O’ZBEKISTON RESPUBLIKASI
OLIY VA O’RTA MAXSUS TA’LIM VAZIRLIGI

SAMARQAND DAVLAT UNIVERSITETI

AMALIY MATEMATIKA VA INFORMATIKA FAKULTETI

“OPTIMAL BOSHQARUV USULLARI” KAFEDRASI

Ro’yxatga olindi:	"TASDIQLAYMAN"
	SamDU o’quv ishlari
№_____________	bo’yicha prorektori
____	prof. Soleev A.
2019 yil	_______________________
“__”________	2019 yil “__”_________

Bilim sohasi: 4100000 – Gumanitar soha
Ta’lim sohasi: 130000 – Matematika
Ta’lim yonalishi: 5130200 – Amaliy matematika va informatika

«OPTIMALLASHTIRISH USULLARI»
fanidan
O’QUV–USLUBIY MAJMUA
(Moodle tizimi rejasi asosida)

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Samarqand – 2019

S. Axatqulov va T. Radjabov. “Optimallashtirish usullari” fanidan o’quv-uslubiy majmua.-Samarqand: SamDU nashri -2019. 215 bet.

Ushbu o’quv-uslubiy majmua oliy o’quv yurtlari amaliy matematika va informatika, 5130200 –amaliy matematika va informatika ta’lim yo’nalishi talabalriga mo’ljallangan

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SamDU o’quv-uslubiy kengashining 2019 yil _______________ dagi _____ qarori bilan o’quv-uslibiy majmua sifatida nashrga tavsiya etilgan

Samarqand 2019

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”Optimal boshqaruv usullari” kafedrasining 2019 yil _______________ 1-son yig'ilishida muhokamadan o'tgan va fakultet kengashida muhokama qilish uchun tavsiya etilgan.

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ANNOTATSIYA (MUALLIFDAN)

Optimallashtirish usullari kursi funksionallarning minimumlarini (maksimumlarini) topishning nazariyasi va usullarini o‘rganadi. Mazkur kursni o‘rganish uchun kamida matematik analiz, analitik geometriya, oliy algebra va differensial tenglamalar kabi fanlarga oid asosiy tushuncha va usullarni bilish kerak bo‘ladi. Fanning maqsadi - talabalarga cheksiz o‘lchovli fazolarda berilgan funksionallarning shartsiz va shartli ekstremumlarini topish masalalarini yechish hamda bunday masalalarni yechishning sonli usullarini qo‘llashni o‘rgatishdan iborat. Fanning vazifalari.

- talabalarni, nazariy va amaliy masalalarni yechish uchun zarur bo‘lgan, ekstremal masalalarning nazariy asoslari bilan tanishtirish;
- talabalarda Optimallashtirish usullari bo‘yicha o‘quv, ilmiy adabiyotlarni mustaqil o‘rganish ko‘nikmalarini hosil qilish;
- talabalarda mantiqiy va algoritmik fikrlash qobiliyatini rivojlantirish;
- talabalarda o‘z fikrlarini abstraktlashtirish va xulosalarni lo‘nda (qat’iy) ifodalash ko‘nikmalarini tarbiyalash;
- talabalarda ekstremumga qo‘yilgan amaliy masalalarni matematik nuqtai nazardan aniq ifodalash hamda ularni tahlil qilish ko‘nikmalarini hosil qilish.

MUNDARIJA

MA’RUZA MASHG‘ULOTLARI

1-ma’ruza. Variatsion hisobning predmeti, funksional tushunchasi va klassik masalalar. Asosiy funksional fazolar

Reja .

1. Variatsion hisobning klassik masalalari.
2. Variatsion hisob predmeti va uning rivojlanishidan qisqa tarixiy ma’lumotlar.
3. Asosiy funksional fazolar.
4. Funksionalning variatsiyalari.
5. Funksionalning ekstremumi.
6. Ekstremumning zaruriy va yetarli shartlari.

Tayanch iboralar. *Braxistoxrona, geodezik chiziq,izoperimetrik shart, funksional, uzluksizlik, differensiallanuvchanlik, funksional fazo, norma, metrika, yaqinlik tartibi, ε-atrof, chiziqli va kvadratik funksionallar, variatsiya,variatsion masala,*
global va lokal ekstremumlar,ekstremumning zaruriy sharti,yetarli shart

Matematikada, tabiiy va texnik fanlarda, iqtisodiyotda va boshqa sohalarda uchraydigan ko’pgina amaliy masalalar cheksiz o’lchovli funksional fazolardagi ekstremal masalalarga olib keladi. Bunday masalalar bilan klassik variatsion hisob va optimal boshqaruv masalalari bo’limlarida tanishamiz.

Ushbu va bundan keyingi bir necha ma’ruzalarimiz variatsion hisob masalalariga bag’ishlanadi.

1.**Variatsion hisobning klassik masalalari . Variatsion hisob predmeti**. Dastlab variatsion hisob predmetini yaxshiroq anglab olishga imkon beruvchi hamda matematikada bu yo’nalishning paydo bo’lishi va rivojlanishida muhim ahamiyatga ega bo’lgan masalalardan quyidagilarni keltiramiz.

Braxistoxrona haqidagi masala. 1696 yilda I.Bernulli tomonidan qo’yilgan bu masalada bir vertikal to’g’ri chiziqda yotmagan ikkita A va B nuqtalarni tutashtiruvchi shunday chiziqni topish talab qilinadiki, material nuqta o’z og’irlik kuchi ta’siri ostida shu chiziq bo’ylab harakat qilib, A nuqtadan B nuqtaga eng qisqa vaqtda yetib kelsin.(1-chizma).

Masalaning nomi grekcha “braxistos” –eng qisqa, “xronos” –vaqt so’zlaridan kelib chiqqan.

Braxistoxrona haqidagi masalani hozirgi zamon matematikasi tilida ifodalash uchun to’g’ri burchakli O*xy* kooordinatalar sistemasini 1-chizmada ko’rsatilganidek, ya’ni O*y* o’qni pastga yo’naltirib, qaraymiz. A nuqtani koordinata-lar boshiga joylashtiramiz. B nuqtaning koordinatalari (x_1,y) bo’lsin.

A$(0,0)$ va B(x_1,y_1) nuqtalarni ixtiyoriy $y=y(x)$ silliq chiziq bilan tutashtiramiz.

Shu chiziq bo’ylab og’irlik kuchi ta’sirida harakatlanuvchi material nuqtaning massasi m ga, t vaqt momentidagi tezligi v ga teng bo’lsin. U holda, t vaqtda harakatdagi nuqtaning kinetik energiyasi $K=\frac{mv^2}{2}$, potensial energiyasi P=-*mgy* bo'ladi, bu yerda *g≈9.8 m/c²*-erkin tushish tezlanishi o'zgarmasi. Fizikadan yaxshi ma'lum bo'lgan energiyaning saqlanish qonuniga ko'ra,

$$-mgy+\frac{mv^2}{2}=0$$

tenglikni olamiz. Bu yerdan $v=\sqrt{2gy}$. Endi

$$v=\frac{ds}{dt},\ ds=\sqrt{dx^2+dy^2},dy=y'dx$$

ekanligini hisobga olsak, $dt=\frac{ds}{v}=\sqrt{\frac{1+y'^2}{2gy}}dx$ bo'ladi.

Demak, *y(x)* chiziq bo'ylab A nuqtadan B nuqtaga ko'chish uchun sarflangan T=T[*y*] vaqt uchun

$$T[y]=\int_0^{x_1}\sqrt{\frac{1+y'^2}{2gy}}dx \qquad (1)$$

ifodaga ega bo'lamiz. (1) ko'rinishdagi *T=T[y]* miqdor *y=y(x), x∈[0,x₁]* uzluksiz differensiallanuvchi funksiyalar fazosida aniqlangan bo'lib, braxistoxrona haqi-dagi masala esa, *T[y]* funksionalning, $y(0)=0$, $y(x_1)=y_1$ shartlarni qanoatlantiruvchi uzluksiz differensiallanuvchi funksiyalar to'plamida, minimumini topish masalasidan iboratdir. Bu masala I.Bernulli, I.Nyuton, G.Leybnislar tomonidan yechilgan bo'lib, *eng tez o'tish (sirpanish)* chizig'i *sikloida* deb ataluvchi chiziqdan iborat bo'lar ekan (bunga biz keyinroq ishonch hosil qilamiz).

Geodezik chiziqlar haqidagi masala. Masala quyidagicha qo'yiladi: Berilgan S sirtda yotuvchi va sirtning A_0 va hamda A_1 nuqtalarini tutashtiruvchi eng qisqa uzunlikka ega bo'lgan chiziq topilsin (2-chizma).

Bunday eng qisqa uzunlikka ega chiziqlar geodezik chiziqlar deb ataladi. Masalaning matematik modelini tuzish uchun, S sirt $\varphi(x,y,z)=0$ tenglama bilan berilgan, A_0 va A_1 nuqtalarning koordinatalari, mos ravishda, (x_0,y_0,z_0) va (x_1,y_1,z_1) bo'lsin, deb faraz qilamiz. Qaralayotgan nuqtalarni tutashtiruvchi ixtiyoriy *y=y(x), z=z(x)*, $x_0 \le x \le x_1$ silliq chiziqni qaraymiz. Matematik analiz kursidan yaxshi ma'lumki, bu chiziqning uzunligi

$$L=L[y,z]=\int_{x_0}^{x_1}\sqrt{1+y'^2+z'^2}dx \qquad (2)$$

formula orqali topiladi. Masalaning qo'yilishiga ko'ra,

$$y(x_0)=y_0,z(x_0)=z_0,y(x_1)=y_1,z(x_1)=z_1,\varphi(x,y(x),z(x))=0,x_0\le x\le x_1 \qquad (3)$$

munosabatlarga ega bo'lamiz. Shunday qilib, geodezik chiziqlar haqidagi masala (2) ko'rinishdagi *L[y,z]* o'zgaruvchi miqdorni (3) shartlarni qanoatlantiruvchi uzluksiz differensiallanuvchi, *y=y(x), z=z(x)*, $x_0 \le x \le x_1$ funksiyalar to'plamida minimallash-tirish masalasidan iborat. Bu masala, 1968 yilda Ya.Bernulli tomonidan yechilgan.

Klassik izoperimetrik masala. Bu masalada berilgan l uzunlikka ega bo'lgan barcha yopiq chiziqlar ichida maksimal *S* yuzani chegaralovchi chiziqni topish talab qilinadi.

Bunday chiziqning aylanadan iborat ekanligi qadimgi Yunonistonda ma'lum edi. Izoperimetrik masala deb ataluvchi bu masalani xozirgi zamon matematikasi tilida ifodalash uchun, yopiq chiziq $x=x(t),\ y=y(t),\ t\in[t_0,t_1]$ parametrik tenglamalar bilan berilgan, deb faraz qilamiz.

U holda, shu yopiq chiziq bilan chegaralangan yuza,

$$S[x,y]=\int_{t_0}^{t_1} x y_t' \, dt \qquad (4)$$

formula orqali topiladi. Chiziq uzunligi l ga tengligi va chiziqning yopiqligini hisobga olsak,

$$\int_{t_0}^{t_1} \sqrt{x'^2+y'^2}\, dt = l \qquad (5)$$

izoperimetrik shartga va

$$x(t_0)=x(t_1), \quad y(t_0)=y(t_1) \qquad (6)$$

chegaraviy shartga ega bo'lamiz. Shunday qilib, qaralayotgan masala (4) ko'rinishdagi o'zgaruvchi miqdorning, (5), (6) shartlarni qanoatlantiruvchi $x=x(t), y=y(t), t_0\le t\le t_1$ uzluksiz differensiallanuvchi funksiyalar to'plamida, minimumini topishdan iboratdir.

Yuqorida keltirilgan masalalarda (1), (2) va (4) ko'rinishdagi o'zgaruvchi miqdorlarga ega bo'ldik. Ular funksional tipidagi o'zgaruvchi miqdorlarga misol bo'la oladi. Funksionallar esa, funksional analiz kursining asosiy tushunchalaridan biri bo'lib, berilgan *W* funksional fazo *V* to'plamining har bir elementiga biror *J(u)* haqiqiy sonni mos qo'yuvchi akslantirishni bildiradi. Funksionallar, odatda, cheksiz o'lchovli funksional fazolarda berilgan bo'ladi. Ularning eng katta (maksimal) va eng kichik (minimal) qiymatlarini topish haqidagi masalalar cheksiz o'lchovli ekstremal masalalar bo'lib, bunday masalalarni o'rganish variatsion hisob predmetini tashkil etadi.

XVII asrning oxiridan XX asr o'rtalarigacha bo'lgan davr klassik variatsion hisobning paydo bo'lishi va rivojlanishini o'z ichiga oladi. Bu davrda dastlabki fundamental tadqiqotlar L.Eyler va J..Lagranj tomonidan bajarildi. XVIII asrning oxirlarida Eyler, Lagranj va Lejandrlarning ilmiy tadqiqotlari natijasida variatsion hisob birinchi variatsiyani tekshirish qismi bo'yicha tugallangan shaklga ega bo'ldi.

XIX asrda esa, avval ma'lum bo'lgan variatsion masalalarni umumlashtirish boshlandi va variatsion hisobning tadbiqlari bo'yicha natijalar olindi (M.I.Ostrogradskiy tomonidan 1834 yilda karrali integralli variatsion masalalar uchun zaruriy shartlar olindi, variatsion hisobning mexanikaga tadbiqi qaraldi).

XIX asrning ikkinchi yarmida funksionallar ekstremumlarining yetarli shartlari olindi (K.Veyershtrass tomonidan, 1879 yilda).

XX asrda variatsion hisobning to'g'ri usullari yuzaga keldi. Ular variatsion masalalarni taqribiy yechish uchun, hamda ularda yechimning mavjudligini isbotlash uchun juda muhimdir.

XX asrning boshlarida matematikada yangi yo'nalish – funksional analiz yuzaga keldi va aniq tabiatshunoslikning turli sohalarida, jumladan, kvant mexanikasida keng qo'llanila boshladi. Variatsion hisob «chiziqsiz» funksional analizning tarkibiy qismiga aylandi.

XX asrnig ikkinchi yarmiga kelib optimal boshqaruvning matematik nazariyasiga asos solinishi va uning jadal rivojlanishi variatsion hisob taraqqiyotida yangi davrni boshlab berdi. Bu yangi yo'nalishda, sobiq Ittifoqda akademik L.S.Pontryaginning «maksimum prinsipi», amerikalik R.Bellmanning dinamik pogrammalashtirish usuli asosiy natijalar hisoblanadi.

2. Asosiy funksional fazolar.

Bizga funksional analiz kursidan yaxshi ma'lum bo'lgan va variatsion hisob masalalarini o'rganishda keng foydalaniladigan eng muhim funksional fazolarni eslatib o'tamiz.

a) *C[a,b]* fazo. Bu fazo chiziqli normalangan fazo bo'lib, u *[a,b]* kesmada aniqlangan uzluksiz funksiyalardan tashkil topgan. *C[a,b]* fazo $f=f(x)$ elementining normasi,

$$\|f\| = \max_{a\le x\le b} |f(x)| \qquad (7)$$

formula bo'yicha aniqlangan.

1-t a ' r i f. *C[a,b]* fazodan olingan ikkita $y_1=y_1(x)$ va $y_2=y_2(x)$ funksiyalar orasidagi *nolinchi tartibli masofa* deb,

$$P_0(y_1, y_2) = \max_{a\le x\le b} |y_1(x) - y_2(x)| \qquad (8)$$

tenglik bilan aniqlanadigan P_0 songa aytiladi. Demak, ikkita funksiya orasidagi nolinchi tartibli masofa – ular ayirmasining normasiga tengdir. (8) tenglik bilan aniqlangan P_0 ga, *C[a,b]* fazodagi metrika ham deyiladi. *C[a,b]* – metrik fazodir.

Nolinchi tartibli metrikaga asoslangan holda,

$$V_0(y_0,\varepsilon) = \{y \in C[a,b]:\} \; p_0(y_0, y) < \varepsilon\} \qquad (9)$$

tenglik orqali, markazi $y_0 \in C[a,b]$ elemenitda bo'lgan nolinchi tartibli ε - atrofni qarash mumkin.

Markazi y_0 da bo'lgan nolinchi tartibli ε-atrof, geometrik nuqtai nazardan, shunday $y=y(x)$ funksiyalar to'plamidan iboratki, ularning grafiklari $y_0=y_0(x)$ ni o'zida saqlovchi 2ε kenglikdagi (vertikal bo'yicha) yo'lakning ichida to'la yotadi (3-chizma).

b) C^1*[a,b]* fazo. Bu fazo *[a,b]* kesmada o'zining birinchi tartibli hosilalari bilan birga uzluksiz funksiyalaridan iborat. Bu holda metrikani ikki xil yo'l bilan aniqlash mumkin.

2-t a ' r i f. Ikkita $y_1=y_1(x)$ va $y_2=y_2(x)$ funksiyalar orasidagi *birinchi tartibli masofa* deb, quyidagi

$$p_1(y_1, y_2) = \max_{a\le x\le b} |y_2(x) - y_1(x)| + \max_{a\le x\le b} |y_2'(x) - y_1'(x)| \qquad (10)$$

tenglik bilan aniqlanadigan ρ_1 songa aytiladi.

Birinchi tartibli masofa tushunchasi orqali birinchi tartibli ε- atrof ushbu

$$V_1(y_0,\varepsilon) = \{y \in C[a,b]:\} \; p_1(y_0, y_1) < \varepsilon\} \qquad (11)$$

tenglik yordamida kiritiladi, bunda y_0 – atrofning markazi.

Birinchi tartibli atrofnig ta'rifiga ko'ra, bir vaqtning o'zida $\left|y(x)-y_0(x)\right|<\varepsilon$ *ва* $\left|y'(x)-y_0'(x)\right|<\varepsilon,\ a\le x\le b$, tengsizliklar bajarilishi shart. Bu yerdan, nolinchi tartibli ε – atrofning, birinchi tartibli ε – atrofdan kengroq ekanligi, ya'ni

$$V_1(y_0,\varepsilon)\subset V_0(y_0,\varepsilon) \qquad (12)$$

mansublik o'rinli bo'lishi kelib chiqadi.

$C^{(1)}[a,b]$ ham chiziqli normalangan fazodir. Unda norma funksiya va uning hosilasi modullari maksimumlarining yig'indisi kabi aniqlanadi:

$$\|f\|_{C^{(1)}[a,b]}=\max_{a\le x\le b}\left|f(x)\right|+\max_{a\le x\le b}\left|f(x)\right| \qquad (13)$$

c) $C^{(n)}[a,b]$ fazo. Bu fazo [a,b] kesmada aniqlangan va o'zining *n*-tartibgacha hosilalari bilan uzluksiz fuknsiyalaridan iborat (n≥1).

$C^{(n)}[a,b]$ fazodagi $f=f(x)$ elementning normasi

$$\|f\|_{C^{(1)}[a,b]}=\sum\max_{a\le x\le b}\left|f^{(1)}(x)\right| \qquad (14)$$

tenglik bilan aniqlanadi. Bu fazoning ikki elementi orasidagi masofa deganda, ular ayirmalarining normasini tushunamiz. Bu masofani *n*-tartibli masofa deb ataymiz va $p_n(y_1,y_2)$ deb belgilaymiz:

$$\mathrm{p_n}(y_1,y_2)=\sum_{i=0}^{n}\max_{a\le x\le b}\left|y_2{}^i(x)-y_1{}^i(x)\right| \qquad (15)$$

$C^{(n)}[a,b]$ metrik fazo, ρ_n - undagi metrikadir.

3. Chiziqli va kvadratik funksionallar. Funksionalning variatsiyalari.

Funksionalning variatsiyasi ta'rifini berishdan oldin chiziqli va kvadratik funksionallar tushunchalarini eslatib o'tamiz.

Agar biror chiziqli normalangan W fazoda aniqlangan *J[u]* funksional bir jinsli va additiv bo'lsa, ya'ni:

1) $J[cu]=cJ[u],\ \forall u\in W,\ c$-ixtiyoriy o'zgarmas.
2) $J[u_1+u_2]=J[u_1]+J[u_2],\ \forall u_1,u_2\in W;$

shartlar bajarilsa, *J[u]-chiziqli funksional* deyiladi. Masalan, agar *p(x)* va *q(x)* lar $[x_0,x_1]$ kesmada uzluksiz funksiya bo'lsa,

$$J[y]=\int_{x_0}^{x_1}[p(x)y(x)+q(x)y'(x)]dx$$

tenglik orqali aniqlangan *J[y]* funksional $W=C^1[x_0,x_1]$ da chiziqli funksional bo'ladi.

W – chiziqli normalangan fazo, *J=J[u,v]* funksional har bir o'zgaruvchisi bo'yicha chiziqli bo'lsin. Agar *u=v* deb olsak, hosil bo'lgan *J[u,u]* funksionalga *kvadratik funksional* deyiladi. Masalan, agar $a(x)$-$[x_0,x_1]$ oraliqda aniqlangan uzluksiz funksiya bo'lsa,

$$J[u,v]=\int_{x_0}^{1}a(x)u(x)v(x)\,dx$$

funksional $W=C[x_0,x_1]$ fazoda har bir *u=u(x)* va *v=v(x)* elementlar bo'yicha chiziqli funksionaldir. Bu yerda *u=v* deb olib, $C[x_0,x_1]$ da aniqlangan

$$J[u,u]=\int_{x_0}^{x_1} a(x)u^2(x)\,dx$$

kvadratik funksionalga ega bo'lamiz.

3-t a ' r i f. Agar *J[y]* funksional *W* chiziqli normalangan fazoda berilgan bo'lsa,

$$\Delta J = J[y+h]-J[y],\quad h\in W$$

ayirmaga *J[y]* funksionalning *orttirmasi* deyiladi.

4-t a ' r i f. Agar *W* chiziqli normalangan fazoda berilgan *J[y]* funksionalning ΔJ orttirmasi uchun,

$$J[y+h]-J[y]=L[y,h]+\beta[y,h] \qquad (16)$$

yoyilma o'rinli bo'lib, bunda $L[y,h]$-h ga nisbatan chiziqli funksional, $\beta[y,h]$ esa, $\|h\|\to 0$ da $\beta[y,h]/\|h\|\to 0$ munosabatni qanoatlantirsa , *J[y]* funksional $y\in W$ nuqtada *differensiallanuvchi* yoki *birinchi variatsiyasiga ega* deyiladi.

(16) yoyilmaning bosh qismidan iborat *L[y,h]* ga esa, *J[y]* funksionalning *birinchi variatsiyasi* deyiladi va u $\delta J=\delta J[y,h]$ kabi belgilanadi: $\delta J = L[y,\delta y]$.

Keltirilgan ta'rif bo'yicha variatsiyaga ega funksionallarga adabiyotlarda *Freshe ma'nosida* (yoki *kuchli ma'noda*) differensiallanuvchi funksionallar ham deyiladi.

