz).

С диссертацией можно ознакомиться в Информационно-ресурсном центре Самаркандского государственного университета имени Шарофа Рашидова (зарегистрирована за № 37). (Адрес: 140104, г. Самарканд, Университетский бульвар, 15. Тел.: (+99866) 231-06-32, факс: (99866) 235-19-38).

Автореферат диссертации разослан «1» 08 2023 года.
(реестр протокол рассылки № _____ от «1» 08 2023 года).

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ВВЕДЕНИЕ (аннотация диссертации доктора наук (DSc))

Цель исследования – получить оценку для обобщенного осцилляторного интегралов и найти точную показатель суммируемости.

Объект исследования. Обобщенные осцилляторные интегралы с полиномиальной фазой.

Научная новизна исследования определяется следующими пунктами:

получены гладкие оценки осциллирующих интегралов с разрывом амплитудной функции методом перевала с точностью ε;

оценивается для обобщенных осцилляторных интегралов, связанных с функцией Миттаг-Леффлера, фазовая функция которой имеет особенности D_{∞}, $D_4^{\pm}$ и A_r;

получены гладкие оценки для обобщенного и классического осцилляторного интегралов, связанных с функцией Миттаг-Леффлера от двух переменных, фазовая функция которой полиномом;

для осцилляторных интегралов, фаза которых является полиномом третьего порядка, оценка получается через инварианты классической группы движений на евклидовой плоскости;

найден точный показатель суммирования в пространстве коэффициентов классических осцилляторных интегралов, где фазовая функция является полиномом второго и третьего порядка.

Внедрение результатов исследования. По результатам оценок обобщенных осцилляторных интегралов и результатов, связанных с их приложениями:

научные результаты по инвариантным оценкам колебательных интегралов с полиномиальной фазой в ведущих зарубежных журналах (Nonlinear Analysis, том 207, июнь 2021, 112292, Analysis and Mathematical Physics, том 12, Номер статьи: 130 (2022), Журнал Сибирского федерального университета. физика 15.4 (2022): 459-466.) использовались при оценить решение частных дифференциальным уравнениям.

оценки обощенных осцилляторных интегралов были применены в проекте №01M01021 «Анализ и частный дифференциальный уравнение а также Мезусаламским программе Гентского университета» (справка №01M01021 Гентского университета от 18 апреля 2023 года) были применены в исследованиях осцилляторных интегралов зависящих от функции Миттаг-Леффлера. Применение научных результатов получено априорных оценках решений дифференциальных уравнений с дробными производными;

оценки осцилляторных интегралов с полиномиальной фазой, а также утверждение о точном показателе суммируемости были применены в проекте №AP08052046 «Некоторые нелокальные аналоги нелинейных дифференциальных уравнении в частных производных» (справка №01-06/060 Института математики и математического моделирования от 17 апреля 2023 года) были применены в исследованиях осцилляторных интегралов зависящих от функции Миттаг-Леффлера. Применение научных результатов

позволило оценочно решить задачу Коши для уравнений в частных производных с дробными производными.

Структура и объем диссертации. Диссертация состоит из введения, четрех глав, заключения и списка литературы. Объем диссертации составил 167 страницы.

E’LON QILINGAN ISHLAR RO‘YXATI
LIST OF PUBLISHED WORKS
СПИСОК ОПУБЛИКОВАННЫХ РАБОТ

I bo‘lim (Part I; Часть I)

1. I.Ikromov, A.R.Safarov, Estimates for oscillatory integrals with discontinuous amplitude // Scientific reports of Samarkand state university. - Samarkand, 2015.-Vol. 89.-No 1.-P.28-32. (01.00.00; №02).

2. A.Safarov, On the L^p-bound for trigonometric integrals // Analysis mathematica.- Hungary, 2019.-Vol. 45.-P.153-176. (№3, Scopus, IF=1.2).

3. A.Safarov, Invariant estimates of two-dimensional oscillatory integrals // Math. Notes.-Russia, 2018.-Vol. 104.-P.293–302. (№3, Scopus, IF=0.864).

4. A.Safarov, On a problem of restriction of Fourier transform on a hypersurface // Russian Mathematics.-Russia, 2019.-Vol. 63.-No 4.-P.57-63. (№3, Scopus, IF=0.8).

5. M.Ruzhansky, A.R.Safarov, G.A.Khasanov, Uniform estimates for oscillatory integrals with homogeneous polynomial phases of degree 4 // Analysis and Mathematical Physics.-USA, 2022.-Vol. 12.-No 130.-P.1-15. (№3, Scopus, IF=2.4).

6. A.R.Safarov, Estimates for Mittag-Leffler Functions with Smooth Phase Depending on Two Variables // J. Sib. Fed. Univ. Math. Phys.-Russia, 2022.-Vol. 15.-No 4.-P.459-466. (№3, Scopus, IF=0.9).

7. A.Safarov, Invariant estimates for two-dimensional oscillatory integrals with polynomial phase third order // Reports of Academy of Sciences of Uzbekistan.-Tashkent, 2017.-No 3.-P.15-19. (01.00.00; №07).