6- t a ' r i f. W chiziqli normalangan fazoning *y* elementi va uning ixtiyoriy $h\in W$ elementi uchun, funksionalning ΔJ orttirmasi,

$$J[y+h]-J[y]=L_1[y,h]+\frac{1}{2}L_2[y,h]+\beta_1(y,h) \qquad (17)$$

ko'rinishdagi yoyilmaga ega bo'lsin, bu yerda $L_1[y,h]-h$ ga nisbatan chiziqli funksional, $L_2[y,h]$ esa, δy ga nisbatan kvadratik funksional, $\beta_1(y,h)/\|h\|^2\to 0,\ \|h\|\to 0.$ U holda, *J[y]* funksional $y\in W$ nuqtada *ikkinchi variatsiyaga ega* deyiladi. *h* ga nisbatan kvadratik funksional $L_2[y,h]$ esa, *J[y]* funksionalning *ikkinchi variatsiyasi* deyiladi hamda bu variatsiya $\delta^2=\delta^2 J[y,h]$ kabi belgilanadi: $\delta^2 = L_2[y,h]$.

M i s o l. $J[y]=\int_{x_0}^{x_1} y^2(x)dx$ bo'lsin.

Bu funksional uchun (17) yoyilma

$$\Delta J = \int_{x_0}^{x_1} 2y(x)hdx+\frac{1}{2}\int_{x_0}^{x_1} 2h^2dx$$

ko'rinishda bo'ladi. Demak,yuqorida keltirilgan ta'riflarga ko'ra,

$$\delta J = \int_{x_0}^{x_1} 2y(x)hdx, \qquad \delta^2 J = \int_{x_0}^{x_1} 2h^2dx.$$

Funksionalning Freshe bo'yicha kuchli ma'noda differensiallanuvchiligi Bilan bar qatorda Lagranj bo'yicha kuchsiz ma'noda differensiallanuvchiligi tushunchasi ham mavjud.

W chiziqli normalangan fazoning biror V to'plamida aniqlangan *J[y]* funksional berilgan bo'lsin. V to'plam yoki $M(y)=\{h\in W: y+h\in V\}$ to'plam W ning chiziqli qism fazosi bo'lsin.

7-t a ' r i f. *J[y]* funksionalning $y\in W$ nuqtadagi *Lagranj bo'yicha birinchi variatsiyasi* deb, $\varphi(\alpha)=J[y+\alpha h]$ funksiyaning $\alpha=0$ nuqtadagi hosilasiga aytiladi:

$$\varphi J=\varphi'(0)=\frac{d}{d\alpha}J[y+\alpha h]\Big|_{\alpha=0}.$$

$\varphi(\alpha)$ funksiyaning $\alpha=0$ nuqtada ikkinchi tartibli hosilasiga esa, *J[y]* funksionalning *Lagranj bo'yicha ikkinchi variatsiyasi* deyiladi:

$$\delta^2 J=\varphi'(0)=\frac{d^2}{d\alpha^2}J[y+\alpha h]\Big|_{\alpha=0}$$

M i s o l. $J[y]=\int_{x_0}^{x_1}[y^2(x)+y'^2(x)]dx$

Bu funksional $W=C^1[x_0,x_1]$ da aniqlangan. Uning Lagranj bo'yicha birinchi va ikkinchi variatsiyalarini hisoblaymiz.

$$\varphi(\alpha)=J[y+\alpha h]=\int_{x_0}^{x_1}[(y+\alpha h)^2+(y'+\alpha h')^2]dx$$

Demak, ta'rifga ko'ra

$$\delta J=\frac{d}{d\alpha}J[y+\alpha h]\Big|_{\alpha=0}=\int_{x_0}^{x_1}(2y(x)h-2y'h')dx,$$

$$\delta^2 J=\frac{d^2}{d\alpha^2}J[y+\alpha h]\Big|_{\alpha=0}=\int_{x_0}^{x_1}(2h^2-2h'^2)dx$$

Funksionalning Freshe bo'yicha dfferensiallanuvchiligidan, Lagranj bo'yicha ham differensiallanuvchiligi kelib chiqadi va bunda mos variatsiyalar o'zaro tengdir

4. Funksionalning ekstremumi. Ekstremumning zaruriy va yetarli shartlari. Yuqorida ta'kidlaganidek, funksionallarning eng katta yoki eng kichik qiymatlarini topishga keltiriluvchi amaliy masalalar juda ko'p uchraydi va matematikaning bunday masalalarni o'rganadigan bo'limi – variatsion hisobdir.

Endi funksionalning ekstremumi tushunchasining aniq matematik ta'rifini keltiramiz va funksional variatsiyasidan foydalanib, ekstremumning umumiy ko'rinishidagi zaruriy hamda yetarli shartlarini bayon qilamiz.

Cheksiz o'lchovli W fazoning biror V to'plamida aniqlangan *J[y]* funksional berilgan bo'lsin.

8-t a ' r i f. Agar ixtiyoriy $y\in V$ uchun $J[y^*]\le J[y] (J[y^*]\ge J[y])$ tengsizlik bajarilsa, $y^*\in V$ nuqta *J[y]* funksionalning V to'plamidagi global minimum (maksimum) nuqtasi, $J[y^*]$ esa funksionalning minimal (maksimal) qiymati deyiladi:

$$J[y^*]=\min_{y\in V}J[y]\quad (J[y^*]=\max_{y\in V}J[y])$$

Funksionalning minimum va maksimum nuqtalarini, umumiy nom bilan, ekstremum nuqtalari deb ataymiz.

Masalan, $W=C[0,1]$ da aniqlangan

$$J[y]=\int_{x_2}^{x_1}[1-y(x)]^2\,dx$$

$$J[y]\geq 0=J[y^*],\quad \forall y\in C[0,1].$$

Endi W – chiziqli normalangan fazo, $J[y]$ funksional $V\subset W$ to'plamda aniqlangan bo'lsin deb faraz qilamiz.

<u>9-t a ' r i f.</u> Agar biror $\varepsilon>0$ son topilib, $\|y-y^*\|_W<\varepsilon$ shartni qanoatlantiruvchi barcha $y\in V$ nuqtaga $J[y^*]\leq J[y]$ $(J[y^*]\geq J[y])$ tengsizlik bajarilsa, $y^*\in V$ nuqtaga $J[y]$ funksionalning V to'plamdagi lokal minimum (lokal maksimum) nuqtasi deyiladi.

Yuqorida keltirilgan ta'riflardan funksionalning global ekstremumi uning lokal ekstremumi ham bo'lishi kelib chiqadi. Bu tasdiqning aksinchasi esa, to'g'ri emas.

<u>M i s o l.</u> $J[y]=\int_{x_0}^{x_1}y'^2(y^2-1)dx$, funksionalni qaraymiz. U, $W=C^1[0,1]$ da aniqlangan.

Shu funksional $V=\{y\in C^1[0,1]:\ y(x)=0\}$ to'plamda global maksimumga ega emas: $\sup J[y]=+\infty$. Haqiqatan ham, agar $y_n=nx,\ n=0,1,\ldots$ $(y_n\in V)$ funksiyalarni qarasak,

$$J[y_n]=\int_{x_0}^{x_1}n^2(n^2x^2-1)dx=\frac{1}{3}n^4-n^2\to+\infty,\ \ n\to\infty.$$

Ammo, $y^*=0$ funksiya, $J[y]$ funksional uchun lokal maksimum nuqtasi bo'ladi.

Haqiqatan ham: $J[y^*]=0$, $\|y-y^*\|_{C[0,1]}<\varepsilon$ $(0<\varepsilon<1)$ bo'lganda, $y^2-1\leq 0$, shuning uchun,

$$J[y]=y'^2(y^2-1)dx\leq 0=J[y^*].\quad \forall y\in V.$$

$J[y]$ funksionalning cheksiz o'lchovli W fazoning V qism to'plamidagi minimumini (yoki maksimumini) topish haqidagi masala, cheksiz o'lchovli ekstremal masaladir. Bu masalani *variatsion masala* deb ataymiz va

$$J[y]\to\min\ (\max),\quad y\in V \qquad (18)$$

yoki

$$J[y]\to extr,\qquad y\in V$$

ko'rinishda belgilaymiz.

Keyingi ma'ruzalarda qaraladigan variatsion masalalarda $J[y]$ funksional, W fazo va uning V to'plami aniqlashtiriladi. Odatda, V to'plam funksiyalar (yoki ularning geometrik talqini sifatida chiziqlar, sirtlar) to'plamidan iborat bo'ladi.

Shuning uchun, (18) ekstremal masalada V to'plam elementlariga *joyiz funksiyalar* (chiziqlar, sirtlar) deb ataymiz.

Chiziqli normalangan W fazoning biror V to'plamida aniqlangan $J[y]$ funksional berilgan bo'lsin ($V=W$ bo'lishi ham mumkin). V – chiziqli qism fazo yoki biror $y_0\in V$ uchun qurilgan $M(y_0)=\{h\in W:\ y+h\in V\}$ to'plam chiziqli qism fazodan iborat bo'lsin.

Shu farazlarda, (18) masalalarda ekstremumning zaruriy va yetarli shartlari quyidagi teoremalarda ifodalangan.

1- t e o r e m a. Agar $y_0 \in V$ nuqta *J[y]* funksionalning lokal minimum (maksimum) nuqtasi bo'lsa va shu nuqtada δJ birinchi variatsiya hamda $\delta^2 J$ ikkinchi variatsiya mavjud bo'lsa,

$$\delta J = 0 \qquad \delta^2 J \geq O(\delta^2 J \leq 0) \qquad (19)$$

shartlar bajariladi.

I s b o t i. Isbotni minimum uchun keltiramiz. Maksimum uchun ham shunga o'xshash isbotlanadi.

J[y] funksionalning $y_0 \in V$ nuqtada δJ variatsiyaga ega ekanligidan,

$$J[y_0 + th] - J[y_0] = t\,\delta J[y_0, h] + O(t), \quad t \in R \qquad (20)$$

tenglik bajariladi, bu yerda O(t)h $\to 0$. t $\to 0$. y_0- lokal minimum nuqtasi bo'lganligidan, yetarli kichik $t > 0$ uchun, (20) dan,

$$\delta J + \frac{O(t)}{t} \geq 0$$

yoki $t \to +0$ da limitga o'tib, $\delta J \geq 0$ munosabatni olamiz. Xuddi shuningdek, (20) yoyilmadan yetarli kichik $t<0$ uchun foydalanib, $\delta J \leq 0$ munosabatni olamiz. Demak, $\delta J = 0$.

Endi *J[y]* funksionalning y_0 nuqtada $\delta^2 J$ ikkinchi variatsiyasining mavjudligini va birinchi variatsiya δJ =0 ekanligini hisobga olib,

$$J[y_0 + th] - J[y_0] = \frac{t^2}{2}\delta^2 J[y_0, h] + O(t^2), \quad t \in R \qquad (21)$$

yoyilmaga ega bo'lamiz, bu yerda $O(t^2)h^2 \to 0,\ t \to 0, y_0 -$lokal minimum nuqtasi bo'lganligi uchun, (21) dan,

$$\frac{1}{2}\delta^2 J + \frac{O(t^2)}{t^2} \geq 0$$

tengsizlik o'rinli bo'lishi kelib chiqadi. Bu tengsizlikda $t \to 0$ da limitga o'tib, $\delta^2 J \geq 0$ munosabatni olamiz. Teorema isbotlandi.

2-t e o r e m a. Agar *J[y]* funksional $y_0 \in V$ nuqtada birinchi va ikkinchi variatsiyalarga ega bo'lib, ular

$$\delta J,\ \delta^2 J \geq \alpha h^2\ (\delta^2 J \leq -\alpha h^2), \ \ \forall h \in W \qquad (22)$$

(bu yerda $\alpha>0$ – biror o'zgarmas) shartlarni qanoatlantirsa, y_0 – lokal minimum (lokal maksimum) nuqtasi bo'ladi.

I s b o t i. (22) munosabatlardan va ikkinchi variatsiya ta'rifidan foydalanib,

$$J[y_0 + h] - J[y_0] = \frac{1}{2}\delta^2 J + o(\|h\|^2) \geq \frac{\alpha}{2}\|h\|^2 + $$
$$+ o(\|h\|^2) \geq \|h\|^2 [\frac{\alpha}{2} + \frac{o(\|h\|^2)}{\|h\|^2}] \qquad (23)$$

munosabatni olamiz.

$$\lim_{h \to 0} \frac{o(\|h\|^2)}{\|h\|^2} = 0$$

bo’lgani uchun, $\alpha>0$ ni hisobga olib, (23) dan, yetarli kichik $\|h\|$ larda,

$$J[y_0+h]-J[y_0]>0$$

munosabatga ega bo’lamiz. Demak, y_0 – lokal minimum nuqtasidir. Teorema isbotlandi.

1-ma’ruza bo’yicha savollar

1. Funksional deb nimaga aytiladi?
2. Funksional aniqlangan funksiyalarning yaqinlik tartiblari haqida bilganlarinigizni yozing.
3. Funksionalning uzluksizligi ta’rifini keltiring.
4. Funksionalning orttirmasi ta’rifini keltiring.
5. Fuksionalning variatsiyasi (Lagranj bo’yicha) ta’rifini bering
6. Chiziqli funksionalning ta’rifini bering va misollar keltiring.
7. Kvadratik funksionalning ta’rifini bering va misollar keltiring.
8. Differensiallanuvchi funksional tarifini keltiring.
9. Funksionalning global maximumi (minimumi) ta’rifini bering.
10. Funksionalning lokal maximumi (minimumi) ta’rifini bering.
11. Funksionalning ekstremumini zaruriy sharti nimadan iborat (teoremani keltiring).
12. Variatsion masalaning qo’yilishini keltiring.

Masalalar.

1. Quyidagi $y_1(x)=x^2$ va $y_2(x)=x^3$ funksiyalar orasidagi masofani: *a*) $C[0,1]$ fazo normasida, *b*) $C^1[0,1]$ fazo normasida hisoblang.

2. $C^1[0,1]$ fazoda aniqlangan

$$J(y)=\int_0^1\left(y-y^1\right)dx$$

funksionalning $C^1[0,1]$ fazo normasida $y_0(x)=x^3$ funksiyada uzluksizligini ko’rsating.

3. Ushbu

$$J(y)=\int_0^1\sqrt{1+y'^2}\,dx$$

funksional $C^1[0,1]$ da aniqlangan bo’lsin. Uning $y_0(x)\equiv 0$ funksiyada: *a*) $C[0,1]$ normasi bo’yicha uzluksizlikka tekshiring.

2-ma'ruza. Funksionalning variatsiyasi, ekstremumi. Variatsiya terminidagi ekstremum shartlari

REJA:

1. Masalaning qo'yilishi. Kuchli va kuchsiz lokal ekstremumlar.
2. Asosiy lemmalar (Lagranj, Dyubua-Reymon lemmalari).

1. Masalaning qo'yilishi. Quyidagilar berilgan bo'lsin:

a) $Q - R^3$ dagi biror ochiq to'plam (soha);

b) $S = \{(x, y) \in R^2 : (x, y, z) \in Q\} - Q$ to'plamning R^2 ga proyeksiyasi;

v) $P_0(x_0, y_0)$, $P_1(x_1, y_1) - S$ to'plamning belgilangan nuqtalari, $x_0 < x_1$;

g) $F(x, y, z)$: $Q \to R^1$ – uzluksiz funksiya.

$C^1[x_0, x_1]$ fazoning

$$V = \{y(x) \in C^1[x_0, x_1] : y(x_0) = y_0,\ y(x_1) = y_1\ (x, y(x), y'(x)) \in Q, x \in [x_0, x_1]\} \quad (1)$$

to'plamida aniqlangan,

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')dx \quad (2)$$

funksionalning ekstremumini topish masalasini qaraymiz. Bu masalaga *variatsion hisobning asosiy masalasi* deyiladi va u

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')dx \to \min(\max),\ y(x_0) = y_0,\ y(x_1) = y_1,\ y(x) \in C^1[x_0, x_1] \quad (3)$$

ko'rinishda belgilanadi. (1) ko'rinishdagi to'plamga (3) masalaning *joyiz funksiyalari* (chiziqlari) to'plami deyiladi. (3) masalada joyiz chiziqlarning uchlari berilgan P_0 va P_1 nuqtalarda mahkamlangan, ya'ni qo'zg'olmasdir.

Variatsion hisobning asosiy masalasi - chegaralari qo'zg'olmas eng sodda variatsion masaladir.

Qaralayotgan (3) masalaning yechimi – (2) funksionalning (1) to'plamdagi global ekstremum nuqtasidan iborat. Yechimni aniqlashda esa, lokal ekstremum tushunchasi ham muhim rol o'ynaydi, chunki ular uchun funksional variatsiyasidan foydalaniladigan zaruriy va yetarli shartlar mavjud (1-ma'ruzaga q.).

(2) funksional qaralayotgan $C^1[x_0, x_1]$ fazoda funksiyaning nolinchi va birinchi tartibli atroflari tushunchalaridan foydalanib, lokal ekstremum nuqtalarini ham, shularga mos holda, aniqlash mumkin.

1-t a ' r i f. $y^0 = y^0(x)$ – joyiz funksiya bo'lsin $(y^0 \in V)$. Agar y^0 ning shunday $V_0(y^0, \varepsilon)$ nolinchi tartibli ε – atrofi mavjud bo'lib, shu atrofga tegishli barcha $y = y(x)$ joyiz funksiyalar uchun,

$$J[y^0] \le J[y] \quad (J[y^0] \ge J[y])$$

munosabat bajarilsa, $y^0(x)$ funksiya - (2) funksionalning *kuchli lokal minimum (maksimum) nuqtasi* deyiladi.

2-t a ’ r i f. Agar $y^0 = y^0(x)$ joyiz funksiyaning shunday $V_1(y^0,\varepsilon)$ birinchi tartibli ε - atrofi mavjud bo’lsaki, $J[y^0] \le J[y] \quad (J[y^0] \ge J[y])$

$\forall y \in V_1(y^0,\varepsilon) \cap V$ munosabat bajarilsa, $y^0(x)$ funksiya (2) funksionalning *kuchsiz lokal minimum (maksimum) nuqtasi* deyiladi.

(2) funksionalning kuchli (kuchsiz) lokal ekstremum nuqtalariga variatsion hisob asosiy masalasida *kuchli (kuchsiz) ekstremallar* ham deyiladi.

Demak, agar $y^0(x)$ -- kuchli ekstemal bo’lsa, bu funksiya, unga faqat qiymatlari bo’yicha yaqin bo’lgan barcha joyiz funksiyalar ichida, funksionalga minimal (yoki maksimal) qiymat beradi. $y^0(x)$ -- kuchsiz ekstremal bo’lganda esa, bu funksiya, unga nafaqat qiymatlari, balki hosilasining qiymatlari bo’yicha ham yaqin bo’lgan joyiz funksiyalar ichida, funksionalga ekstremal qiymat beradi.

Nolinchi tartibli atrofning birinchi tartibli atrofdan kengroqligini, ya’ni $V_1(y^0,\varepsilon) \subset V_0(y^0,\varepsilon)$ ekanligini hisobga olsak, yuqoridagi ta’riflardan har bir kuchli ekstremalning kuchsiz ekstremal ham bo’lishi kelib chiqadi. Bu tasdiqning aksi esa, to’g’ri emas.

Misol. $$\left.\begin{array}{l} J[y] = \int_0^\pi y^2(1-y'^2)dx \to \min \\ y(0) = y(\pi) = 0, \ y(x) \in C^1[0,\pi] \end{array}\right\}$$

Ravshanki, $y^0 = y^0(x) \equiv 0$ funksiya bu masalada kuchsiz minimal bo’ladi. Haqiqatan ham,

$$\|y - y^0\| = \|y\|_{C^1[0,\pi]} < \varepsilon \qquad (0 < \varepsilon < 1)$$

shartni qanoatlantiruvchi har bir $y = y(x) \in C^1[0,\pi]$ funksiya uchun $[y'(x) < 1,] \quad x \in [0,\pi]$, bo’lganligidan,

$$J[y] = \int_0^\pi y^2(1-y^2)dx \ge 0 = J[y^0,]$$

$$\|y - y^0\|_{C^1[0,\pi]} < \varepsilon, \quad y(x) = y(\pi) = 0$$

munosabat bajariladi. Ammo $y^0(x)$-- kuchli minimal bo’la olmaydi. Haqiqatan ham, $\varepsilon < 0$ istalgancha kichik bo’lganda ham, $y_n = y_n(x) = \frac{1}{\sqrt{n}} \sin nx$ joyiz funksiya yetarli katta n lar uchun, $\|y_n - y^0\|_{C^1[0,\pi]} < \varepsilon$ shartni qanoatlantiradi, ammo

$$J(y_n) = \frac{1}{n}\int_0^\pi \sin^2 nx(1 - n\cos^2 nx)\,dx = \frac{1}{n}\int_0^\pi \sin^2 nx\,dx -$$

$$-\frac{1}{n}\int_0^\pi \sin^2 2nx\,dx = \frac{\pi}{2n} - \frac{\pi}{8} < 0 = J[y^0], \ n > 4.$$

Variatsion hisob asosiy masalasida yechimning mavjudligini tekshirishda quyidagi yetarli shartdan foydalanish mumkin. Biz bu tasdiqni isbotsiz keltiramiz.

Faraz qilaylik: 1) $F(x,y,z)\in C^1$; 2) $F(x,y,z)$ funksiya z bo'yicha qavariq (botiq); 3) $F(x,y,z)\geq \Phi(z)$ $\quad (F(x,y,z)\leq \Phi(z))$, $\Phi(z)/z\to +\infty,\ z\to\infty$ bo'lsin. U vaqtda, shunday $y^0(x)$, $x\in[x_0,x_1]$ absolyut uzluksiz funksiya mavjudki, u $y^0(x_0)=y_0$, $y^0(x_1)=y_1$ shartlarni qanoatlantiradi va bu funksiyada (2) funksional kuchli minimum (maksimum) ga erishadi.

2. Asosiy lemmalar. Avvalgi ma'ruzamizda funksional ekstemumining zaruriy sharti – birinchi variatsiyaning ekstremum nuqtasida nolga teng bo'lishi ekanligi ko'rsatildi. Ushbu ma'ruzada shu natija asosida variatsion hisob asosiy masalasida eksremumning birinchi tartibli zaruriy sharti aniqlashtiriladi.

Dastlab, variatsion hisobning asosiy lemmalari deb ataluvchi, ba'zi yordamchi tasdiqlarni keltiramiz.

1-l e m m a (Lagranj)**.** Agar $a(x)$ funksiya $[x_0,x_1]$ kesmada aniqlangan va uzluksiz bo'lib, shu $[x_0,x_1]$ da uzluksiz differensiallanuvchi hamda $h(x_0)=h(x_1)=0$ shartni qanoatlantiruvchi barcha $h(x)$ funksiyalar uchun,

$$\int_{x_0}^{x_1} a(x)\,h(x)\,dx=0 \tag{4}$$

tenglik bajarilsa, $[x_0,x_1]$ kesmada $a(x)=0$ bo'ladi.

I s b o t i. Lemmaning tasdig'i o'rinli bo'lmasin, deb faraz qilamiz, ya'ni qandaydir $x^*\in[x_0,x_1]$ nuqtada $a(x^*)\neq 0$ bo'lsin. $a(x)$ funksiyaning uzluksizligidan foydalangan holda, x^* nuqtani kesmaning ichki nuqtasi deb hisoblash mumkin. Aniqlik uchun, $a(x^*)>0$ deb olamiz. $a(x)$ funksiyaning uzluksizlik xossasiga ko'ra, x^* nuqtaning shunday $(x^*-\varepsilon,\ x^*+\varepsilon)$ $(\varepsilon>0)$ atrofi topiladiki, unda $a(x)$ funksiya o'z ishorasini saqlaydi, ya'ni musbat bo'lib qola veradi. Endi $h(x)$ funksiyani quyidagicha aniqlaymiz:

$$h(x)=\begin{cases}(\varepsilon+x^*-x)(\varepsilon-x^*+x), x\in[x^*-\varepsilon,x^*+\varepsilon]\\ 0,\ x\notin[x^*-\varepsilon,x^*+\varepsilon].\end{cases} \tag{5}$$

(5) funksiya uchun,

$$\int_{x_0}^{x_1} a(x)h(x)dx=\int_{x^*-\varepsilon}^{x^*+\varepsilon} a(x)(\varepsilon-x^*+x)dx>0,$$

ya'ni (4) ga zid munosabat bajariladi. Olingan qarama-qarshilik lemmani isbotlaydi.

2-l e m m a (Dyubua-Reymon). Faraz qilaylik, $a_0(x)$ va $a_1(x)$ funksiyalar $[x_0,x_1]$ kesmada uzluksiz bo'lsin. Agar $h(x_0)=h(x_1)=0$ shartni qanoatlantiruvchi barcha $h(x)\in C^1[x_0,x_1]$ funksiyalar uchun

$$\int_{x_0}^{x_1}[a_0(x)h(x)+a_1(x)h'(x)]dx=0 \qquad (6)$$

tenglik bajarilsa, $a_1(x)$ -uzluksiz differensiallanuvchi funksiya bo'ladi va barcha $x\in[x_0,x_1]$ nuqtalarda

$$a_0(x)-\frac{d}{dx}a_1(x)=0 \qquad (7)$$

munosabat o'rinli bo'ladi.

I s b o t i. Quyidagi,

$$p(x)=\int_{x_0}^{x_1}a_0(t)dt+c,\ \ c=const,$$

funksiyani qaraymiz. Bu funksiya $[x_0,x_1]$ kesmada uzluksiz differensiallanuvchidir. O'zgarmas c sonni shunday tanlaymizki,

$$\int_{x_0}^{x_1}p(x)dx=\int_{x_0}^{x_1}a_1(x)dx \qquad (8)$$

tenglik bajarilsin. Endi $dp(x)=a_0(x)dx$ ekanligini va $h(x_0)=h(x_1)=0$ shartni hisobga olib, (6) tenglikni quyidagicha yozamiz:

$$\begin{aligned}&\int_{x_0}^{x_1}[a_1(x)h'(x)+a_0(x)h(x)]dx=\int_{x_0}^{x_1}a_1(x)h'(x)dx+\int_{x_0}^{x_1}h(x)dp(x)=\\&=\int_{x_0}^{x_1}[a_1(x)-p'(x)]h'(x)dx+h(x)p(x)\Big|_{x_0}^{x_1}=\int_{x_0}^{x_1}[a_1(x)-p(x)]h'(x)dx=0.\end{aligned} \qquad (9)$$

$[x_0,x_1]$ kesmada uzluksiz differensiallanuvchi

$$h(x)=\int_{x_0}^{x_1}[a_1(t)-p(t)]dt \qquad (10)$$

funksiyani qaraymiz. Bu funksiyaning aniqlanishidan, $h(x_0)=0$ ekanligi ravshan. (8) ni hisobga olsak, $h(x_1)=0$ ekanligini ko'ramiz. Demak, (10) funksiya lemmaning barcha shartlarini qanoatlantiradi. Bu funksiya uchun, (9) shart,

$$\int_{x_0}^{x_1}[a_1(x)-p(x)]^2dx=0$$

ko'rinishni oladi. Bu yerdan, $a_1(x)\equiv p(x)$, ya'ni $a_1(x)\in C^1[x_0,x_1]$ ekanligi hamda

$$a_0(x)-\frac{d}{dx}a_1(x)=0,\ \forall x\in[x_0,x_1]$$

tenglik kelib chiqadi. Lemma isbotlandi.

Isbotlangan lemmadan $a_0(x)\equiv 0$ bo'lganda quyidagi natijada ega bo'lamiz.

3- l e m m a**.** Agar $a(x)$ funksiya $[x_0,x_1]$kesmada uzluksiz bo'lib, $h(x_0)=h(x_1)=0$ shartni qanoatlantiruvchi ixtiyoriy $h(x)\in C^1[x_0,x_1]$ funksiya uchun,

$$\int_{x_0}^{x_1} a(x)h'(x)dx = 0$$

bo'lsa, $a(x) \equiv const$, $x \in [x_0, x_1]$ bo'ladi.

Bu tasdiq ham, ba'zi adabiyotlarda, Dyubua-Reymon lemmasi deb yuritiladi.

Mustaqil ishlash uchun savollar.

1. Variasion hisob asosiy masalasi qanday qo'yiladi?
2. Funksionalning kuchli lokal minimumi (maksimumi) deb qanday nuqta (chiziq, sirt va h.k) ga aytiladi?
3. Funksionalning kuchli lokal minimumi (maksimumi) deb qanday nuqta (chiziq, sirt va h.k.) ga aytiladi.
4. Variasion hisob asosiy masalasida yechim mavjud bo'lishining yetarli sharti nimadan iborat?
5. Variasion hisobning asosiy lemmalar; Lagranj lemmasini yekltiring.
6. Variasion hisobning asosiy Lemmalari Dyuba-Riman lemmasini yekltiring.

3-ma'ruza. Variatsion hisobning asosiy masalasi. Kuchsiz ekstremumning birinchi tartibli zaruriy sharti, Eyler tenglamasi. Eyler tenglamasining xususiy hollari

Reja:

1. Eyler tenglamasi.
2. Eyler tenglamasining xususiy hollari
3. Gilbert teoremasi
4. Eylerning integro-differensial tenglamasi

1. Eyler tenglamasi. Ushbu va bundan keyingi ma'ruzalarimizda, $F(x, y, y')$ funksiyaning xususiy hosilalari uchun, quyidagi belgilashlardan foydalanamiz:

$$F_x = \frac{\partial F}{\partial x}, \quad F_y = \frac{\partial F}{\partial y}, \quad F_{y'} = \frac{\partial F}{\partial y'},$$

$$F_{xy'} = \frac{\partial^2 F}{\partial x \partial y'}, \quad F_{yy} = \frac{\partial^2 F}{\partial y^2}, \quad F_{yy'} = \frac{\partial^2 F}{\partial y \partial y'}, \quad F_{y'y'} = \frac{\partial^2 F}{\partial y'^2}$$

Avvalo, $F(x, y, y') \in C^{(2)}(Q)$, $y^0(x) \in C^{(2)}[x_0, x_1]$, deb faraz qilamiz va $y^0 = y^0(x) \in V$ nuqtada (2) funksionalning birinchi variatsiyasini hisoblaymiz. Buning uchun,

$$M(y^0) = \{h \in C^{(1)}[x_0, x_1] : y^0 + h \in V\} = \{h \in C^{(1)}[x_0, x_1] : h(x_0) = h(x_1) = 0\}$$

to'plamni qaraymiz. Bu to'plam, $C^{(2)}[x_0, x_1]$ fazoning chiziqli qism fazosi bo'lgani uchun, variatsiyaning ta'rifiga ko'ra,

$$\delta J = \frac{d}{dx} J\left[y^0 + \alpha h\right]_{\alpha=0}, \; h \in M\left(y^0\right).$$

$Q \subset R^3 -$ ochiq to'plam, $F(x,y,y') \in C^{(2)}(Q)$ bo'lgani uchun, $F\left(x, y^0(x) + \alpha h(x),\right.$ $\left.y^0{}'(x) + \alpha h'(x)\right)$ funksiya $\{(x,\alpha): x \in [x_0, x_1] |\alpha| < \varepsilon\}$ to'plamda (bu yerda $\varepsilon > 0$) – etarlicha kichik α bo'yicha uzluksiz xususiy hosilaga ega, shuning uchun,

$$\varphi(\alpha) = J\left[y^0 + \alpha h\right] = \int_{x_0}^{x_1} F\left(x, y^0(x) + \alpha h(x), \; y^0{}'(x) + \alpha h'(x)\right) dx$$

funksiya $\alpha = 0$ nuqta atrofida uzluksiz differensiallanuvchidir. Shunday qilib,

$$\delta J = \varphi'(\alpha)\Big|_{\alpha=0} = \int_{x_0}^{x_1} \frac{\partial}{\partial \alpha} F\left(x, y^0(x) + \alpha h(x), \; y^0(x) + \alpha h'(x)\right) dx_{\;\alpha=0} =$$
$$\int_{x_0}^{x_1} \left[F_y\left(x, y^0(x), \; y^0{}'(x)\right) h(x) + F_{y'}\left(x, y^0(x), y^0{}'(x)\right) h'(x)\right] dx. \tag{11}$$

$F(x,y,y') \in C^{(2)}(Q), \quad y^0(x) \in C^{(2)}[x_0, x_1]$ bo'lgani uchun, $F_{y'}\left(x, y^0(x), \; y^0{}'(x)\right)$ funksiya $[x_0, x_1]$ kesmada uzluksiz differensiallanuvchi va

$$\frac{d}{d\alpha} F_{y'}\left(x, y^0(x), y^0{}'(x)\right) = F_{y'y'}\left(x, y^0(x), \; y^0{}'(x)\right) y^0{}''(x) +$$
$$+ F_{yy'}\left(x, y^0(x), \; y^0{}'(x)\right) y^0{}'(x) + F_{xy'}\left(x, y^0(x), y^0{}'(x)\right) \tag{12}$$

tenglik o'rinlidir. Endi (11) dagi ikkinchi qo'shiluvchini bo'laklab integrallab va $h(x_0) = h(x_1) = 0$ shartni hisobga olib,

$$\delta J = \int_{x_0}^{x_1} \left[F_y\left(x, y^0(x), y^0{}'(x)\right) - \frac{d}{dx} F_{y'}\left(x, y^0(x), y^0(x)\right)\right] h(x) dx \tag{13}$$

formulaga ega bo'lamiz.

Faraz qilaylik, $y^0 = y^0(x) -$ (3) masalada kuchsiz ekstremal bo'lsin. U vaqtda, ekstremumning zaruriy shartiga ko'ra, shu nuqtada hisoblangan birinchi variatsiya nolga teng, ya'ni (13) ga asosan,

$$\int_{x_0}^{x_1} \left[F_y\left(x, y^0(x), y^{0'}(x)\right) - \frac{d}{dx} F_{y'}\left(x, y^0(x), y^0(x)\right)\right] h(x) dx = 0 \tag{14}$$

bo'ladi. Hosil qilingan (14) munosabatga Lagranj lemmasini qo'llaymiz va (12) ni hisobga olib, $y^0(x)$ funksiya uchun,

$$F_{y'y'}\left(x, y^0(x), y^0{}'(x)\right) + y^0{}''(x) + F_{yy'}\left(x, y^0(x), \; y^0{}'(x)\right) y^0{}'(x) +$$
$$+ F_{xy'}\left(x, y^0(x), \; y^0{}'(x)\right) - F_{y'}\left(x, y^0(x), y^0{}'(x)\right) = 0$$

tenglik, barcha $x \in [x_0, x_1]$ nuqtalarda bajarilishini ko'ramiz. Shunday qilib, quyidagi tasdiqqa ega bo'ldik.

1-t e o r e m a. $F(x,y,y') \in C^{(2)}(Q)$ bo'lsin. Agar $y^0(x) \in C^{(2)}[x_0, x_1]$ (3) masalada kuchsiz ekstremal bo'lsa, u

$$F_{y'y'} y'' + F_{yy'} y' + F_{xy'} - F_y = 0 \tag{15}$$

tenglamani qanoatlantiradi.

Hosil qilingan (15) differensial tenglamaga, *Eyler tenglamasi* deyiladi.

Shunday qilib, ekstremumning zaruriy sharti bo'lgan (15) Eyler tenglamasi odatdagidan kuchliroq talabda, ya'ni $F(x,y,y')$ funksiya va $y^0(x)$ ekstremalga ikki marta uzluksiz differensiallanuvchilik sharti qo'yilganda keltirib chiqarildi. Ammo variatsion hisob asosiy masalasining qo'yilishida $y(x)$ joyiz funksiyalarni $C^{(1)}[x_0,x_1]$ sinfdan deb va $F(x,y,y')$ funksiyani esa faqatgina uzluksiz deb hisoblaganmiz, xolos. Shuning uchun ekstremumning zaruriy sharti bo'lgan Eyler tenglamasini $F(x,y,y')$ integrantga va $y^0(x)$ ekstremalga kuchsizroq talab qo'yilgan hol uchun umumlashtirish muhimdir.

2-t e o r e m a. $F(x,y,y') \in C^{(1)}(Q)$ bo'lsin. Agar $y^0(x) \in C^{(1)}[x_0,x_1]$ joyiz funksiya (3) masalada kuchsiz ekstremal bo'lsa, u

$$F_y(x,y,y') - \frac{d}{dx}F_{y'}(x,y,y') = 0 \qquad (16)$$

tenglamani qanoatlantiradi.

I s b o t i. Funksionalning variatsiyasi uchun yuqorida hosil qilingan (11) formuladan foydalanib, ekstremumning zaruriy sharti $\delta J = 0$ ni yozamiz :

$$\int_{x_0}^{x_1} \left[F_y\left(x, y^0(x), y^0{}'(x)\right)h(x) + F_{y'}\left(x, y^0(x), y^0{}'(x)\right)h'(x)\right]dx = 0,$$

bu yerda $h(x) \in C^{(1)}[x_0,x_1]$, $h(x_0) = h(x_1) = 0$.

Endi $a_0(x) = F_y\left(x, y^0(x), y^0{}'(x)\right)$, $a_1(x) = F_{y'}\left(x, y^0(x), y^0{}'(x)\right)$ deb olsak, (17) dan Dyubua – Reymon lemmasiga ko'ra,

$$F_y\left(x, y^0(x), y^0{}'(x)\right) - \frac{d}{dx}F_{y'}\left(x, y^0(x), y^0{}'(x)\right) = 0, \quad \forall x \in [x_0,x_1]$$

munosabatni olamiz, ya'ni $y^0(x)$ ekstremal (16) tenglamani qanoatlantiradi. Teorema isbotlandi.

$F(x,y,y') \in C^{(2)}(Q)$ bo'lganda (16) differensial tenglama (15) ko'rinishni oladi. Ekstremumning zaruriy sharti sifatida olingan (16) tenglama ham, Eyler tenglamasi deb ataladi.

3-t a ' r if. Eyler tenglamasini qanoatlantiruvchi $y = y(x)$ joyiz funksiyalarga (2) *funksionalning joyiz stasionar funksiyalari* deyiladi.

Yuqorida isbotlangan teoremalarga ko'ra, (2) funksionalning joyiz stasionar funksiyalari ekstremumga shubhali funksiyalardir.

M i s o l l a r. 1) $\left.\begin{array}{l} J[y] = \int_1^2 \left(y'^2 + 4xy\right)dx \to extr \\ y(1) = 0, \quad y(2) = -1 \end{array}\right\}$

masalani qaraymiz. Uning uchun Eyler tenglamasini tuzamiz:

$$F = y'^2 + 4xy, \; F_y = 4x, \; F_{y'} = 2y', \; \frac{d}{dx}F_{y'} = 2y''$$

$$F_y - \frac{d}{dx}F_{y'} = 4x - 2y'' = 0.$$

Demak, Eyler tenglamasi $y''-2x=0$ tenglamadan iborat. Bu tenglamaning umumiy yechimi

$$y=y(x)=\frac{x^3}{3}+c_1x+c_2$$

ko’rinishda bo’ladi. $y(1)=0,\ y(2)=-1$ chegaraviy shartlarga ko’ra,

$$c_1+c_2=-\frac{1}{3},\ \ 2c_1-c_2=-\frac{11}{3}.$$

Bu yerdan, $c_1=-\frac{10}{3}$, $c_2=3$. Shunday qilib, $y=\frac{x^3}{3}-\frac{10}{3}x+3$ funksiya – yagona joyiz stasionar funksiyadir.

2) $$\left.\begin{array}{l} J[y]=\int_1^{2\pi}\left(y'^2-y^2\right)dx\to extr \\ y(0)=1,\quad y(2\pi)=1 \end{array}\right\}$$

masalani qaraymiz. Unda $F='^2-y^2$, $F_y=2y$, $F_{y'}=2y''$. Demak, (16) tenglama $y''+y=0$ ko’rinishda bo’ladi. $y=c_1\cos x+c_2\sin x$ funksiya hosil qilingan Eyler tenglamasining umumiy yechimidir. $y(0)=y(2\pi)=1$ chegaraviy shartlarga asosan, $y=\cos x+c\sin x$ funksiyaga ega bo’lamiz, bu yerda c – ixtiyoriy o’zgarmas. Demak, qaralayotgan masalada cheksiz ko’p joyiz stasionar funksiyalar mavjud.

3) $$\left.\begin{array}{l} J[y]=\int_1^{3}(3x-y)y\,dx\to extr \\ y(1)=1,\quad y(3)=4. \end{array}\right\}$$

Bu masalada $F=(3x-y)y,\ F_y=3x-2y,\ F_y=0.$ Eyler tenglamasi, $F_y=0$ tenglamadan iborat. Bu yerdan $y=\frac{3x}{2}$. Bu funksiya, $y(1)=1,\ y(3)=4$ chegaraviy shartlarni qanoatlantirmaydi. Qaralayotgan funksional joyiz stasionar funksiyalarga va, demak, ekstremumga ham ega emas.

2.Eyler tenglamasining xususiy hollari. $F(x,y,y')$ integrant o’z argumentlarining «to’liq» funksiyasi bo’lmagan hollarda Eyler tenglamasini soddalashtirish yoki uning birinchi integralini aniqlash mumkin.

a) F funksiya faqat y' ga bog’liq, ya’ni $F=F(y')$ bo’lsin. Bu holda Eyler tenglamasi $\frac{d}{dx}F_{y'}(y')=0$ yoki $F_{y''}=0$ bo’ladi. Bu yerdan $y''=0$ yoki $F_{y''}=0$ bo’lishi kelib chiqadi. Agar $y''=0$ bo’lsa, $y=C_1x+C_2$ ikki parametrli to’g’ri chiziqlar oilasiga ega bo’lamiz. Agar $F_{y'y'}(y')=0$ tenglama bir yoki bir necha $y'=k_i$ ildizga ega bulsa, $y=k_ix+C$ bir parametrli to’g’ri chiziqlar oilasiga ega bo’lamiz. Shunday qilib, $F=F(y')$ bo’lganda Eyler tenglamasining yechimlari $y=C_1x+C_2$ chiziqli funksiyalardan iboratdir.

b) $F=F(x,y')$, ya’ni F funksiya faqat x va y' larga bog’liq bo’lsin. Bu holda $F_y=0$ bo’ladi va Eyler tenglamasi $\frac{d}{dx}F_{y'}=0$ ko’rinishni oladi. Bu yerda uning $F_{y''}(x,y')=C$

birinchi integraliga ega bo’lamiz. Bu olingan tenglama Eyler tenglamasiga teng kuchlidir. Uni y' ga nisbatan yechish yoki biror parametr kiritish yo’li bilan integrallash mumkin.

c) F faqat y va y' ga bog’liq bo’lsin: $F = F(y, y')$. Bu holda $F \in C^{(2)}$ deb faraz qilib, (15) Eyler tenglamasini yozamiz: $F_{y'y'}y'' + F_{y'y}y' - F_y = 0$ $(F_{xy'} = 0)$. Agar bu tenglamaning ikkala tomonini y' ga ko’paytirsak, uni $\frac{d}{dx}\left(F - y'F_{y'}\right) = 0$ ko’rinishda yozish mumkin. Natijada Eyler tenglamasi

$$F - y'F_{y'} = C \qquad (18)$$

ko’rinishdagi birinchi integralga ega bo’ladi. Eyler tenglamasining har qanday yechimi (18) tenglamani qanoatlantirishi ravshan. Aksincha, agar (18) tenglama faqat chekli sondagi nuqtalarda hosilasi nolga teng bo’lgan $y = y[x] \in C^{(2)}[x_0, x_1]$ yechimga ega bo’lsa, bu yechim Eyler tenglamasini ham qanoatlantiradi. (18) tenglamani y' ga nisbatan yechish, yoki parametr kiritish yo’li bilan integrallash mumkin.

Misol sifatida, braxistoxrona haqidagi masalada hosil qilingan

$$J[y] = \int_{x_0}^{x_1} \sqrt{\frac{1 + y^2}{2gy}}\, dx$$

funksionalni qaraymiz. Bu yerda $F = \sqrt{\frac{1 + y^2}{2gy}}$ faqat y va y' ga bog’liq. Demak, Eyler tenglamasining birinchi integralini ((18) tenglama) yozamiz:

$$F_{y'} = \frac{y'}{\sqrt{2gy\left(1 + y'^2\right)}}; \qquad \sqrt{\frac{1 + y'^2}{2gy}} - y' \cdot \frac{y'}{\sqrt{2gy\left(1 + y'^2\right)}} = C; \qquad \frac{1}{\sqrt{2gy\left(1 + y'^2\right)}} = C.$$

Bu yerdan, $y\left(1 + y'^2\right) = C_1$, $C_1 = \frac{1}{2gc^2}$, yoki $y' = \sqrt{\frac{C_1 - y}{y}}$ tenglamani olamiz. Endi $y = C_1 \sin^2 \frac{t}{2}$ almashtirish olsak, $dx = C_1 \sin^2 \frac{t}{2} dt = \frac{C_1}{2}(1 - \cos t)\, dx$ bo’ladi. Natijada: $x = C_2 + \frac{C_1}{2}(t - \sin t)$, $y = \frac{C_1}{2}(t - \cos t)$. Agar $y(x)$ joyiz chiziqning uchlari $A(0,0)$ va $B(x_1, y_1)$ nuqtalarda ekanligidan foydalansak, $C_2 = 0$ bo’lishini olamiz.