8. A.Safarov, About uniform estimates for model trigonometrical integrals with discontinuous amplitude // Uzbek Mathematical Journal.-Tashkent, 2016.-No 1.-P. 123-134. (01.00.00; №06).

9. I.Ikromov, A.Safarov, About estimates for duble trigonometrical integrals // Uzbek Mathematical Journal.-Tashkent, 2013.-No 4.-P.7-16. (01.00.00; №06).

10. A.Safarov, Normal forms of some types of Arnold singularities // Scientific reports of Samarkand state university.-Samarkand, 2021.-Vol. 125.-No 1.-P.11-16. (01.00.00; №02).

11. I.Ikromov, A.Safarov, D.Vakhabova, Estimates for trigonometric integrals with incomplete quadratic phase // Scientific reports of Samarkand state university.-Samarkand, 2022.-Vol.133.-No 3.-P.74-81. (01.00.00; №02).

12. I.Ikromov, A.Safarov, Uniform estimates for oscillatory integrals with smooth phase, Bulleten of the Institut of Mathematics.-Tashkent, 2022.-No 5(3).-P. 211-219. (01.00.00; №04).

13. I.A.Ikromov, A.R.Safarov, A.T.Absalamov, On the convergence exponent of the special integral of the Tarry problem for a quadratic polynomial // J. Sib. Fed. Univ. Math. Phys.-Russia, 2023.-Vol. 16.-No 4.-P.488-497, (No3, Scopus, IF=0.9). (in press)

14. I.A.Ikromov, A.R.Safarov, D. Khudoyberdiev, Invariant estimates for generalized oscillatory integrals with polynomial phase // Scientific reports of Samarkand state university.-Samarkand, 2023.-Vol.139.-No 3.-P.150-159. (01.00.00; №02).

II bo‘lim (Part II; Часть II)

15. A.R.Safarov, Invariant estimates duble oscillatory integrals, International scientific conference of students, Phd students and young scientists "Lomonosov-2017", Conference Proceedings, Moscow, 10-14 Aprel, 2017.

16. I.Ikromov, A.R.Safarov, On estimates for trigonometric integrals with quadratic phase, International conference "P. Chebyshev Mathematical Ideas and Their Applications to Natural Sciences" commemorating the 200th anniversary of P. Chebyshev, the great Russian mathematician, Conference Proceedings, Kaluga, 14-18 may, 2021.

17. I.Ikromov, A.R.Safarov, Estimates for two-dimensional integrals with Mittag-Leffler functions International scientific and practical conference "Modern problems of applied mathematics and information technologies", Bukhara, 11-12 may, 2022.

18. I.Ikromov, A.R.Safarov, Estimates for generalizated oscillatory integrals with polynomial phase Scientific conference "Operator algebras, non-associative structures and related problems", Tashkent, 14-15 september, 2022.

19. M.Ruzhansky, A.R.Safarov, G.A.Khasanov, Uniform estimates for oscillatory integrals with homogeneous polynomial phases of degree 4, International conference "Mathematical analysis and its applications in modern mathematical physics", Samarkand, 23-24 September, 2022.

20. I.Ikromov, A.R.Safarov, Uniform estimates for oscillatory integrals, International scientific and practical conference "XII Lomonosov’s readings", Conference Proceedings, Dushanbe, 29-30 Aprel, 2022.

21. A.Safarov, Invariant estimates for trigonometric integrals with polynomial phase, Scientific and practical conference "The 21 st century is the century of the intellectual generation", Samarkand, 2-3 June, 2016.

22. A.Safarov, Estimates generelized function of Airy, The international scientific conference "Modern problems of applied mathematics and information technologies - Al-Khorezmiy 2014", Tashkent, 15-17 September, 2014.

23. A.Safarov, On a problem of restriction of Fourier transform on a hypersurface, The Republic scientific conference "New results on mathematics and their apllications", Samarkand, 14-15 May, 2018.

24. A.Safarov, On an analogue of V.N. Karpushkin’s theorem on uniform estimates in the three-dimensional case, The international scientific conference "Modern problems of applied mathematics and information technologies - Al-Khorezmiy 2014", Tashkent, 19-22 December, 2012.

25. A.Safarov, Uniform estimates of oscillatory integrals and invariant of Aronkhold, Scientific and practical conference "Ill-posed and non-classical problems of mathematical physics and analysis", Samarkand, 5-6 July, 2012.

26. A.Safarov, Invariant estimates oscillatory integrals with polynomial phase, The international scientific conference "Modern problems of applied mathematics and information technologies - Al-Khorezmiy 2014", Tashkent, 9-10 November, 2016.

27. A.Safarov, Invariant estimates duble oscillatory integrals with polynomial phase third degree The republician scientific conference with participiation of foreign scientists "Modern problems of dynamical systems and their applications", Tashkent, 1-3 May, 2017.

28. A.Safarov, Summation of oscillatory integrals with polynomial phase of third degree, The republician scientific conference "Modern problems of analysis", Karshi, 22-23 Aprel, 2016.

Avtoreferat Sharof Rashidov nomidagi Samarqand davlat universitetining
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