$x = \frac{C_1}{2}(t - \sin t)$, $y = \frac{C_1}{2}(t - \cos t)$ tenglamalar sistemasi sikloida deb ataluvchi chiziqning parametrik shakldagi tenglamalaridan iborat. Demak, braxistoxrona haqidagi masalaning yechimi faqat sikloida yoyidan iborat bo’la olar ekan.

3. Gilbert teoremasi. Yuqorida ko’rsatildiki, agar $F(x, y, z) \in C^{(2)}$ bo’lsa, Eyler tenglamasi, $y = y(x) \in C^{(2)}$ funksiyaga nisbatan ikkinchi tartibli differensial tenglamadan iborat. Ammo, (3) masalada ekstremal $C^{(1)}[x_0, x_1]$ dan izlanadi. Shuning uchun, ekstremalning $C^{(2)}[x_0, x_1]$ ga tegishli bo’lishini ta’minlovchi quyidagi tasdiqning ahamiyati kattadir.

3-t e o r e m a (Gilbert). $F(x,y,y')\in C^{(2)}(Q)$ bo'lsin. Agar (3) masalaning $y^0(x)$ kuchsiz lokal ekstremali uchun $F\big(x,y^0(x),\ y^0{}'(x)\big)\neq 0,\ x\in[x_0,x_1]$ bo'lsa, $y^0(x)\in C^{(2)}[x_0,x_1]$ bo'ladi.

I s b o t i. Quyidagi,

$$\Phi(x,z)=F_{y'}\big(x,y^0(x),t\big)-\int_{x_0}^{x_1}F_y\big(t,y^0(t),y^0{}'(t)\big)dt-C$$

funksiyani qaraymiz, bunda $C=const$. 2-teoremaga ko'ra ,

$$\frac{d}{dx}F_{y'}\big(x,y^0(x),\ y^0{}'(x)\big)=F_y\big(x,y^0(x),\ y^0{}'(x)\big),\quad \forall x\in[x_0,x_1],$$

ya'ni $F_{y'}\big(x,y^0(x),\ y^0{}'(x)\big)$ funksiya $F_y\big(x,y^0(x),\ y^0{}'(x)\big)$ funksiya uchun boshlang'ich funksiyadir. Shuning uchun,

$$F_{y'}\big(x,y^0(x),y^0{}'(x)\big)=\int_{x_0}^{x_1}F_y\big(t,y^0(t),y^0{}'(t)\big)\,dt+c,\ \ c=const\,.$$

Demak, $z^0=y^0{}'(x),\ x\in[x_0,x_1]$ funksiya uchun $\Phi(x,z^0)=0$.

Teoremaning shartiga ko'ra

$$\Phi_z\big(x,z^0\big)\Big|_{z^0=y^{0'}}=F_{y'y'}\big(x,y^0(x),y^0{}'(x)\big)\neq 0,\ \ x\in[x_0,x_1].$$

U vaqtda oshkormas funksiyaning mavjudligi haqidagi teoremaga asosan, $\Phi(x,z)=0$ tenglamaning $z^0=y^0{}'(x)$ yechimi $\Phi(x,z)$ funksiya o'z argumentlari buyicha nechanchi tartibli xosilaga ega bo'lsa, shunday tartibli xosilaga egadir. $F(x,y,z)\in C^{(2)}$ bo'lgani uchun $\Phi(x,z)\in C^{(1)}$. Demak, $z^0{}'=y^{y}{}''\in C[x_0,x_1]$, ya'ni $y^0(x)\in C^{(2)}[x_0,x_1]$. Teorema isbotlandi.

4. Eylerning integro-differensial tenglamasi. Variatsion hisobning asosiy masalasi silliq joyiz funksiyalar sinfida yechimga ega bo'lmasligi mumkin. Shuning uchun, *V* joyiz funksiyalar to'plamini kengaytirib, uni quyidagicha aniqlaymiz:

$$V=\{y=y(x)\in D^{(1)}[x_0x_1]:\ y(x_0)=y_0,\ y(x_1)=y_1\}\qquad (19)$$

Bu yerda $D^1[x_0,x_1]-[x_0,x_1]$ kesmada uzluksiz, hosilalari esa faqat chekli sondagi uzilish nuqtalariga ega $y=y(x)$ funksiyalar to'plamidan iborat. Har bir joyiz funksiya uchun $y'(\xi_i+0)\neq y'(\xi_i-0)$ shartni qanoatlantiruvchi bir necha $\xi_i,\ i=\overline{1,k}$ nuqtalar mavjud. Bunday ξ_i nuqtalarga bo'lakli – silliq joyiz funksiyaning egilish nuqtalari deyiladi.

Endi joyiz funksiyalari bo'lakli-silliq funksiyalar sinfidan olingan variatsion hisob asosiy masalasini qaraymiz:

$$\left.\begin{aligned}&J[y]=\int_{x_0}^{x_1}F(x,y,y')dx\to\min(\max)\\&y(x_0)=y_0,\,y(x_1)=y_1,\ y=y(x)\in D^{(1)}[x_0,x_1]\end{aligned}\right\}\qquad (20)$$

$D^{(1)}[x_0,x_1]$ dan olingan $y=y(x)$ va $y^0=y^0(x)$ funksiyalar orasidagi birinchi tartibli masofani ham $C^1[x_0,x_1]$dagi metrika kabi aniqlash mumkin:

$$\rho_1(y,y^0)=\max_{x\in[x_0,x_1]}\left|y(x)-y^0(x)\right|+\sup_{x\in A}\left|y'(x)-y^0{}'(x)\right|,$$

bu yerda $A \subset [x_0, x_1]$ to'plam – $y'(x)$ va $y^0{}'(x)$ hosilalar mavjud bo'lgan $x \in [x_0, x_1]$ nuqtalar to'plamidir.

$D^{(1)}[x_0, x_1]$ dagi $y^0 = y^0(x)$ funksiyaning birinchi tartibli ε – atrofini

$$V_1(y^0, \varepsilon) = \{y \in D^{(1)}[x_0, x_1] : \rho_1(y, y^0) < \varepsilon\} \qquad (21)$$

kabi aniqlab, (2) funksionalning (19) to'plamidagi kuchsiz lokal ekstremumi ta'rifini berish mumkin. Buning uchun, 2-ta'rifdagi $V(y^0, \varepsilon)$ ni (21) dagi $V_1(y^0, \varepsilon)$ bilan almashtirish kifoya. Shunday aniqlangan kuchsiz ekstremal uchun quyidagi teorema o'rinlidir.

4-t e o r e m a. $F(x, y, y') \in C^{(2)}(Q)$ bo'lsin. Agar $y^0(x)$ – (20) masalada kuchsiz ekstremal bo'lsa, u quyidagi

$$F_{y'}(x, y, y') = \int_{x_0}^{x_1} F_y(t, y, y')\, dt + c, \ \ c = const \qquad (22)$$

tenglamani qanoatlantiradi.

I s b o t i. 1-teorema isbotidagiga o'xshash mulohaza yuritib, funksional variatsiyasining (11) ko'rinishdagi ifodasiga ega bo'lamiz:

$$\delta J = \int_{x_0}^{x_1} \left[F_y(x, y^0(x), y^0{}'(x))h(x) + F_{y'}(x, y^0(x), y^0{}'(x))h'(x)\right] dx,$$

bu yerda $h(x) \in C^1[x_0, x_1]$, $h(x_0) = h(x_1) = 0$. Integral ostidagi birinchi qo'shiluvchini bo'laklab integrallaymiz:

$$\int_{x_0}^{x_1} F_y(x, y^0(x), y^0{}'(x))h(x)dx = \int_{x_0}^{x_1} F_y(t, y^0(t), y^0{}'(t))\, dx \cdot h(x)\Big|_{x_0}^{x_1} -$$

$$- \int_{x_0}^{x_1} \int_{x_0}^{x_1} F_y(t, y^0(t), y^0{}'(t)) dx \cdot h'(x) dx = \int_{x_0}^{x_1} \int_{x_0}^{x_1} F_y(t, y^0(t), y^0{}'(t))\, dx \cdot h'(x)\, dx.$$

U vaqtda, $\delta J = 0$ shart,

$$\int_{x_0}^{x_1} \left[F_y(x, y^0(x), y^0{}'(x)) - \int_{x_0}^{x_1} F_y(t, y^0(t), y^0{}'(t)) dt \right] h'(x) dt = 0$$

ko'rinishini oladi. Endi 3-lemmani qo'llab,

$$F_{y'}(x, y^0(x), y^0{}'(x)) = \int_{x_0}^{x_1} F_y(t, y^0(t), y^0{}'(t))\, dt + c$$

tenglikni olamiz, bu yerda $x \in [x_0, x_1]$ nuqta $y^0{}'(x)$ hosila mavjud bo'lgan nuqtadir. Shunday qilib, $y^0(x)$ funksiya (22) tenglamani qanoatlantirishi ko'rsatildi. Teorema isbotlandi.

(22) ga *Eylerning integro-differensial tenglamasi* deyiladi.

Isbotlangan teoremadan ko'rinadiki, agar $y^0(x)$ ekstremal ξ_i, $i = \overline{1, k}$ bukilish nuqtalariga ega bo'lsa, (23) ning o'ng tomoni (ξ_i, ξ_{i+1}) oraliklarda uzluksiz differensiallanuvchi, demak, $y^0(x)$ funksiya shu oraliklarda (16) Eyler tenglamasini qanoatlantiradi. Joyiz funksiyalari silliq funksiyalar sinfidan olingan (3) masala uchun esa, (22) tenglama, (16) tenglamaga teng kuchlidir.

(23) tenglikning o'ng tomoni barcha $x\in[x_0,x_1]$ nuqtalarda uzluksiz. Demak, bu tenglikning $F_{y'}(x,y^0(x),y^0{}'(x)$ chap tomoni ham uzluksiz funksiyadir. Natijada, agar $\xi - y^0(x)$ ekstremalning bukilish nuqtasi bo'lsa,

$$F_{y'}\left(y^0(\xi),y^0{}'(\xi+0)\right)=F_{y'}\left(\xi,y^0(\xi),y^0{}'(\xi-0)\right) \qquad (24)$$

tenglik o'rinlidir. (24) ga *Veyershtrass – Erdman sharti* deyiladi. Bunday tipdagi shartlardan yana biri,

$$H^0=F\left(x,y^0(x),y^0{}'(x)\right)-y^0{}'(x)F_{y'}\left(x,y^0(x),y^0{}'(x)\right)$$

funksiyaning $x=\xi$ bukilish nuqtasida uzluksizligidir, ya'ni

$$H^0(\xi+0)=H^0(\xi-0) \qquad (25)$$

ko'rinishdagi Veyershtrass – Erdman shartining isboti optimal boshqaruv nazariyasi natijalaridan kelib chiqadi.

Ma'ruzamiz oxirida, yechimi bo'lakli-silliq funksiyalar sinfida bo'lgan variatsion masalaga misol sifatida quyidagini keltiramiz.

Misol. $J[y]=\int_{-1}^{1}y^2(1-y')^2\,dx\to\min,\ y(-1)=0,\ y(1)=1$.

Bu masalada joyiz funksiyalarni $C^1[-1,1]$ sinfidan olsak, $\inf\limits_{y\in U}J[y]=0$ bo'ladi, silliq yechim mavjud emas. Haqiqatan ham, agar

$$y(x)=\begin{cases}0,\ x\in\left[-1,\frac{1}{n}\right],\\ \frac{n}{V_2(1+V_2)}\left(x+\frac{1}{n}\right)^2\quad x\in\left(-\frac{1}{n},\frac{1}{\sqrt{2n}}\right),\\ x,\qquad x\in[-]\end{cases}$$

funksiyalar ketma-ketligini qarasak, bu funksiyalar joyiz funksiyalar bo'ladi, ya'ni $y_n(-1)=0,\ y_n(1)=1$. Bu $y_n(x)$ ketma-ketlikda $|y_n(x)|\le 1,\ |y'_n(x)|\le 1$ bo'lgani uchun,

$$0\le J[y_n]=\int_{-\frac{1}{n}}^{\frac{1}{\sqrt{2n}}}y^2{}_n(x)(1-y'_n(x))^2\,dx<\frac{2}{n},$$

va $\lim\limits_{n\to\infty}J[y_n]=0$. Demak, $J[y_n]\ge 0,\ \forall y\in C^1[-1,1]$ ekanligini hisobga olsak, $J[y]=0$. Ammo, $J[\bar{y}]=0,\ \bar{y}\in C^1[-1,1]$ bo'lishi uchun, $\bar{y}(x)$ funksiya $[-1,1]$ kesmada yo aynan nolga teng, yoki $\bar{y}=const$ bo'lishi kerak. Biroq, bu funksiyalar, $y(-1)=0,\ y(1)=1$ chegaraviy shartlarni qanoatlantirmaydi.

Bundan ko'rinadiki, qaralayotgan masala, silliq joyiz funksiyalar sinfida yechimga ega emas. Masalaning bo'lakli-silliq funksiyalar sinfida yechimi mavjud. Shunday yechim sifatida,

$$y^0(x)=\begin{cases}0,x\in(-1,0]\\ x,\ x\in(0,1]\end{cases},,$$

funksiyani olish mumkin $\left(y\in D^1[-1,1], J\left[y^0\right]=0\right)$. Bu funksiya uchun, $\xi=0-$burchak nuqta (bukilish nuqtasi) bo'ladi. Ushbu burchak nuqtada Veyershtrass – Erdman shartlari bajariladi.

Mustaqil ishlash uchun savollar.

1. Variasion hisob asosiy masalasi qanday qo'yiladi?
2. Funksionalning kuchli lokal minimumi (maksimumi) deb qanday nuqta (chiziq, sirt va h.k) ga aytiladi?
3. Funksionalning kuchli lokal minimumi (maksimumi) deb qanday nuqta (chiziq, sirt va h.k.) ga aytiladi.
4. Variasion hisob asosiy masalasida yechim mavjud bo'lishining yetarli sharti nimadan iborat?
5. Variasion hisobning asosiy lemmalar; Lagranj lemmasini yekltiring.
6. Variasion hisobning asosiy Lemmalari Dyuba-Riman lemmasini yekltiring.
7. Variasion hisobning asosiy masalasida ekstremumning zaruriy sharti – Eyler tenglamasi qanday ko'rinishga ega?
8. Eyler tenglamasining xususiy hollarini keltiring.
9. Gilbert teoremasini keltiring.
10. Eylerning integro-differensial tenglamasini yozing.

4-ma'ruza. Variatsion hisobning asosiy masalasida ikkinchi tartibli zaruriy shartlar va yetarli shartlar. Ikkinchi variatsiyani tekshirish. Lejandr-Klebsh, Yakobi shartlari. Ekstremumning yetarli sharti

Reja.

1. Ikkinchi variatsiyani hisoblash. Lejandr sharti.
2. Yakobi tenglamasi. Yakobi sharti.
3. Kuchsiz ekstremumning yetarli shartlari.
4. Kuchli ekstremumning zaruriy va yetarli shartlari.
5. Kvadratik funksional uchun natija.
6. Misollar.

Tayanch iboralar: *ikkinchi variatsiya, Lejandr sharti, qo'shib olingan variatsion masala, Yakobi sharti, kuchaytirilgan Yakobi va Lejandr shartlari, Veyershtrass funksiyasi, Veyershtrass sharti, kuchli ekstremumning yetarli sharti, Veyershtrass-Erdman shartlari, kvadratik funksionalning ekstremumi.*

Avvalgi ma'ruzamizda variatsion hisob asosiy masalasi uchun ekstremumning birinchi tartibli zaruriy shartlari bayon qilingan edi.

Bu yerda esa, shu masala uchun ekstremumning ikkinchi tartibli zaruriy shartlari va yetarli shartlari qaraladi.

$$J[y]=\int_{x_0}^{x_1}F(x,y,y')dx \qquad (1)$$

funksionalning

$$V=\{y=y(x)\in C^{(1)}[x_0,x_1]: y(x_0)=y_0, y(x_1)=y_1\} \qquad (2)$$

to'plamdagi ekstremumini topish masalasi, ya'ni variatsion hisob asosiy masalasi, berilgan bo'lsin. Bu yerda $F(x,y,y')$ funksiyani R^3 ning biror ochiq Q to'plamida aniqlangan, $P_0(x_0,y_0)$ va $P_1(x_1,y_1)$ nuqtalarni esa, $S=\{(x,y):(x,y,z)\in Q\}$ to'plamga tegishli, deb hisoblaymiz.

1. Lejandr sharti. $y^0=y^0(x)$ joyiz funksiya bo'lsin $(y^0\in V)$. Shu nuqtada (1) funksionalning ikkinchi variatsiyasini hisoblaymiz. Ta'rifga ko'ra, bu variatsiya,

$$\delta^2 J[y^2,h]=\frac{d^2J[y^0+\alpha h]}{d\alpha^2}\ /_{\alpha=0}\text{—}$$

formula bo'yicha hisoblanadi, bu yerda

$$h=h(x)\in C^{(1)}[x_0,x_1], h(x_0)=h(x_1)=0\ .$$

Agar $F(x,y,y')\in C^{(2)}(Q)$ deb faraz qilsak, $\varphi(\alpha)=J[y^0+\alpha h]$ funksiya α=0 nuqta atrofida uzluksiz ikkinchi tartibli hosilaga ega. Demak,

$$\delta^2 J[y^0,h]=\frac{d^2}{d\alpha^2}\int_{x_0}^{x_1}F(x,y^0(x)+\alpha h(x),y^{0'}(x)+\alpha h'(x))dx\,|_{\alpha=0}=$$

$$=\int_{x_0}^{x_1}\frac{\partial^2}{\partial\alpha^2}F(x,y^0(x)+\alpha h(x),y^{0'}(x)+\alpha h'(x))dx\,|_{\alpha=0}=$$

$=$

$$=\int_{x_0}^{x_1}[F_{yy}(x,y^0(x),y^{0'}(x))h^2(x)+2F_{yy'}(x,y^0(x),y^{0'}(x))h(x)h'(x)+ \qquad (3)$$

$$+F_{y'y'}(x,y^0(x),y^{0'}(x))h'^2(x)]dx, \quad h=h(x)\in C^{(1)}[x_0,x_1], \quad h(x_0)=h(x_1)=0$$

1-t e o r e m a (Lejandr). $F(x,y,y')\in C^{(2)}(Q)$ bo'lsin. Agar $y^0(x)\in C^{(1)}[x_0,x_1]-(1)$ funksionalning (2) to'plamdagi kuchsiz minimali (maksimali) bo'lsa,

$$F_{y'y'}(x,y^0(x),y^{0'}(x))\ge 0 \quad (\le 0), \quad \forall x\in[x_0,x_1] \qquad (4)$$

tengsizlik bajariladi. (4) munosabatga Lejandr sharti deyiladi.

I s b o t i. Faraz qilaylik, (4) bajarilmasin, ya'ni biror $\xi\in[x_0,x_1]$ uchun,

$$F_{y'y'}(\xi,y^0(\xi),y^{0'}(\xi))<0 \qquad (5)$$

bo'lsin. $F_{y'y'}(x,y^0(x),y^{0'}(x))$ funksiyaning uzluksizligiga asosan, $\xi\in(x_0,x_1)$ deb olish mumkin. Bundan tashqari, bu funksiyaning uzluksizligidan va (5) tengsizlikdan, shunday ε>0 sonning mavjudligi kelib chiqadiki,

$$\sup \quad F_{y'y'}(x, y^0(x), y^{0'}(x)) = \beta < 0 \qquad x \in (\xi - \varepsilon, \xi + \varepsilon)$$
(6)
bo'ladi. Quyidagi,

$$h_\varepsilon(x) = \begin{cases} \sin^2 \dfrac{\pi(x - \xi + \varepsilon)}{2\varepsilon}, x \in (\xi - \varepsilon, \xi + \varepsilon) \\ 0, \qquad x \notin (\xi - \varepsilon, \xi + \varepsilon) \end{cases}$$

funksiyani qaraymiz, bu funksiya $C^{(1)}[x_0, x_1]$ ga tegishli va

$$h_\varepsilon'(x) = \begin{cases} \dfrac{\pi}{2\varepsilon} \sin \dfrac{\pi(x - \xi + \varepsilon)}{\varepsilon}, x \in (\xi - \varepsilon, \xi + \varepsilon) \\ 0, \qquad x \notin (\xi - \varepsilon, \xi + \varepsilon) \end{cases}$$

Endi (3) formulada $h = h_\varepsilon(x), h' = h_\varepsilon'(x)$ deb olib, quyidagiga ega bo'lamiz:

$$\delta^2 J[y^0, h_\varepsilon] = \int_{\xi-\varepsilon}^{\xi+\varepsilon} [F_{yy}(x, y^0(x), y^{0'}(x)) \sin^4 \frac{\pi(x-\xi+\varepsilon)}{2\varepsilon} +$$
$$+ F_{yy'}(x, y^0(x), y^{0'}(x)) \frac{\pi}{\varepsilon} \sin^2 \frac{\pi(x-\xi+\varepsilon)}{2\varepsilon} \sin \frac{(x-\xi+\varepsilon)}{\varepsilon} +$$
$$+ F_{y'y'}(x, y^0(x), y^{0'}(x)) \frac{\pi^2}{4\varepsilon^2} \sin^2 \frac{\pi(x-\xi+\varepsilon)}{2\varepsilon}] dx \le$$
$$\le \frac{1}{\varepsilon^2} \int_{\xi-\varepsilon}^{\xi+\varepsilon} \left[\beta \frac{\pi^2}{4} \sin^2 \frac{\pi(x-\xi+\varepsilon)}{\varepsilon} + \varepsilon\pi F_{yy'}(x, y^0(x), y^{0'}(x)) \sin^2 \frac{\pi(x-\xi+\varepsilon)}{2\varepsilon} \sin \frac{\pi(x-\xi+\varepsilon)}{\varepsilon} + \right]$$
$$+ \varepsilon^2 F_{yy}(x, y^0(x), y^{0'}(x)) \sin^4 \frac{\pi(x-\xi+\varepsilon)}{2\varepsilon}] dx$$

Bu yerdan (6) ni e'tiborga olib, yetarli kichik $\varepsilon>0$ uchun $\delta^2 J[y^0, h_\varepsilon]<0$ tengsizlikni olamiz. Ammo, lokal minimumning zaruriy shartiga ko'ra, $\delta^2 J[y^0, h_\varepsilon]\ge 0$ munosabat bajarilishi kerak. Bu qarama-qarshilik, teoremani minimum uchun isbotlaydi. U maksimum uchun ham, shunga o'xshash isbotlanadi.

2.Yakobi sharti. Yuqorida isbotlangan Lejandr sharti, lokal minimum (maksimum)ning, funksional ikkinchi variatsiyasi yordamida ifodalanadigan,i $\delta^2 J[y^0, h] \ge 0 \quad (\le 0)$ shartidan foydalanib keltirib chiqariladi. Funksional ikkinchi variatsiyasining ekstremum nuqtasida ishorasini saqlashini ifodalovchi bu shartdan yana bitta ikkinchi tartibli zaruriy shart-Yakobi shartini keltirib chiqarish mumkin.

$F(x, y, y') \in C^{(2)}(Q)$ deb hisoblab, $y^0(x)$ joyiz funksiya uchun,

$$\omega(x, h, h') = F_{y'y'}(x, y^0(x), y^{0'}(x)) h'^2 +$$
$$+ 2F_{yy'}(x, y^0(x), y^{0'}(x)) hh' + F_{yy}(x, y^0(x), y^{0'}(x)) h^2$$

funksiyani qaraymiz. U vaqtda, (3) formulaga ko'ra,

$$\delta^2 J[y^0, h\ ] = \int_{x_2}^{x_1} \omega(x, h, h')dx \tag{7}$$

bo'ladi. Agar $y^0(x)$ kuchsiz minimal (maksimal) bo'lsa, $\delta^2 J[y^0, h\ ] \geq 0 \quad (\leq 0)$ shart barcha $h(x) \in C^{(1)}[x_0, x_1]$, $h(x_0) = h(x_1) = 0$ funksiyalar uchun bajariladi. $h^0(x) = 0$ uchun esa $\delta^2 J[y^0, h^0] = 0$ bo'lishi ravshan. Demak, qaralayotgan variatsion hisob masalasiga, *qo'shib olingan ekstremal masala* deb ataluvchi,

$$\left.\begin{aligned} &\delta^2 J[y^0, h] = \int_{x_0}^{x_1} \omega(x, h, h')dx \to \min(\max) \\ &\mathrm{h(x_0) = h(x_1) = 0, \quad h(x) \in C^{(1)}[x_0, x_1]} \end{aligned}\right\} \tag{8}$$

Masala, $h^0(x) = 0$ yechimga ega.

Faraz qilaylik, $F(x, y, y') \in C^{(3)}(Q), y^0(x) \in C^{(2)}[x_0 x_1]$ - joyiz stasionar funksiya $F_{y'y'}(x, y^0(x), y^{0'}(x) \neq 0\ \forall x \in [x_0 x_1]$ bo'lsin. U vaqtda, (8) masala uchun tuzilgan,

$$\omega_h(x, h, h') - \frac{d}{dx}\omega_{h'}(x, h, h') = 0$$

Eyler tenglamasiga, variason hisob asosiy masalasi uchun *Yakobi tenglamasi* deyiladi. $\omega(x, h, h')$ funksiyaning ko'rinishini hisobga olib, Yakobi tenglamasini

$$A(x)h'' + B(x)h' + C(x)h = 0 \tag{9}$$

ikkinchi tartibli bir jinsli chiziqli differensial tenglama ko'rinishida yozish mumkin, bu yerda

$$A(x) = F_{y'y'}(x, y^0(x), y^{0'}(x)), \quad B(x) = \frac{d}{dx}F_{y'y'}(x, y^0(x), y^{0'}(x)),$$

$$C(x) = \frac{d}{dx}F_{yy'}(x, y^0(x), y^{0'}(x)) - F_{yy}(x, y^0(x), y^{0'}(x))$$

Differensial tenglamalar kursidan ma'lumki, (9) tenglama $h(x_0) = 0, h'(x_0) = 1$ chegaraviy shartlarni qanoatlantiruvchi (aynan noldan farqli) yagona yechimga ega. Shu yechimning x_0 dan farqli nollariga, *x_0 nuqtaga qo'shma nuqta* deyiladi. Qo'shma nuqtaga yana quyidagi ekvivalent ta'rifni ham berish mumkin.

T a' r i f. Agar (9) Yakobi tenglamasi $h(x_0) = 0, h(x^*) = 0$ $x^* \neq x_0$ shartlarni qanoatlantiruvchi trivial (aynan nol) bo'lmagan $h(x)$, $x \in [x_0, x_1]$ yechimga ega bo'lsa, x^* nuqtaga $y^0(x)$ joyiz chiziq bo'ylab x_0 nuqtaga qo'shma nuqta deyiladi.

2-t e o r e m a (Yakobi). Faraz qilaylik:

a) $F(x, y, y') \in C^{(3)}(Q)$, б) $y^0(x) \in C^{(2)}[x_0 x_1]$ - kuchsiz minimal (maksimal) $F_{y'y'}(x, y^0(x), y^{0'}(x)) > 0 \quad (< 0) \quad \forall x \in [x_0 x_1]$ bo'lsin. U holda, $y^0(x)$ funksiya Yakobi shartini qanoatlantiradi: (x_0, x_1) intervalda $y^0(x)$ chiziq bo'ylab x_0 nuqtaga qo'shma bo'lgan nuqta mavjud emas.

I s b o t i. Teskarisini faraz qilamiz. $y^0(x)$, $x\in[x_0,x_1]$ joyiz chiziq bo’ylab x_0 ga qo’shma bo’lgan $x^*\to x\in[x_0,x_1]$ nuqta mavjud bo’lsin. $h^*(x)\neq 0$, $x\in[x_0,x_1]$ esa, Yakobi tenglamasining unga mos yechimi bo’lsin, ya’ni

$$\begin{aligned}&\omega_h(x,h^*(x),h^{*'}(x))-\frac{d}{dx}\omega_{h'}(x,h^*(x),h^{*'}(x))\equiv 0, \quad \forall x\in[x_0,x_1]\\ &h^*(x_0)=h^*(x^*)=0\end{aligned} \qquad (10)$$

shartlar bajarilsin. Qo’shma nuqta ta’rifiga ko’ra, $h^{*'}(x^*-0)\neq 0$ shart bajariladi.Quyidagi

$$h(x)=\begin{cases}h^*(x), x\in[x_0,x^*],\\ 0, x\in(x^*,x_1].\end{cases} \qquad (11)$$

funksiyani tuzamiz.

Endi $2\omega(x,h,h')=h\omega_h+h'\omega_{h'}$ formulani, hamda (10), (11) larni hisobga olib, (8) dan quyidagini olamiz:

$$\delta^2 J[y^0,h]=\int_{x_0}^{x_1}\omega(x,h,h')dx=\int_{x_0}^{x^*}\omega(x,h,h')dx=\frac{1}{2}\int_{x_0}^{x^*}(h^*\omega_h+h^{*'}\omega_{h'})dx=$$

$$=\frac{1}{2}\int_{x_0}^{x^*}(h^*\frac{d}{dx}\omega_h+h^{*'}\omega_{h'})dx=\frac{1}{2}\int_{x_0}^{x^*}\frac{d}{dx}(h^*\omega_{h'})dx=$$

$$=\frac{1}{2}h^*(x)\omega_h(x,h^*(x),h^{*'}(x))|_{x_0}^{x^*}=0.$$

Shunday qilib, (11) funksiya - (8) masalaning yechimidir. teng munosabatlar ko’rsatadiki, h(x) funksiya uchun x^* - bukilish nuqtasidir. Natijida, $x=x^*$ nuqtada

$$\omega_h(x,h,h')|_{x=x^*-0}=\omega_{h'}(x,h,h')|_{x=x^*+0} \qquad (12)$$

Veyershtrass - Erdman sharti bajariladi. $\omega(x,h,h')$ funksiyaning ko’rinishini hisobga olib ,(12) tenglikni quyidagicha yozamiz:

$$\begin{aligned}&[h(x)F_{yy}(x,y^0(x),y^{0'}(x))+h'(x)F_{y'y'}(x,y^0(x),y^{0'}(x))]|_{x=x^*-0}=\\ &=[h(x)F_{yy}(x,y^0(x),y^{0'}(x))+h'(x)F_{y'y'}(x,y^0(x),y^{0'}(x))]|_{x=x^*+0}\end{aligned} \qquad (13)$$

$h(x^*-0)=h(x^*+0)=0$ va $F_{y'y'}(x,y^0(x),y^{0'}(x))$ uzluksiz bo’lgani uchun, (13) tenglik

$$[h'(x^*-0)-h'(x^*+0)]F_{y'y'}(x^*,y^0(x^*),y^{0'}(x^*))=0 \qquad (14)$$

ko’rinishga keladi. Teoremaning shartiga ko’ra $F_{y'y'}(x^*,y^0(x^*),y^{0'}(x^*))\neq 0$. U vaqtda (14) dan $h'(x^*-0)=h'(x^*+0)=0$ bo’lishi kelib chiqadi. Bu esa x^* ning bukilish nuqtasi ekanligiga ziddir. Olingan qarama-qarshilik teoremani isbotlaydi.

3.Kuchsiz ekstremumning yetarli shartlari. Shunday qilib, variatsion hisob asosiy masalasida ekstremumning birnchi tartibli zaruriy sharti - joyiz funksiyaning stasionarligi (Eyler tenglamasining bajarilishi) bo’lsa, Lejandr va Yakobi shartlari –ikkinchi tartibli zaruriy shartlardir.

Sodda misollar ko'rsatadiki, bu uchala shartning birortasi ham alohida olganda ekstremumning yetarli sharti bo'la olmaydi. Ammo ular birgalikda kuchsiz ekstremumning yetarli shartiga yaqinroqdir.

Quyida Lejandr va Yakobi shartlarini kuchaytirish natijasida kelib chiqadigan yetarli shartlarni keltiramiz.

3-t e o r e m a. Faraz qilaylik:

a) $F(x,y,y')\in C^{(3)}(Q), y^0(x)\in C^{(2)}[x_0x_1]$ -joyiz stasionar funksiya bo'lsin;

b) kuchaytirilgan Lejandr sharti bajarilsin:

$$F_{y'y'}(x,y^0(x),y^{0'}(x))>0 \quad (<0) \quad \forall x\in[x_0x_1];$$

c) kuchaytirilgan Yakobi sharti o'rinli bo'lsin: $y^0(x)$ joyiz chiziq bo'ylab $(x_0,x_1]$ da x_0 nuqtaga qo'shma bo'lgan x_1 nuqta mavjud emas.

U holda, $y^0(x)$ - variatsion hisobning asosiy masalasida kuchsiz lokal minimal (maksimal) bo'ladi.

I s b o t i. Funksionalning ikkinchi variatsiyasi uchun hosil qilingan (3) formuladagi ikkinchi qo'shiluvchini bo'laklab integrallaymiz:

$$\int_{x_0}^{x_1}2hh'F_{yy}dx=\int_{x_0}^{x_1}F_{yy'}\frac{d}{dx}h^2dx=F_{yy}h^2(x)\,|_{x_0}^{x_1}-\int_{x_0}^{x_1}h^2\frac{d}{dx}F_{yy'}dx=$$

$$=-\int_{x_0}^{x_1}h^2\frac{d}{dx}F_{yy'}dx \qquad (15)$$

$$u(x)\in C^{(2)},u(x)>0,x\in[x_0,x_1] \qquad (16)$$

shartlarni qanoatlantiruvchi ixtiyoriy $u(x)$ funksiyani qaraymiz. Bu funksiya uchun,

$$\int_{x_0}^{x_1}\frac{d}{dx}(\frac{u'}{u}F_{y'y}h^2)dx=\frac{u'}{u}F_{y'y}h^2\,|_{x_0}^{x_1}=0 \qquad (17)$$

munosabat bajariladi. Endi (15) ni hisobga olib, (3) va (17) dan,

$$\delta^0J[y^0,h]=\int_{x_0}^{x_1}\{[F_{yy}-\frac{d}{dx}F_{y'y'}-\frac{d}{dx}(\frac{u'}{u}F_{y'y'})]h^2dx-$$
$$-2[\frac{u'}{u}F_{y'y'}]hh'+[F_{y'y'}]h'^2\}dx \qquad (18)$$

tenglikni olamiz. $u(x)$, $x\in[x_0,x_1]$ funksiyani shunday tanlaymizki, (18) dagi integral ostidagi ifoda h va h' ga nisbatan to'la kvadrat bo'lsin. Buning uchun, quyidagi:

$$[\frac{u'}{u}F_{y'y'}]^2=[F_{yy}-\frac{d}{dx}F_{y'y'}-\frac{d}{dx}(\frac{u'}{u}F_{y'y'})]F_{y'y'} \qquad (19)$$

ayniyatning bajarilishi zarur va yetarli.

$$\frac{d}{dx}(\frac{u'}{u}F_{y'y'})=(\frac{u''u-u'^2}{u^2})F_{y'y}+\frac{u'}{u}\frac{d}{dx}F_{y'y'}$$

bo'lgani uchun, (19) ayniyatni,

$$F_{y'y'}\{-u''F_{y'y'}-u'\frac{d}{dx}F_{y'y'}+(F_{yy}-\frac{d}{dx}F_{y'y'})u\}/u=0 \qquad (20)$$

ko'rinishda yozish mumkin. Teoremaning shartiga ko'ra,

$F_{y'y'}=F_{y'y'}(x,y^0(x),y^{0'}(x))\neq 0,\ x\in[x_0,x_1]$. Demak, (20) dan ko'rinadiki, $u(x)$ funksiya,

$$(A(x)u''+B(x)u'+C(x)u)/u=0$$

tenglamani qanoatlantirishi kerak, bu yerda *A(x),B(x),C(x)*- (9) Yakobi tenglamasidagi kabi aniqlanadi.Teoremaning c) shartiga ko'ra,

$$A(x)u''+B(x)u'+C(x)u=0 \qquad (21)$$

Yakobi tenglamasi, $u(x_0)=0,\ u(x^*)=0,\ x\in[x_0,x_1]$ shartlarni qanoatlantiruvchi trivial bo'lmagan yechimga ega emas. $[x_0,x_1]$ ning kompaktligi va differensial tenglama yechimining boshlang'ich shartlaridan uzluksiz bog'liqligi haqidagi teoremadan, shunday yetarli kichik $\varepsilon>0$ sonning mavjudligi kelib chiqadiki, (21) tenglamaning $u(x_0-\varepsilon)=0,\ u'(x_0-\varepsilon)=1$ boshlang'ich shartlardagi yechimi $[x_0,x_1]$ da musbat bo'ladi. Shunday qilib (16), (19) shartlarni qanoatlantiruvchi $u(x)$ funksiya mavjud. Shu funksiya yordamida, (18) ifodani,

$$\delta^0 J[y^0,h]=\int_{x_0}^{x_1}\{[F_{y'y'}]^{\frac{1}{2}}h'-[F_{yy}-\frac{d}{dx}F_{yy'}-\frac{d}{dx}(\frac{u'}{u}F_{y'y'})]^{\frac{1}{2}}h^2dx \qquad (22)$$

ko'rinishda yozamiz:

(22) dan ko'rinadiki, barcha $h(x)\in C^{(1)}x\in[x_0,x_1],h(x_0)=h(x_1)=0,$ funksiyalar uchun $\delta^2 J[y^0,h^*]\geq 0$ bajariladi. Agar biror $h^*(x)\neq 0,x\in[x_0,x_1]$ funksiya uchun $\delta^2 J[y^0,h^*]=0$ deb faraz qilsak, (22) dan

$$[F_{y'y'}]^{\frac{1}{2}}h^{*}{}'(x)=[F_{yy}-\frac{d}{dx}F_{yy'}-\frac{d}{dx}(\frac{u'}{u}F_{y'y'})]^{\frac{1}{2}}h^*(x) \qquad \forall x\in[x_0,x_1]$$

bo'lishi kelib chiqadi. Bu yerdan $h^*(x_0)=0,\ F_{y'y'}|_{x=x_0}>0$ shartlarni hisobga olib, $h^*(x_0)=0$ ga ega bo'lamiz. Ammo $h^*(x)$ funksiya (8) minimallashtirish masalaning yechimi bo'lgani uchun, $\delta^2 J[y^0,h^*]=0$, (9) Yakobi tenglamasini qanoatlantiradi va $h^*(x_0)=h^{*\prime}(x_0)=0$ boshlang'ich shartlardan, $h^*(x)\equiv 0,\ x\in[x_0,x_1]$ bo'lishi kelib chiqadi. Bu olingan qarama-qarshilik ko'rsatadiki, barcha $h^*(x)\neq 0,\ x\in[x_0,x_1],\ h(x_0)=h(x_1)=0$ funksiyalar uchun, $\delta^2 J[y,h]>0$ bajariladi.

Endi quyidagi funksionalni qaraymiz:

$$\Phi(y^0,h)=\delta^2 J[y^0,h]-\frac{q}{2}\int_{x_0}^{x_1}h^2(x)dx, \qquad q>0\ . \qquad (23)$$

Bu funksional uchun Eyler tenglamasi,

$$(A(x)-q)h''+B(x)h'+C(x)h=0 \qquad (24)$$

ko'rinishda bo'ladi. $A(x) = F_{y'y'}(x, y^0(x), y^{0'}(x)) > 0,\ x \in [x_0, x_1]$ .bo'lgani uchun, shunday $q>0$ topiladiki, $\cdot A(x)q > 0,\ x \in [x_0, x_1]$ bajariladi. Farazga ko'ra, (9) Yakobi tenglamasining ,

$$h(x_0)=0,\ h'(x_0)=1 \qquad (25)$$

boshlang'ich shartlardagi yechimi (x_0, x_1) intervalda nolga aylanmaydi.

Differensial tenglamalar yechimining parametrdan uzluksiz bog'liqligi haqidagi teoremaga ko'ra, (24) tenglamaning (25) shartlardagi yechimi ham, shunday xossaga ega bo'ladi. (23) funksional uchun yuqoridagi kabi mulohaza yuritib,

$$\Phi(y^0, h) \ge 0, \forall h(x) \in C^{(1)}[x_0, x_1],\ h(x_0) = h(x_1) = 0$$

munosabatni olamiz. Bu esa, (23) ga ko'ra,

$$\delta\ J[y^0, h] \ge \frac{q}{2}\int_{x_0}^{x_1} h^2(x)dx \qquad (26)$$

tengsizlik bajarilishini bildiradi.

$h(x) = \int_{x_0}^{x} h'(t)dt$ bo'lgani uchun, Koshi-Bunyakovskiy tengsizligini qo'llab, quyidagini olamiz:

$$h^2(x) = (\int_{x_0}^{x} h'(t)dt)^2 \le (x - x_0)\int_{x_0}^{x_1} h'^2(t)dt \le (x_1 - x_0)\int_{x_0}^{x_1} h'^2(t)dt. \qquad x \in [\lambda_0, \lambda_1];$$

$$\int_{x_0}^{x_1} h^2(t)dt \le (x_1 - x_0)^2 \int_{x_0}^{x_1} h'^2(t)dt.$$

Natijada, (26) tengsizlikdan,

$$\delta^2 J[y^0, h] \ge \frac{q}{2(x_1 - x_0)^2}\int_{x_0}^{x_1} h^2(x)dx \qquad (27)$$

tengsizlik kelib chiqadi.

Ikkinchi variatsiya ta'rifiga ko'ra,

$$\Delta J[y^0] = J[y^0 + \varepsilon h] - J[y^0] = \varepsilon\delta J[y^0, h] + \frac{\varepsilon^2}{2}\delta^2 J[y^0, h] + O(y^0, \varepsilon^2 // h //^2) \qquad (28)$$

tenglik bajariladi,bu yerda

$$\frac{O(y_0, \varepsilon^2 \| h \|^2)}{\varepsilon^2} \to 0,\ \varepsilon \to 0 \qquad (29)$$

$y^0(x)$- E yler tenglamasini qanoatlantirgani uchun, $\delta J[y^0, h] = 0$ bo'ladi. U vaqtda, (28) dan,

$$\Delta J[y^0] = \frac{\varepsilon^2}{2}[\delta^2 J[y^0, h] + \frac{O(y_0, \varepsilon^2 \| h \|^2)}{\varepsilon^2}] \qquad (30)$$

kelib chiqadi.

Endi (27) va (29) ni hisobga olgan holda, (30) munosabatdan shunday $\varepsilon > 0$ sonning mavjudligi kelib chiqadiki, barcha $0 < \varepsilon < \varepsilon_0$ va $h(x) \in C^{(1)}[x_0, x_1], \int_{x_0}^{x_1} h^2(x)\,dx \le \alpha < +\infty$ uchun, $\Delta J[y^0] \ge 0$ bo'ladi, ya'ni $y^0(x)$ kuchsiz lokal minimaldir. Teorema isbotlandi.

4. **Kuchli ekstremumning zaruriy va yetarli shartlari.** Bu yerda kuchli ekstremumning zaruriy va yetarli shartlarini isbotsiz keltirish bilan cheklanamiz (isbotlarini ,masalan [2] dan qarash mumkin).

$Q = S \times R^2, S \subset R^2$ berilgan ochiq to'plam, $F(x, y, y') \in C'(Q)$ bo'lsin.

Quyidagi

$$E(x, y, y', u,) = F(x, y, u) - F(x, y, y') - (u - y')F_{y'}(x, y, y'), (x, y, y', u) \in Q \times R^1 \qquad (31)$$

funksiyani qaraymiz. $E(x, y, y', u)$ funksiyaga Veyershtrass funksiyasi deyiladi.

4- t e o r e m a.Agar $y^0(x) = D^1[x_0, x_1]$- (1) funksionalning $\tilde{V} = \{y(x) \in D'[x_0, x_1] : y(x_0) = y_0, y(x_1) = y_1\}$

to'plamdagi kuchli lokal minimum (maksimum) nuqtasi bo'lsa, $y^0(x)$ mavjud bo'lgan barcha $x \in [x_0, x_1]$ nuqtalarda

$$E(x, y^0(x), y^0{}'(x), u) \ge 0 \quad (\le 0) \quad \forall u \in R^1 \qquad (32)$$

Veyershtrass sharti bajariladi. $y^0(x)$ ning ξ burchak nuqtalarida esa, (32) shart,

$$E(\xi, y^0(\xi), y^0(\xi \pm 0), u) \ge 0 (\le 0) \forall u \in R \qquad (33)$$

ko'rinishda bo'ladi.

5-t e o r e m a**.** Quyidagi shartlar bajarilsin:

1). $F(x, y, y') \in C^{(4)}(Q), Q = S \times R^1,$

2). $F_{y'y'}(x, y, y') \ge 0 \;\; (\le 0), \;\; \forall (x, y, y') \in Q;$

3). $y^0(x) \in C^{(3)}[x_0, x_1] -$ (1) funksionalning joyiz stasionar funksiyasi;

4). $y^0(x)$ funksiya uchun kuchaytirilgan Lejandr va Yakobi shartlari o'rinli.

U holda, $y^0(x)$- (1) funksionalning (2) to'plamdagi kuchli lokal minimum (maksi-mum) nuqtasi bo'ladi.

Bu yerda shuni ta'kidlash joyizki, yuqorida keltirilgan Veyershtrass shartidan muhim natijalar olish mumkin. Shulardan biri, avvalgi ma'ruzamizda aytib o'tilgan, Veyershtrass-Erdman shartlarining ikkinchisidir. Haqiqatan ham, Veyershtrass-Erdman shartlarining birinchisiga ko'ra, $y^0(x)$ kuchli ekstremalning ξ burchak nuqtasida.

$$F_{y'}(\xi, y^0(\xi), y^0{}'(\xi - 0)) = F_{y'}(\xi, y^0(\xi), y^0{}'(\xi + 0)) \qquad (34)$$

bajariladi. (33) Veyershtrass shartida $u = y^{0\prime}(\xi \mp 0)$ deb olsak,

$$E(\xi, y^0(\xi), y^0{}'(\xi \pm 0), y^0{}'(\xi \mp 0)) \ge 0 \qquad (35)$$

tengsizlik bajariladi. Endi Veyershtrass funksiyasining (31) ko'rinishini hisobga olib, (35) ni qayta yozsak, va bunda (34) dan foydalansak,

$$\begin{aligned} &F(\xi, y^0(\xi), y^0{}'(\xi - 0)) - y^0{}'(\xi - 0)F_{y'}(\xi, y^0(\xi), y^0{}'(\xi - 0) = \\ &= F(\xi, y^0(\xi), y^0{}'(\xi + 0)) - y^0{}'(\xi + 0)F_{y'}(\xi, y^0(\xi), y^0{}'(\xi + 0)) \end{aligned} \qquad (36)$$

munosabatni olamiz. Bu esa, Veyershtrass-Erdman shartlarining ikkinchisidir.

Veyershtrass-Erdman shartlarini qisqacha

$$F_y\Big|_{x=\xi-0} = F_{y'}\Big|_{x=\xi+0}, (F - y'F_{y'})\Big|_{x=\xi-0} = (F - y'F_{y'})\Big|_{x=\xi+0},$$

ko'rinishda yozish mumkin.

5. Kvadratik funksional bo'lgan hol. Faraz qilaylik, (1) funksional quyidagi

$$J[y] = \int_{x_0}^{x_1}\left[p(x)y^2 + 2q(x)yy' + r(x)y'^2\right]dx \tag{37}$$

ko'rinishdagi kvadratik funksionaldan iborat bo'lsin. Bunday funksional ekstremumining zaruriy va yetarli shartlari quyidagi teoremadan aniqlashtiriladi.

6- t e o r e m a.

$$p(x)\in C[x_0,x_1]q(x)\in C^1[x_0,x_1], r(x)\in C^1[x_0,x_1], r(x)>0(<0)\ \forall x\in[x_0,x_1]$$

bo'lsin. Agar Yakobi sharti bajarilmasa, u holda,

$$\inf_{y\in V} J[y] = -\infty (\sup_{u\in V} J[y] = +\infty),$$

ya'ni (37) funksionalning (2) to'plamda global ekstremumi mavjud emas.Agar kuchaytirilgan Yakobi sharti bajarilsa, (37) funksionalning yagona joyiz stasionar funksiyasi mavjud va bu stasionar funksiya funksionalning global minimum (maksimum) nuqtasi bo'ladi.

6. Misollar.

$$1) \int_0^b (y'^2 - y^2 + 2y\sin x)dx \to \min,\ y(0)=0,\ y(b)=1. \tag{38}$$

$$F(x,y,y') = y'^2 - y^2 + 2\sin x,\ F_y = -2y + 2\sin x,\ F_{y'} = 2y'$$

Eyler tenglamasini yozamiz:

$$y'' + y - \sin x = 0$$

Bu tenglamaning umumiy yechimi $y = -\frac{1}{2}x\cos x + c_1\cos x + c_2\sin x$ ko'rinishda bo'ladi.

y(0)=0, y(b)=1

shartlardan foydalanib c_1 va c_2 o'zgarmaslarni topamiz:

$$c_1 = 0,\quad c_2 = \frac{1 + \frac{b}{2}\cos b}{\sin b}, b \neq \pi k, k = 1,2,...$$

Demak, $b\neq\pi k$, $k=1,2,..$ bo'lganda ,

$$y^0(x) = -\frac{1}{2}x\cos x - \frac{2 + b\cos b}{2\sin b}\sin x, x\in[0,b] \tag{39}$$

Funksiya (38) masalada yagona joyiz stasionar funksiyadir. Agar $b=\pi k$, $k=1,2,..$ bo'lsa, masalada stasionar funksiyalar yo'q. Demak, bu holda ekstremum mavjud emas.

$F_{y'y'} = 2 > 0$,demak, kuchaytirilgan Lejandr sharti bajariladi.

Yakobi tenglamasini tuzamiz: $A(x) = F_{y'y'} = 2 > 0$,

$$B(x)\frac{d}{dx}F_{y'y'}=0, C(x)\frac{d}{dx}F_{y'y'}-F_{yy}=2.$$

$h^{*}+h=0$ - Yakobi tenglamasi, $h(x)=\gamma_1\cos x+\gamma_2\sin x-$ uning umumiy yechimi. Demak, $x^{*}=\pi$ nuqta $x_0=0$ nuqtaga qo'shma nuqtadir. $(0,\pi)$ oraliqda $x_0=0$ ga qo'shma nuqta yo'q. Shunday qilib, $b\le\pi$ bo'lganda Yakobi sharti, $b<\pi$ bo'lganda esa kuchaytirilgan Yakobi sharti bajariladi. $b>\pi$ bo'lganda esa Yakobi sharti bajarilmaydi. 5-teoremaga ko'ra, $0<b<\pi$ bo'lganda (39) formula bilan aniqlanuvchi $y^0(x)$ funksiya (38) masalada kuchli minimal bo'ladi.

2) $F=F(y')=y'^3-3y'^2$ (40)

bo'lgani uchun Eyler tenglamasi $(y'-1)y''=0$ ko'rinishda bo'ladi. Bu yerdan $y=x+c$ va $y=c_1x+c_2$ ko'rinishdagi yechimlarga ega bo'lamiz. $y=x+c$ funksiya chegaraviy shartlarni qanoatlantirmaydi. $y=c_1x+c_2$ funksiya uchun $y(0)=y(2)=0$ chegaraviy shartlardan $c_1=c_2=0$ bo'lishi kelib chiqadi. Demak, $y^0(x)=0, x\in[0.2]$ qaralayotgan masalada yagona silliq joyiz stasionar fnksiyadir.

$F_{y'y'}(y^0(x))=6(y^{0\prime}(x)-1)=-6,$ ya'ni kuchaytirilgan Lejandr sharti bajariladi. Endi Yakobi tenglamasini tuzamiz:

$$A(x)=F_{y'y'}|_{y=y^0}=-6,\ B(x)=\frac{d}{dx}F_{y'y'}|_{y=y^0}=0,\ C(x)=\frac{d}{dx}F_{y'y'}-F_{yy}=0.$$

$h''=0$ - Yakobi tenglamasi, $h(x)=\gamma_1x+\gamma_2-$ uning umumiy yechimidir. $h(0)=0, h(x^*)=0, x^*>0$ shartlardan $h(x)\equiv0$ bo'lishi kelib chiqadi. Demak, $[0,2]$ da $x_0=0$ nuqtaga qo'shma nuqta mavjud emas, ya'ni kuchaytirilgan Yakobi sharti bajariladi. Demak, 3-teoremaga ko'ra , $y^0(x)\equiv0$ kuchsiz lokal maksimal bo'ladi.

Ammo $y^0(x)\equiv0$ uchun Veyershtrass sharti bajarilmaydi, ya'ni

$$E(x,y^0,y^{0\prime},u)=F(x,y^0,u)-F(x,y^0,y^{0\prime})-(u-y^0)F_{y'}(x,y^0,y^{0\prime})=u^3-3u^2$$

funksiya ishorasini o'zgartiradi. 4-teoremaga asosan, $y^0(x)\equiv0$- kuchli lokal maksimal bo'la olmaydi.

Mustaqil ishlash uchun savollar.

1. Variatsion hisobning asosiy masalasida ikkinchi variatsiyani hisoblash. Lejandr sharti.
2. Yakobi tenglamasini keltirib chiqarish. Yakobi sharti.
3. Kuchsiz ekstremumning yetarli shartlari haqidagi teorema.
4. Kuchli ekstremumning zaruriy sharti (Veyershtrass sharti) va yetarli shartlari.
5. Kvadratik funksionalli variatsion hisob masalasi. Ekstremum shartlari.

5-ma'ruza. Variatsion hisob asosiy masalasining umumlashmalari. Eyler tenglamalari sistemasi, Eyler-Puasson tenglamasi

REJA:

1. Bir necha funksiyalarga bog'liq bo'lgan funksionalning ekstremumi.
   a)Masalaning qo'yilishi. Kuchli va kuchsiz ekstremallar.
   b)Eyler tenglamalari sistemasi. Stasionar funksiyalar.
2. Yuqori tartibli hosilalarga bog'liq funksionalning ekstremumi.
   a)Kuchli va kuchsiz ekstremumlar. Eyler-Puasson tenglamasi.
   b)Ekstremumning yetarli shartlari. Misol.

3. Bir necha o`zgaruvchili funksiyalarga bog`liq funksionallarning ekstremumi.

<u>Tayanch iboralar.</u> *Vector funksiya,* $C_n^{\,1}[x_0, x_1]$ *fazo, Eyler tenglamalari sistemasi, kvadratik forma, Eyler –Puasson tenglamasi.*

1. **<u>Bir necha funksiyalarga bog'liq bo'lgan funksionalning ekstremumi.</u>**

Variatsion hisob asosiy masalasining umumlashmasi sifatida, dastlab, bir necha, $y_1 = y_1(x),..., y_n = y_n(x)$ funksiyalarga bog'liq funksionalning ekstremumi haqidagi masalani qaraymiz.

Faraz qilaylik, $Q \in R^{2n+1}$ – biror ochiq to'plam (soha), $F(x, y_1,..., y_n, z_1,..., z_n) - Q$ da aniqlangan uzluksiz funksiya, $P(x_0, y_{01},..., y_{0n})$ va $P(x_1, y_{11},..., y_{1n})$ lar, $S = \{(x, y_1,..., y_n) : (x, y_1,..., y_n, z_1,..., z_n) \in Q\}$ to'plamning belgilangan nuqtalari, $x_0 < x_1$ bo'lsin.

Qabul qilingan belgilashlar asosida, quyidagi

$$J[y_1,..., y_n] = \int_{x_0}^{x_1} F(x, y_1..., y_n, y_1'..., y_n')dx \to \min(\max) \qquad (1)$$

$$\left.\begin{array}{c} y_1(x_0) = y_{01}, y_2(x_0) = y_{02},..., y_n(x_0) = y_{0n}, \\ y_1(x_1) = y_{11}, y_2(x_1) = y_{12},..., y_n(x_1) = y_{2n}, \\ (x, y_1(x),..., y_n(x), y_1'(x),..., y_n'(x)) \in Q, x \in [x_0, x_1] \\ y_i(x) \in C^1[x_0, x_1], i = 1,2,..., n. \end{array}\right\} \qquad (2)$$

ekstremal masalani qaraymiz.

Qaralayotgan masalani ixchamroq shaklda yozish uchun quyidagi belgilashlarni kiritamiz: $y = (y_1,..., y_n)$, $y' = (y_1',..., y_n')$, $y_0 = (y_{01},..., y_{0n})$, $C_n^{(1)}[x_0, x_1] - [x_0, x_1]$ kesmada uzluksiz differensiallanuvchi $y(x) = (y_1(x),..., y_n(x)) - n -$ vektor funksiyalar fazosi. U holda (1), (2) masalani

$$J[y] = \int_{x_0}^{x_1} F(x, y, y')dx \to \min(\max), \qquad (1^/)$$

$$y(x_0) = x_0, y(x_1) = x_1, \; y(x) \in C_n^{(1)}[x_0, x_1] \qquad (2^/)$$

ko'rinishda yozish mumkin. ($2^/$) munosabatlarni qanoatlantiruvchi $y(x) = (y_1(x),..., y_n(x))$ funksiyalarga (1)-(2) masalaning joyiz funksiyalari (chiziqlari) deyiladi. Qaralayotgan masalada joyiz chiziqlarning uchlari R^{n+1} fazoning $P_0(x_0,y_0)$ va $P_1(x_1,y_1)$ nuqtalarida mahkamlangan.

Biz $C_n^{(1)}[x_0,x_1]$ bilan bir qatorda, $[x_0,x_1]$ kesmada uzluksiz $y(x)-n-$vektor funksiyalar fazosi $C_n[x_0,x_1]$ dan ham foydalanamiz. Ma'lumki, $C_n[x_0,x_1]$ va $C_n^1[x_0,x_1]$ fazolar chiziqli normalangan fazolar bo'lib, ularda normalar, mos ravishda,

$$\|y\|_{C_n[x_0,x_1]} = \max_{1\le i\le n}' \max_{x\in[x_0,x_1]} |y_i(x)|,$$

$$\|y\|_{C_n[x_0,x_1]} = \max_{1\le i\le n}\left[\max_{x\in[x_0,x_1]} |y_i(x)| + \max_{x\in[x_0,x_1]} |y_i'(x)|\right]$$

kabi aniqlanadi. Shuning uchun, $y^0(x)=(y_1^0(x),...,y_n^0(x))$ joiz chiziqning nolinchi va birinchi tartibli $\varepsilon-$ atroflarini, mos ravishda, quyidagicha aniqlaymiz:

$$V_0(y^0,\varepsilon)=\left\{y(x)\in C_n[x_0,x_1]: \|y-y^0\|_{C_n[x_0,x_1]}<\varepsilon\right\}=\left\{(y_1(x),...,y_n(x)): \|y_i-y_i^0\|_{C_n[x_0,x_1]}<\varepsilon,\ i=1,2,...,n\right\};$$

$$V_1(y^0,\varepsilon)=\left\{y(x)\in C_n^1[x_0,x_1]: \|y-y^0\|_{C_n^1[x_0,x_1]}<\varepsilon\right\}=\left\{(y_1(x),...,y_n(x)): \|y_i-y_i^0\|_{C_n^1[x_0,x_1]}<\varepsilon,\ i=1,2,...,n\right\};$$

1-t a' r i f. Agar $y^0(x)$ joiz funksiyaning shunday $V_0(y^0,\varepsilon)$ nolinchi tartibli $\varepsilon-$ atrofiga tegishli barcha $y=y(x)$ joiz funksiyalar uchun

$$J[y^0]\le J[y] \quad (J[y^0]\ge J[y]) \qquad (3)$$

munosabat bajarilsa, $y^0(x)$ funksiya (1) funksionalning kuchli lokal minimum (maksimum) nuqtasi deyiladi.

2-t a' r i f. Agar (3) munosabat $y^0(x)$ joiz funksiyaning biror $V_1(y^0,\varepsilon)$ birinchi tartibli ε-atrofiga tegishli $y=y(x)$ joiz funksiyalar uchun bajarilsa, $y^0(x)-$ (1) funksionalning *kuchsiz lokal minimum (maksimum)* nuqtasi deyiladi.

Demak, (1),(2) masala uchun kuchli va kuchsiz ekstremumlar variatsion hisobning asosiy masalasidagiga o'xshash aniqlanadi.

Keltirilgan ta'riflardan ravshanki , kuchli ekstremum nuqtasi kuchsiz ekstremum nuqtasi ham bo'ladi. Buning teskarisi esa, to'g'ri emas. Shuning uchun, avvalo, kuchsiz ekstremumning zaruriy shartlarini keltiramiz.

1-t e o r e m a. $F(x,y,y')\in C^1(Q)$ bo'lsin. Agar (1) funksional $y^0(x)=(y_1^0(x),...,y_n^0(x))\in C_n^{(1)}[x_0,x_1]$ joiz funksiyada kuchsiz lokal ekstremumga erishsa, $[x_0,x_1]$ kesmada

$$F_{y_i}\left(x,y^0(x),y^{0'}(x)\right)-\frac{d}{dx}F_{y_i'}\left(x,y^0(x),y^{0'}(x)\right)=,\ i=\overline{1,n} \qquad (4)$$

tengliklar bajariladi.

I s b o t i. ($1^{/}$) funksionalning $y^0=y^0(x)$ nuqtadagi birinchi variatsiyasini hisoblaymiz:

$$\delta J\left[y^0,h\right]=\frac{\partial}{\partial\alpha}J\left[y^0+\alpha h\right]_{\alpha=0}=\frac{\partial}{\partial\alpha}\int_{x_0}^{x_1}F\left(x,y^{0/}+\alpha h,y^{0/}+\alpha h'\right)dx\Bigg|_{\alpha=0}=$$

$$=\int_{x_0}^{x_1}\frac{\partial}{\partial\alpha}F\left(x,y_1{}^0(x)+\alpha h_1(x),...,y_n{}^0(x)+\alpha h_n(x),y_1{}^{0'}(x)+\alpha h_1{}'(x),...,y_n{}^{0'}(x)+\alpha h_n{}'(x)\right)dx\Bigg|_{\alpha=0}=$$

$$=\int_{x_0}^{x_1}\sum_{i=1}^{n}\left[F_{y_i}\left(x,y^0(x),y^{0'}(x)\right)h(x)+F_{y_i{}'}(x,y^0(x),y^{0'}(x))h_i{}'(x)\right]dx,$$

bu yerda $h_i(x)\in C^1\left[x_0,x_1\right],\ h_i(x_0)=h_1(x_1)=0,\ i=\overline{1,n}.$

Funksional ekstremumining zaruriy shartiga ko’ra, $\delta J[y^0,h]=0$, $\forall h\in C_n{}^1\left[x_0,x_1\right],\ h(x_0)=h(x_1)=0,$ ya’ni

$$\int_{x_0}^{x_1}\sum_{i=1}^{n}\left[F_{y_i}\left(x,y^0(x),y^{0'}(x)\right)h(x)+F_{y_i{}'}(x,y^0(x),y^{0'}(x))h_i{}'(x)\right]dx=0,\ i=\overline{1,n},\qquad(5)$$

$h_1(x)\in C^1[x_0,x_1],\ i=\overline{1,m}$, funksiyalar sistemasining o’zaro bog’lanmaganligini hisobga olib, (5) dan

$$\int_{x_0}^{x_1}\left[F_{y_i}\left(x,y^0(x),y^{0'}(x)\right)h(x)+F_{y_i{}'}(x,y^0(x),y^{0'}(x))h_i{}'(x)\right]dx=0,\ i=\overline{1,n},\qquad(6)$$

tengliklarga ega bo’lamiz. Endi Dyubua-Reymon lemmasini qo’llab, (6) tengliklardan, talab qilingan, (4) tengliklar sistemasini olamiz. Teorema isbotlandi.

Isbotlangan teorema ko’rsatadiki, (1), (2) masalada $y^0(x)=(y_1{}^0(x),...,y_n{}^0(x))$ kuchsiz ekstremallar,

$$F_{y_i}(x,y,y')-\frac{d}{dx}F_{y_i{}'}(x,y,y')=0,\ \ i=\overline{1,n},\qquad(7)$$

Eyler tenglamalari sistemasini qanoatlantirar ekan.

Bu yerda, hususiy holda, $n=1$ bo’lganda variatsion hisobning asosiy masalasi uchun olingan natija, ya’ni Eyler tenglamasiga ega bo’lamiz.

Agar $F(x,y,y')\in C^{(2)}(Q)$ bo’lsa, (7) dan

$$\sum_{j=1}^{n}F_{y_i'y_j'}(x,y,y')y_j{}''+\sum_{j=1}^{n}F_{y_iy_j}(x,y,y')y_j{}'+F_{xy_i'}(x,y,y')-F_{y_i}(x,y,y')=0,\ \ i=\overline{1,n}\qquad(8)$$

sistemaga ega bo’lamiz. Bu esa, n noma’lumli ikkinchi tartibli n ta differensial tenglamalar sistemasidir.

Bundan buyon quyidagi belgilashlardan foydalanamiz:

$$F^0_{y_iy_j}=F_{y_iy_j}\left(x,y_1^0(x),...,y_n^0(x),y_1^{0'}(x),...,y_n^{0'}(x)\right),$$

$$F^0_{y_iy_j}=F_{y_iy_j'}\left(x,y_1^0(x),...,y_n^0(x),y_1^{0'}(x),...,y_n^{0'}(x)\right),$$

$$F^0_{y_iy_j}=F_{y_i'y_j'}\left(x,y_1^0(x),...,y_n^0(x),y_1^{0'}(x),...,y_n^{0'}(x)\right),\ \ i,j=\overline{1,n}.$$

Elementlari $F^0_{y_i'y_j'}(x)$ lardan tuzilgan $n\times n$ matrisani $F^0_{y'y'}(x)$ deb belgilaymiz. Faraz qilaylik, $F(x,y,y')\in C^{(2)}(Q)$ bo’lsin. Agar $y^0(x)=(y_1{}^0(x),...,y_n{}^0(x))$ kuchsiz lokal ekstremal uchun $\det F^0_{y'y'}(x)\neq 0,\ \ \forall x\in[x_0,x_1]$ bo’lsa, $y^0(x)$ funksiya $[x_0,x_1]$ kesmada (8) tenglamalar sistemasini qanoatlantiradi.

3-t a’ r i f. Eyler tenglamalar sistemasini qanoatlantiruvchi $y(x)=(y_1(x),...,y_n(x))$ joyiz funksiyalarga (1) funksionalning *stasionar funksiyalari* deyiladi.

M i s o l. $J[y_1,y_2]=\int_0^{\frac{\pi}{2}}\left[y'^2_1+y'^2_2+2y_1y_2\right]dx\to\min(\max),$

$y_1(0)=0,\ \ y_2(0)=0,$

$y_1(\frac{\pi}{2})=1,\ \ y_2(\frac{\pi}{2})=-1.$

Bu masalada $F=y'^2_1+y'^2_2+2y_1y_2,\ F_{y_1}=2y_2,\ F_{y_2}=2y_1,\ F_{y_1'}=2y_1',\ F_{y_2'}=2y_2'$ va Eyler tenglamalar sistemasi ikkita tenglamadan iborat bo’ladi:

$$\left.\begin{aligned} F_{y_1}-\frac{d}{dx}F_{y_1'}=0\Leftrightarrow y_1'-y_2=0,\\ F_{y_2}-\frac{d}{dx}F_{y_2'}=0\Leftrightarrow y_2'-y_1=0\end{aligned}\right\}$$

bu yerdan

$$y_2=y_1'',\ \ y_1^{IV}-y_1=0$$

sistemaga kelamiz. Hosil qilingan sistemaning umumiy yechimi

$$y_1=c_1e^x+c_2e^{-x}+c_3\cos x+c_4\sin x,$$
$$y_2=c_1e^x+c_2e^{-x}-c_3\cos x-c_4\sin x$$

ko’rinishda bo’ladi. Chegaraviy shartlardan foydalanib, $C_1=0,\ C_2=0,\ C_3=0,\ C_4=1$ ekanligini topamiz. Demak, qaralayotgan masalada joyiz stasionar funksiya

$$y_1^0(x)=\sin x,\ \ y_2^0(x)=-\sin x$$

bo’ladi. Lejandr sharti kuchaytirilgan shaklda bajariladi:

$$F_{y'y'}=\begin{pmatrix}F_{y_1'y_1'} & F_{y_1'y_2'}\\ F_{y_2'y_1'} & F_{y_2'y_2'}\end{pmatrix}>0 \qquad (13)$$

Yakobe tenglamalar sistemasi esa Eyler tenglamalar sistemasi kabi bo’ladi:

$$h_1''-h_2=0,\ \ h_2''-h_1=0.$$

Uning umumiy yechimini yozamiz:

$$h_1=\gamma_1e^x+\gamma_2e^{-x}+\gamma_3\cos x+\gamma_4\sin x,$$
$$h_2=\gamma_1e^x+\gamma_2e^{-x}-\gamma_3\cos x-\gamma_4\sin x$$

$h_1(0)=0,\ h_2(x_*)=0,\ 0<x_*\leq\frac{\pi}{2}$ chegaraviy shartlardan $\gamma_1=\gamma_2=\gamma_3=\gamma_4=0$, ya’ni $h_1(x)=0,\ h_2(0)=0$ bo’lishi kelib chiqadi. Demak, Yakobi sharti kuchaytirilgan shaklda bajariladi. Shunday qilib, kuchsiz lokal minimumning yetarli shartlari bajariladi. Bundan tashqari, (13) shart $F^0_{y'y'}(x,y,y')>0\ \ \forall(x,y,y')\in R^5$ ekanligini bildirganligidan, 4-teoremaga ko’ra $y^0(x)=(\sin x,-\sin x)$ joyiz funksiya kuchli minimal ham bo’ladi.

2. Yuqori tartibli hosilalarga bog'liq bo'lgan funksionalning ekstremumi.

Faraz qilaylik, $Q \subset R^{n+2}$ berilgan ochiq to'plam (soha),

$S = \{(x, y, z_1, ..., z_{n-1}) : (x, y, z, ..., z_n) \in Q\}$, $F(x, y, z_1, ..., z_n) - Q$ sohada uzluksiz funksiya, $P_0 = (x_0, y_{00}, y_{01}, ..., y_{0,n-1})$, $P_1(x_1, y_{10}, y_{11}, ..., y_{1,n-1}) - S$ to'plamning belgilangan nuqtalari , $x_0 < x_1$ bo'lsin.

Quyidagi

$$J[y] = \int_{x_0}^{x_1} F(x, y, y', ..., y^{(n)})dx \to \min(\max), \qquad (14)$$

$$\left. \begin{array}{l} y(x_0) = y_{00},\ y'(x_0) = y_{01}, ...,\ y^{(n-1)}(x_0) = y_{0.n-1}, \\ y(x_0) = y_{10},\ y'(x_1) = y_{11}, ...,\ y^{(n-1)}(x_1) = y_{1.n-1}, \\ y(x) = \in C^{(n)}[x_0, x_1],\ (x, y(x), y^1(x), ..., y^n(x)) \in Q,\ x \in [x_0, x_1] \end{array} \right\} \qquad (15)$$

ekstremal masalani qaraymiz.

Yuqori tartibli hosilalar qatnashgan (14), (15) masala ham chegaralari qo'zg'olmas variatsion masaladir. Bu masalada joyiz funksiyalar (chiziqlar) (15) shartlar bilan aniqlanadi, ya'ni ularning uchlari berilgan P_0 va P_1 nuqtalarda mahkamlangan.

5-t a' r i f. Agar biror $\varepsilon>0$ son topilib, $\|y - y^0\|_{C^{(n)}[x_0, x_1]} < \varepsilon$ shartni qanoatlantiruvchi barcha $y=y(x)$ joyiz chiziqlar uchun $J[y^0] \le J[y]$ $(J[y^0] \ge J[y])$ tengsizlik bajarilsa, (14) funksional $y^0=y^0(x)$ joyiz chiziqda kuchsiz lokal minimumga (maksimumga) erishadi, deyiladi. Bunda $y^0(x)$ - (15) masalaning kuchsiz minimali (maksimali) deyiladi.

Keltirilgan ta'rifda $C^{(n)}[x_0, x_1]$ fazodagi norma o'rniga $C^{(n-1)}[x_0, x_1]$ fazodagi normadan foydalansak, kuchli ekstremal ta'rifiga ega bo'lamiz. Avvalgi qaralgan variatsion masaladagi kabi bu yerda ham har bir kuchli ekstremalning kuchsiz ekstremal bo'lishi ravshan.

Kuchsiz ekstremumning birinchi tartibli zaruriy sharti quyidagi teoremada berilgan.

6-t e o r e m a. $F(x, y, z_1, ..., z_n) \in C^{(n+1)}(Q)$ bo'lsin. Agar $y^0(x)$ – (14), (15) masalada kuchsiz ekstremal bo'lsa, barcha $x \in [x_0, x_1]$ uchun

$$F_y^0(x) - \frac{d}{dx} F_{y'}^0(x) + \frac{d^2}{dx^2} F_{y''}^0(x) - ... + (-1)\frac{d^n}{dx^n} F_{y^{(n)}}^0(x) = 0 \qquad (16)$$

tenglik bajariladi, bu yerda

$$F_{y^{(i)}}(x) = F_{y^{(i)}}(x, y^0(x), y^{0'}(x), ..., y^{0(n)}(x)),\ i = \overline{0, n}. \qquad (17)$$

I s b o t i. (14) funksionalning $y^0=y^0(x)$ nuqtadaga birinchi variatsiyasini hisoblaymiz:

$$\delta J[y^0, h] = \frac{\partial}{\partial\alpha} J[y^0 + \alpha h]_{\alpha=0} = \frac{\partial}{\partial\alpha} \int_{x_0}^{x_1} F(x, y^0 + \alpha h, y^{0'} + \alpha h', ..., y^{0(n)} + \alpha h^{(n)})dx_{\alpha=0}\Big|_{\alpha=0} =$$

$$= \int_{x_0}^{x_1} \sum_{i=1}^{n} [F_{y^{(i)}}(x, y^0(x), y^{0'}(x)), ..., y^{0(n)}(x) h^{(i)}(x)]dx, \qquad (18)$$

bu yerda $h = h(x) \in C^{(n)}[x_0, x_1]$, $h^{(1)}(x_0) = h^{(i)}(x_1) = 0,\ i = \overline{0, n-1}$

(18) dagi qo'shiluvchilarning ikkinchisini $(i=1)$ bir marta, uchinchisini $(i=2)$ ikki marta, va h.k, oxirgisini $(i=n+1)$ marta bo'laklab integrallaymiz.

$h^{(i)}(x_0)=h^{(i)}(x_1)=0,\ i=\overline{0,n-1}$ shartlarni hisobga olib,

$$\int_{x_0}^{x_1} F_{y^{0'}}(x, y^{0'}(x),..., y^{0(n)}(x))h^{(i)}(x)dx =$$

$$=(-1)^n \int_{x_0}^{x_1} \frac{d^i}{dx^i} F_{y^{(i)}}(x, y^{0'}(x),..., y^{0(n)}(x))h(x)dx,\quad i=\overline{1,n} \qquad (19)$$

tengliklarga ega bo'lamiz. Endi (18) formulada (19) tengliklarni va (17) belgilashlarni e'tiborga olib, ekstremumning zaruriy sharti bo'lgan, $\delta J[y^0, h]=0$ munosabatni

$$\int_{x_0}^{x_1}\left[F_y^0(x)-\frac{d}{dx}F_{y'}^0(x)+\frac{d^2}{dx^2}F_{y''}^0(x)+...+(-1)\frac{d^n}{dx^n}F_{y^{(n)}}^0(x)\right]h(x)dx=0$$

ko'rinishda yozamiz, bu yerda $h(x)$ uzluksiz differensiallanuvchi funksiya, $h(x_0)=h(x_1)=0$. Bu oxirgi tenglikka Lagranj lemmasini(2-ma'ruzaga q.) qo'llaymiz va natijada (17) ni hosil qilamiz. Teorema isbotlandi.

Noma'lum $y=y(x)\in C^{(n)}[x_0, x_1]$ funksiyaga nisbatan

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''}^0 + ... + (-1)\frac{d^n}{dx^n}F_{y^{(n-1)}}^0 = 0 \qquad (20)$$

tenglamaga *Eyler–Puasson tenglamasi* deyiladi. $F(x,y,z_1,...,z_n)\in C^{(n+1)}(Q)$, bo'lganda Eyler-Puasson tenglamasi *2n*– tartibli oddiy differensial tenglamadan iborat.

6-t a ' r i f. Eyler-Puasson tenglamasini qanoatlantiruvchi $y^0(x)$ joyiz funksiyaga (14), (15) masalaning *stasionar funksiyasi* deyiladi.

Endi qaralayotgan masala uchun ekstremumning ikkinchi tartibli zaruriy shartlari va yetarli shartlarini qisqacha bayon qilamiz.Ular to'g'risida to'liq ma'lumotni[2,3] lardan olish mumkin.

Agar $F(x,y,z_1,...,z_n)\in C^{(2)}(Q)$ bo'lsa, (14) funksional har bir $y^0=y^0(x)\in C^{(n)}[x_0,x_1]$ nuqtada ikkinchi variatsiyaga ega va bu variatsiya,

$$\delta^2 J[y^0, h]=\int_{x_0}^{x_1}\sum_{i,j=1}^{n} F_{y^{(i)}y^{(j)}}^0(x), h^{(i)}(x), h^{(j)}(x)dx, \qquad (21)$$

$$h=h(x)\in C^{(n)}[x_0,x_1],\quad h^{(i)}(x_0)=h^{(i)}(x_1)=0,\quad i=\overline{0,n-1}.$$

formula bo'yicha hisoblanadi, bu yerda

$$F_{y^{(i)}y^{(j)}}^0(x)=F_{y^{(i)}y^{(j)}}^0\left(x, y^0(x), y^{0'}(x),..., y^{0(n)}(x)\right).$$

h=h(x) ga bog'liq (21) funksional uchun tuzilgan Eyler-Puasson tenglamasiga (14), (15) masala uchun *Yakobi tenglamasi* deyiladi.

7-t a' r i f. Agar Yakobi tenglamasi $h^{(i)}(x_0)=h^{(i)}(x_*)=0,\ i=\overline{0,n-1}$, shartlarni qanoatlantiruvchi trivial (aynan nol) bo'lmagan yechimga ega bo'lsa, x_* nuqta- $y^0(x)$ joyiz chiziq bo'ylab x_0 nuqtaga *qo'shma nuqta* deyiladi.

7-t e o r e m a. $F(x,y,z_1,...,z_n)\in C^{(n+2)}(Q)$ bo'lsin. Agar $y^0(x)\in C^{(2n)}[x_0,x_1]$ (14), (15) masalada kuchsiz minimal (maksimal) bo'lsa, quyidagi shartlar bajariladi:

a) Lejandr sharti: $F^0_{y^{(n)}y^{(n)}}(x)\geq 0 \ \ (\leq 0), \ \forall x\in[x_0,x_1]$

b) Yakobi sharti: (x_0,x_1) intervalda *y⁰(x)* chiziq bo'ylab x_0 nuqtaga qo'shma nuqta mavjud emas.

8-t e o r e m a. Agar:

a) $F(x,y,z_1,...,z_n)\in C^{(n+2)}(Q)$, $Q=S\times R$, $S\subset R^{n+1}$–ochiq to'plam;

b) $F^0_{y^{(n)}y^{(n)}}(x,y,z_1,...,z_n)\geq 0 \ \ (\leq 0), \ \forall(x,y,z_1,...,z_n)\in Q$;

c) $y^0(x)\in C^{(2n)}[x_0,x_1]$ joyiz stasionar funksiya;

d) kuchaytirilgan Lejandr sharti bajarilsa: $F^0_{y^{(n)}y^{(n)}}(x)>0 \ \ (<0), \ \forall x\in[x_0,x_1]$;

e) kuchaytirilgan Yakobi sharti bajarilsa: $[x_0,x_1]$ oraliqda $y^0(x)$ chiziq bo'ylab x_0 nuqtaga qo'shma nuqta mavjud emas. U holda, $y^0(x)$ – (14), (15) masalada kuchli lokal minimal (maksimal) bo'ladi.

Endi (14) funksional

$$J[y]=\int_{x_0}^{x_1}\sum_{i=0}^{n}P_i(x)\left[y^{(i)}(x)\right]^2dx \qquad (22)$$

ko'rinishdagi kvadratik funksionaldan iborat bo'lsin. U holda, Yakobi tenglamasi,

$$\sum_{i=0}^{n}(-1)^{(i)}\frac{d^i}{dx^i}(P_i(x)h^{(i)})=0$$

ko'rinishda bo'ladi.

9-t e o r e m a. Faraz qilaylik, $P_i(x)\in C^{(i)}[x_0,x_1]$, $P_n(x)>0 \ \ (<0), \ \forall x\in[x_0,x_1]$ bo'lsin. Agar Yakobi sharti bajarilmasa, (22) funksional uchun $\inf J[y]=-\infty$ $(\sup J[y]=+\infty)$ bo'ladi. Agar kuchaytirilgan Yakobi sharti bajarilsa, (22) funksional uchun yagona joyiz stasionar funksiya mavjud, bu funksiya funksionalga global minimum (maksimum) beradi.

M i s o l.

$$\left.\begin{aligned}&J[y]=\int_0^{\pi}(y''^2-16y^2)dx\to\min,\\ &y(0)=0,\ y'(0)=0,\ y(\pi)=0,\ y'(\pi)=1.\end{aligned}\right\}$$

Bu masalada qatnashayotgan hosilalarning yuqori tartibi *n=2* bo'lgani uchun, (20) tenglama,

$$F_y-\frac{d}{dx}F_{y'}+\frac{d^2}{dx^2}F_{y''}=0$$

ko'rinishda yoziladi. $F=y''^2-16y^2$, $F=-32y$, $F_{y'}=0$, $F_{y''}=2y''$ bo'lgani uchun, Eyler-Puasson tenglamasi,

$$-32y+\frac{d^2}{dx^2}(2y'')=0\Leftrightarrow y^{IV}-16=0$$

ko'rinishida bo'ladi. Uning umumiy yechimi,

$$y=c_1e^{2x}+c_2e^{-2x}+c_3\cos 2x+c_4\sin 2x.$$

Chegaraviy shartlardan

$$c_1=\frac{1}{2\left(e^{2n}-1\right)},\quad c_2=\frac{1}{2\left(1-e^{-2n}\right)},\quad c_3=\frac{1+e^{2n}}{2\left(e^{2n}-1\right)},\quad c_4=\frac{1}{2}$$

bo'lishi kelib chiqadi. Qaralayotgan masaladagi funksional (22) ko'rinishdagi kvadratik funksionaldir: $n=2,\ P_0(x)=-16,\ P_1(x)=0,\ P_2(x)=1$. Uning uchun Yakobi tenglamasi $h^{IV}-16h=0$ bo'ladi. Bu tenglamaning umumiy yechimi

$$h=\gamma_1 e^{2x}+\gamma_2 e^{-2x}+\gamma_3\cos 2x+\gamma_4\sin 2x.$$

uchun $h(0)=0,\ h(x_*)=0,\ h'(x)=0,\ h'(x_*)=0,\ x_*>0$ shartlardan $h(x)=0$ bo'lishi kelib chiqadi, ya'ni kuchaytirilgan Yakobi sharti bajariladi. $F_{y''y'}=2>0-$ kuchaytirilgan Lejandr sharti ham bajariladi. 9-teoremaga asosan,

$$y^0(x)=\frac{e^{2x}}{2\left(e^{2n}-1\right)}+\frac{e^{-2x}}{2\left(1-e^{-2n}\right)}-\frac{1+e^{2n}}{2\left(e^{2n}-1\right)}\cos 2x+\frac{1}{2}\sin 2x$$

funksiya masalaning global yechimidir.

Mustaqil ishlash uchun savollar.

1. Bir necha funksiyalarga bog'liq bo'lgan funksionalning kuchli va kuchsiz ekstremallari. Eyler tenglamalar sistemasi. Statsionar funksiyalar.
2. Yuqori tartibli hosilalarga bog'liq funksional ekstremumi. Kuchsiz ekstremumning birinchi tartibli zaruriy sharti. Eyler-Puasson tenglamasi.

6-ma'ruza. Variatsion hisob asosiy masalasining umumlashmalari. Bir necha o'zgaruvchili funksiyalarga bog'liq funksionallarning ekstremumi. Eyler-Ostrogradskiy tenglamasi

Reja:

1.Masalaning qo'yilishi
2. Funksionalning birinchi variasiyasini topish.
3. Ekstremumining zaruriy sharti. Eyler-Ostrogradskiy teglamasi.

1.Masalaning qo'yilishi. Soddalik uchun , ikki o'zgaruvchili funksiyaga bog'liq funksional uchun qo'yilgan,

$$J[z(x,y)]=\iint_D F(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y})dxdy \qquad (23)$$

funksionalning ekstremumini , $C^1(D)$ sohada aniqlangan (uzluksiz va o'zining har bir argumenti bo'yicha uzluksiz birinchi tartibli uzluksiz hosilalarga ega bo'lgan) va

$$z(x,y)\big|_{\partial D}=f(x,y) \qquad (24)$$

chegaraviy shartlarni qanoatlantiruvchi $z(x,y)$ joiz funksiyalar sinfida, topish haqidagi variatsion masalani qaraymiz, buyerda $\partial D-D$ sohaning chegarasi, $f(x,y)$- berilgan funksiya.

Integral tagidagi $F(x, y, z, p, q)$ funksiya(integrant) o'zining z, p, q argumentlari bo'iyicha ikkinchi tartibgacha uzluksiz xususiy hosilalarga ega bo'lsin, deb faraz qilamiz.

2. Funksionalning birinchi variasiyasini topish. Bu yerda ham yuqoridagi shatlarda (23) funksionalning birinchi variatsiyasi uchun formula keltirib chiqaamiz. Buning uchun, variatsion hisob asosiy masalasidagiga o'xshash,

$$z(x, y, \alpha) = z(x, y) + \alpha\delta z(x, y), 0 \le \alpha \le 1 \qquad (25)$$

bir parametirli sirtlar oilasini qaraymiz va unda aniqlangan

$$J[z(x, y, \alpha)] = \iint F\left(x, y, z + \alpha\delta z, \frac{\partial z}{\partial x} + \alpha\frac{\partial}{\partial x}\delta z, \frac{\partial z}{\partial y} + \alpha\frac{\partial}{\partial y}\delta z\right) \qquad (26)$$

funksionalni qaraymiz.

$z = z(x, y) -$ (23) funksionalga kuchsiz ekstremum beruvchi sirt bo'lsin. Quyidagi,

$$\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \delta z = h(x, y), \frac{\partial}{\partial x}\delta z = \frac{\partial h}{\partial x} = h_x, \frac{\partial}{\partial y}\delta z = \frac{\partial h}{\partial y} = hy \qquad (27)$$

belgilashlarni kiritamiz va

$$\delta J[z(x, y)] = \frac{\partial}{\partial \alpha} J[z(x, y, \alpha)]\Big|_{\alpha=0} \qquad (28)$$

ifodani hisoblaymiz. Farazimizga ko'ra, (4) funksionalda integrallash va α parametr bo'yicha diferensiallash amallarining o'rnini almashtirish mumkin bo'lganligidan, birinchi variatsiya formulasi,

$$\delta J[z(x, y)] = \iint_D [F_z \cdot h + F_p \cdot h_x + F_q \cdot hy] dxdy \qquad (29)$$

ko'rinishni oladi, $h = h(x, y)$ - $z(x, y)$ funksiyaning variatsiyasidir

3. Ekstremumining zaruriy sharti. Eyler-Ostrogradskiy teglamasi.

Ma'lumki, agar $z(x, y)$ funksiya ekstremal bo'lsa, ixiyoriy $h = h(x, y)$ variatsiya uchun,

$$\delta J[z(x, y)] \equiv 0 \Leftrightarrow \iint [F_z \cdot h + F_p \cdot h_x + F_q \cdot h_y] dxdy \equiv 0 \qquad (30)$$

munosabat o'rinli bo'ladi. Oxirga munosabatning chap tomonidagi ikkinchi va uchunchi qo'shiluvchilarning shaklini o'zgartiramiz. Buning uchun,

$$\frac{\partial}{\partial ч}[F_p \cdot h] = \frac{\partial}{\partial x}\{F_p\} \cdot h + F_p \cdot h_x, \qquad F_p \cdot h_x = \frac{\partial}{\partial x}[F_p \cdot h] - \frac{\partial}{\partial x}\{F_p\} \cdot h$$

ifodalardan

$$\frac{\partial}{\partial y}[F_q \cdot h] = \frac{\partial}{\partial y}\{F_q\} \cdot h + F_q \cdot h_y \qquad F_q \cdot h_y = \frac{\partial}{\partial y}[F_q \cdot h] - \frac{\partial}{\partial y}\{F_q\} \cdot h$$

ifodalarni olamiz va (30) munosabatga keltirib qo'yamiz:

$$\iint \left[F_z - \frac{\partial}{\partial x}\{F_p\} - \frac{\partial}{\partial y}\{F_q\}\right] \cdot hdxdy + \iint_D \left\{\frac{\partial}{\partial x}[F_p \cdot h] + \frac{\partial}{\partial y}[F_q \cdot h]\right\} dxdy = 0.$$

Oxirgi munosabatdagi ikkinchi integralda ikki karrali integralni egri chiziqli integral orqali ifodalash formulasi- Grin formulasi

$$\iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial y}{\partial y}\right) dxdy = \oint_{\partial D} Px + Qdy \qquad (31)$$

dan foydalanib va (24) chegaraviy shartlardan, $h[(x, y)]_{\partial D} = 0$ ekanligini hisobga olib, (30) shartni,

$$\iint_D \left[F_z - \frac{\partial}{\partial x}\{F_P\} - \frac{\partial}{\partial y}\{F_q\}\right] h(x, y) dxdy \equiv 0 \qquad (32)$$

ko’rinishda yozamiz. Bunda h(x,y) variatsiyaning ixtiyoriyligidan, Lagranj lemmasining, ikki karrali shitegral uchun umumlashmasi barcha shartlari o’rinli bo’lganligidan, (32) ifodaning chap tamonidagi o’rta qavs ichidagi ifodaning D to’plamning barcha nuqtalarida aynan nolga teng bo’lishini olamiz. Demak ,(23) funksional ekstremumga erishadigan $z = z(x, y)$ funksiya quyidagi,

$$F_z - \frac{\partial}{\partial x}\{F_p\} - \frac{\partial}{\partial y}\{F_q\} = 0 \qquad (33)$$

tenglamani qanoatlantirishi zarur. (33) tenglama, *Eyler – Ostrogradskiy tenglamasi* deyiladi va u ikkinchi tartibli xususiy hosilali diffe
rensial tenglamadan iborat,bunda

$$\frac{\partial}{\partial x}\{F_p\} = \frac{\partial}{\partial x}\left\{F_p(x, y, z, p, q)\right\} = F_{xp} + F_{zp\cdot}\cdot p + F_{pp}\cdot\frac{\partial p}{\partial x} + F_{qp}\frac{\partial q}{\partial x},$$

$$\frac{\partial}{\partial y}\{F_q\} = \frac{\partial}{\partial y}\{F_q(x, y, z, p, q,)\} = F_{yq} + F_{zq}\cdot q + F_{pq}\frac{\partial p}{\partial y} + F_{qq}\frac{\partial q}{\partial y},$$

$$\frac{\partial p}{\partial y} = \frac{\partial^2 z}{\partial y dx}, \frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \partial y}.$$

Matematik fizika tenglamalari kursida (33) tenglamaning (24) chegaraviy shartlarni qanoatlantruvchi echimini topish masalasi –*Dirixle masalasi* deb ataladi.
Funksional uch yoki undan ko’p o’zgaruvchili funksiyaga bogliq bo’lganda ham Eyler- O strofadskiy teglamasini keltirib chiqarish mumkin .

3.4. misol. Ushbu

$$J[z] = \iint_D \left(z^2{}_x + z_y{}^2\right)dxdy$$

funksional uchun Eyler-Ostrogradskiy tenglamasini yozing.

E ch i l i sh i. Berilgan funksional ikki o’zgaқuvchili funksiyaga bogliq bo’lib, unda $F = F(p, q) = p^2 + p^2$ bo’lganligidan, $F_z = 0, F_p = 2p, F_q = 2q$ va Eyler-Ostrogradskiy tenglamasi

$$\frac{\partial}{\partial ч}\{2p\} + \frac{\partial}{\partial D}\{2q\} = 0,$$

yoki, $z^2{}_{xx} + z^2{}_{yy} = o \Leftrightarrow \Delta z = o$ Laplas teglamasidan iborat. Ma’lumki, uning echimlari *garmonik funksiyalar* deb ataladi.

Mustaqil ishlash uchun savollar.

1. Bir necha funksiyalarga bog’liq bo’lgan funksionalning kuchli va kuchsiz ekstremallari. Eyler tenglamalar sistemasi. Statsionar funksiyalar.
2. Lejandr va Yakobi shartlari. Yetarli shartlar.
3. Yuqori tartibli hosilalarga bog’liq funksional ekstremumi. Kuchsiz ekstremumning birinchi tartibli zaruriy sharti. Eyler-Puasson tenglamasi.
4. Ikkinchi tartibli zaruriy shartlar.

7-ma'ruza. Chegaralari qo'zg'aluvchan variatsion masalalar. Transversallik shartlari.

Reja

1. Masalaning qo'yilishi.
2. Eyler tenglamasi, transversallik shartlari.
3. Transversallik shartlarining xususiy xollari.
4. Misol.

Tayanch iboralar: *joiz chiziq, qo'zg'oluvchan chegaralar, funksional-ning birinchi variasiyasi, Eyler tenglamasi, transversallik shartlari.*

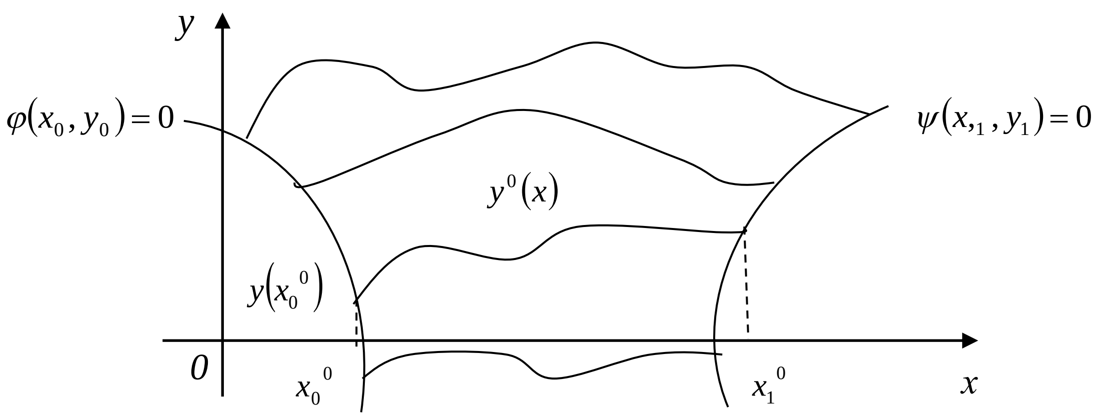
1. Masalaning qo'yilishi. Quyidagilar berilgan bo'lsin:

a) $Q - R^3$ dagi biror aniq to'plam (soha);

v) $S = \{(x,y) \in R^2;\ (x,y,z) \in Q\} - Q$ to'plamning R^2 da proyeksiyasi;

s) $(x_0, y(x_0)), (x_1, y(x_1)) \in S'$; bu yerda x_0, x_1 lar oldindan berilmagan;

d) $y(x)$ funksiyalar uzluksiz differensiallanuvchi, ya'ni $y \in C^1[a,b]$, bo'lib,



5.1-chizma

$x_0, x_1 \in (a,b)$;

ye) $\varphi(x,y), \psi(x,y)$ - uzluksiz differensiallanuvchi funksiyalar;

f) $F(x,y,z)$; $Q \to R^1$ - uzluksiz funksiya.

$C^{(1)}[x_0, x_1]$ fazoning

$V = \{y(x) \in C^{(1)}[x_0, x_1]: \varphi(x_0, y(x_0)) = 0,\ \psi(x_0, y(x_0)) = 0,\ (x, y(x), y'(x)) \in Q,\ x \in [x_0, x_1]\}$

to'plamida aniqlangan

$$J(y) = \int_{x_0}^{x_1} F(x,y,y')dx$$

funksionalning ekstremumini topish masalasini qaraymiz.

Bu masala *chegaralari qo'zg'oluvchan variasion masala* deyiladi va u

$$J(y) = \int_{x_0}^{x_1} F(x,y,y')dx \to \min(\max), \tag{5.1}$$

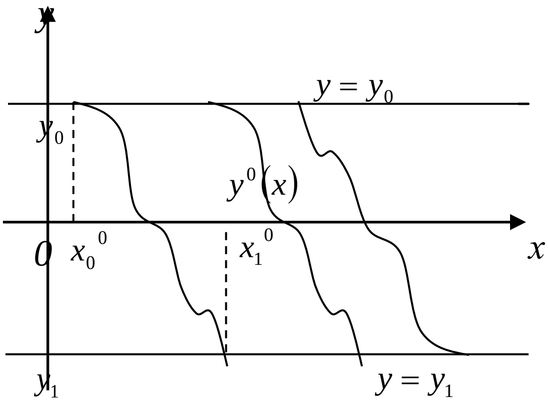
$$\varphi(x_0, y_0) = 0,\ \psi(x_1, y_1) = 0,\ y \in C^{(1)}[a,b] \tag{5.2}$$

ko'rinishda belgilanadi, bu yerda $y_0 = y(x_0), y_1 = y(x_1)$.

1 – izoh. (5.2) shartlar qo'zg'oluvchan chegaralarni aniqlaydi (5.1-chizma). Shunday qilib, qo'yilgan masalada ekstremum, uchlari $\varphi(x_0, y_0) = 0$ (chap uchi uchun), $\psi(x_1, y_1) = 0$ (o'ng uchi uchun) tenglamalar bilan aniqlanadigan berilgan chiziqlar bo'ylab siljiydigan silliq chiziqlar sinfida izlanadi.

Qo'yilgan masalaning quyidagi xususiy hollarini ajratish mumkin.

1) Joiz chiziqlarning uchlari $x = x_0$, $x = x_1$ tenglamalar bilan aniqlanadigan ikkita berilgan vertikal to'g'ri chiziqlar bo'yicha siljiydi (5.2-chizma):



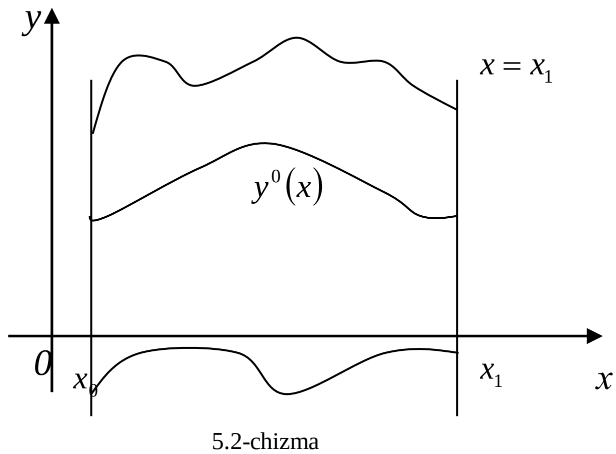
5.3-chizma

2) Joiz chiziqlarning uchlari

$$y = \varphi(x), \quad y = \phi(x) \tag{5.3}$$

tenglamalar bilan ifodalanadigan ikkita berilgan chiziqlar bo'ylab siljiydi. Bu holda chizma 5.1-chizma kabi bo'ladi.Qaralayotgan holda berilgan chiziqlar abssissalar o'qiga parallel $y = y_0$, $y = y_1$ to'g'ri chiziqlardan iborat masalani alohida ajratish mumkin (5.3-chizma):

2-izoh. Qo'yilgan masalada $y^0(x)$ chiziqni izlash bilan birga x_0^0 va x_1^0 qiymatlar ham tanlanadi (5.1, 5.2-chizmalarga q.), ya'ni $(y^0(x), x_0^0, x_1^0)$ uchlik izlanadi. Bunda uning birinchi tartibli ε - atrofi (ε>0)



5.2-chizma

$$\|y - y^0\|_{C^{(1)}[a,b]} < \varepsilon, \ |x_0 - x_0^0| < \varepsilon, \ |x_1 - x_1^0| < \varepsilon$$

shartlarni qanoatlantiradigan $(y(x), x_0, x_1)$ uchliklar orkali hosil qilinadi.

(5.1) funksional aniqroq ko'rinishda

$$J(y, x_0, x_1) = \int_{x_0}^{x_1} F(x, y, y')dx$$

kabi yoziladi.

Agar (5.1) funksional uchun $(y^0(x), x_0^0, x_1^0)$ uchlikning birinchi tartibli ε - atrofida

$$J(y^0(x), x_0^0, x_1^0) \le J(y, x_0, x_1)$$

tengsizlik bajarilsa, funksional $(y^0(x), x_0^0, x_1^0)$ uchlikda kuchsiz minimumga erishadi, deyiladi.

2. Eyler tenglamasi. (5.1), (5.2) masalaning yechimini izlash uchun funksional ekstremumining birinchi tartibli zaruriy sharti $\delta J = 0$ dan foydalanamiz.

Faraz qilaylik, $(y^0(x), x_0^0, x_1^0)$ uchlikda (5.1) funksional ekstremumga erishsin. U holda joiz chiziqlar

$$y(x) = y^0(x) + \alpha h(x), \ y'(x) = y^{0\prime}(x) + \alpha h'(x)$$

munosabatlar bilan aniqlanadi, bu yerda $\alpha \in R -$ parametr, $h(x) -$ tanlangan variasiya; integrallash chegaralarining joiz qiymatlari esa,

$$x_0 = x_0^0 + \alpha\delta x_0,\ x_1 = x_1^0 + \alpha\delta x_1$$

formulalar bilan aniqlanadi.

Funksionalning birinchi variasiyasini topamiz. Ta’rifga ko’ra,

$$\delta J = \frac{d}{d\alpha}\int_{x_0^0+\alpha\delta x_0}^{x_1^0+\alpha\delta x_1} F\left(x,\ y^0 + \alpha h,\ y^0{}' + \alpha h'\right)dx\Bigg|_{\alpha=0}$$

ni topish uchun, ushbu

$$\frac{d}{d\alpha}\int_{u(\alpha)}^{v(\alpha)} f(t,\alpha)dt = \int_{u(\alpha)}^{v(\alpha)} \frac{\partial f(t,\alpha)}{\partial\alpha}dt +$$
$$+ f(v(\alpha),\alpha)\frac{dv}{d\alpha} - f(u(\alpha),\ \alpha)\frac{du}{d\alpha}$$

integralni parametr bo’yicha differensiallash formulasidan foydalanamiz. U holda quyidagiga ega bo’lamiz:

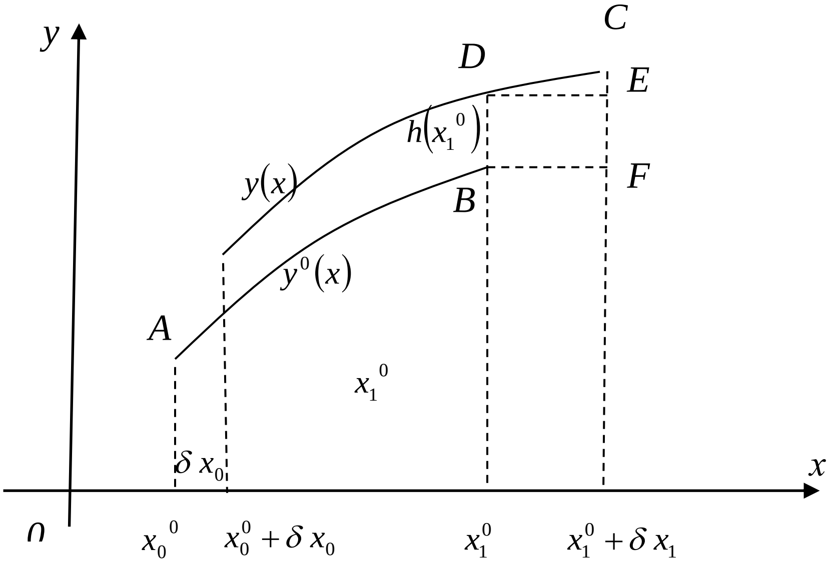
$$\delta J = \left\{\int_{x_0^0+\alpha\delta x_0}^{x_1^0+\alpha\delta x_1}\left[\frac{\partial F\left(x,\ y^0 + \alpha h,\ y^0{}' + \alpha h'\right)}{\partial y}h(x) + \frac{\partial F\left(x,\ y^0 + \alpha h,\ y^0{}' + \alpha h'\right)}{\partial y'}h'(x)\right]dx + \right.$$
$$\left. + F\left(x,\ y^0 + \alpha h,\ y^0{}' + \alpha h'\right)\Big|_{x=x_1^0+\alpha\delta x_1}\delta x_1 - F\left(x,\ y^0 + \alpha h,\ y^0{}' + \alpha h'\right)\Big|_{x=x_0^0+\alpha\delta x_0}\delta x_0\right\}\Bigg|_{\alpha=0} =$$
$$\int_{x_0^0}^{x_1^0}\left[F_y h(x) + F_{y'}h'(x)\right]dt + F\left(x_1^0, y^0\left(x_1^0\right), y^0{}'\left(x_1^0\right)\right)\delta x_1 - F\left(x_0^0, y^0\left(x_0^0\right), y^0{}'\left(x_0^0\right)\right)\delta x_0$$

Integralda ikkinchi qo’shiluvchini bo’laklab integrallab (2-ma’ruzaga q.), ekstremumning zaruriy shartini

$$\delta J = \int_{x_0^0}^{x_1^0}\left[F_y - \frac{d}{dx}F_{y'}\right]h(x)dx + F_{y'}h(x)\Big|_{x_0^0}^{x_1^0} + F\Big|_{x=x_1^0}\delta x_1 - F\Big|_{x=x_0^0}\delta x_0 = 0 \qquad (5.4)$$

ko’rinishda yozamiz.

(5.4) shartdan qo’rinadiki, funksionalning birinchi variasiyasi δJ tanlangan x_0^0 va x_1^0 qiymatlarda $h(x)$ joiz chiziq variasiyasi bilan aniqlanadigan integral qismdan hamda integrallash intervali uchlarining $\delta x_0,\ \delta x_1$ variasiyalari va ekstremal uchlarining $x = x_0^0,\ x = x_1^0$ dagi $h(x)$ variasiyalariga bog’liq uchta qo’shiluvchidan iborat (5.4-chizma).



5.4-chizma

$\delta\ J = 0$ shartdan ikkita tenglik kelib chiqadi:

1. $\int_{x_0^0}^{x_1^0}\left[F_y - \frac{d}{dx}F_{y'}\right]h(x)dx = 0$

ya'ni (5.1), (5.2) masalaning $y^0(x)$ ekstremali

$$F_y - \frac{d}{dx}F_{y'} = 0$$

Eyler tenglamasining yechimi bo'lishi zarur (2-ma'ruzaga q.).

2. $$F_{y'}h(x)\Big|_{x_0^0}^{x_1^0} + F\Big|_{x=x_1^0}\delta x_1 - F\Big|_{x=x_0^0}\delta x_0 = 0 \qquad (5.5)$$

Bunda $h(x)\big|_{x=x_1^0} \neq h(x_1),\ h(x)\big|_{x=x_0^0} \neq h(x_0)$ ekanligini ta'kidlash lozim (5.4-chizma).

5.4-chizmada $$BD = h(x)|_{x=x_1^0},\quad FC = h(x_1),\quad DE = \delta x_1,$$
$$EC \approx y'(x_1^0)\delta x_1,\quad BD = FC - EC,$$

ya'ni

$$h(x)|_{x=x_1^0} \approx h(x_1) - y'(x_1^0)\cdot \delta x_1. \qquad (5.6)$$

Bu yerdagi taqribiy tenglik yuqoriroq tartibli cheksiz kichik miqdorlar aniqligida o'rinli ekanligini ta'kidlab o'tamiz.

Xuddi shunga o'xshash,

$$h(x)|_{x=x_0^0} \approx h(x_o) - y'(x_0^0)\delta x_0$$

Shuning uchun, (5.5) tenglikni,

$$F_y\Big|h(x_1) + [F - y'F_{y'}]_{x=x_1^0}\delta x_1 - F_{y'}\Big|_{x=x_0^0} h(x_0) - [F - y'F_{y'}]_{x=x_0^0}\delta x_0 = 0 \qquad (5.7)$$

ko'rinishda yozish mumkin.

(5.5) ni (5.7) ga almashtirishni hisobga olsak, funksionalning variasiyasi va (5.4) ekstremumning zaruriy sharti

$$\delta J = \int_{x_0^0}^{x_1^0}[F_y - \frac{d}{dx}F_{y'}]h(x)dx + Fy'|_{x=x_1^0} h(x_1) +$$
$$+[F - y'F_{y'}]_{x=x_1^0}\delta x_1 - F_{y'}|_{x=x_0^0}(x) - [F - y'F_{y'}]|_{x=x_0^0}\delta x_0 = 0 \qquad (5.8)$$

ko'rinishda yoziladi.

(5.2) chegaraviy shartlarga asosan, $h(x_1)$ va $\delta(x_1)$, hamda $h(x_0)$ va $\delta(x_0)$ variasiyalar bog'langan:

$$\delta\varphi = \frac{\partial\varphi}{\partial x}|_{x_0^0, y^0(x_0^0)}\delta x_0 + \frac{\partial\varphi}{\partial y}|_{x_0^0, y^0(x_0^0)} h(x_0) = 0, \qquad (5.9)$$
$$\delta\phi = \frac{\partial\phi}{\partial x}|_{x_1^0, y^0(x_1^0)}\delta x_1 + \frac{\partial\phi}{\partial y}|_{x_1^0, y^0(x_1^0)} h(x_1) = 0,$$

lekin $h(x_1)$ va $\delta(x_1)$ variasiyalar, $h(x_0)$ va $\delta(x_0)$ variasiyalar bilan bog'lanmagan. Shuning uchun, (5.7) tenglikni,

$$F_{y'}|_{x=x_1^0} h(x_1) + [F - y'Fy']_{x=x_1^0}\delta x_1 = 0$$
$$F_{y'}|_{x=x_0^0} h(x_0) + [F - y'Fy']_{x=x_0^0}\delta x_0 = 0 \qquad (5.10)$$

ko'rinishda yozish mumkin.

(5.9 va (5.10) shartlar *transversallik shartlari* deyiladi.

Keltirilgan natijalarni quyidagi tasdiq ko'rinishida ifodalaymiz.

1-teorema(ekstremumning zaruriy sharti). (5.1) funksionalning (5.2) chegaraviy shartlarni qanoatlantiruvchi $y^0(x) \in C^{(1)}[a,b]$ kuchsiz ekstremali:

$a)$ $F_y = \frac{d}{dx} F_{y'} = 0$ Eyler tenglamasini;

$b)$ (5,9), (5,10) transversallik shartlarini qanoatlantiradi.

3.Transversallik shartlarining xususiy hollari.

1) Agar joiz chiziqlarning bir uchi mahkamlangan bo'lsa, bu uch uchun transversallik shartlari yozilmaydi, variasiyalar nolga teng.

2) Agar joiz chiziqlarning uchlari ikkita ikkita berilgan $x = x_0, \quad x = x_1$ vertikal to'g'ri chiziqlar bo'ylab siljiydigan masalalar berilgan bulsa, uning uchun $\delta x_0 = 0, \quad \delta x_1 = 0$ bo'ladi.

Demak, (5.10) transversallik shartlari

$$F_{y'} |_{x=x_1^0} = 0 \qquad F_{y'} |_{x=x_0^0} = 0 \qquad (5.11)$$

ko'rinishga ega bo'ladi.

(5.9) shartlar bajariladi, chunki berilgan to'g'ri chiziqlar tenglamalarini

$$\varphi(x_0) = x_0 - x_0^0 = 0, \quad \psi(x_1) = x_1 - x_1^0 = 0$$

ko'rinishda yozish mumkin.

3) Agar joiz chiziqlarning uchlari berilgan ikkita $y = \varphi(x), \quad y = \psi(x)$ chiziqlar bo'ylab siljisa, (5.2) shartlarni

$$\varphi(x_0 y_0) = y_0 - \varphi(x_0) = 0, \quad \psi(x_1, y_1) = y_1 - \psi(x_1) = 0$$

ko'rinishda yozish mumkin. Bu holda (5.9) dan

$$-\varphi'(x_0)\delta x_0 + 1 \cdot h(x_0) = 0,$$
$$-\psi'(x_1^0)\delta x_1 + 1 \cdot h(x_1) = 0$$

tengliklarga ega bo'lamiz yoki:

$$h(x_0) = \varphi'(x_0^0)\delta x_0, \quad h(x_1) = \varphi'(x_1^0)\delta x_1$$

U holda (5.10) dan

$$F + [(\psi' - y')F_{y'}]_{x=x_1^0} \delta x_1 = 0$$
$$F + [(\varphi' - y')F_{y'}]_{x=x_0^0} \delta x_0 = 0$$

tengliklar kelib chiqadi. δx_1 va δx_0 variasiyalarning ixtiyoriyligidan, bu hol uchun

$$\begin{aligned} &[F + (\psi' - y')F_{y'}]_{x=x_1^0} = 0 \\ &[F + (\varphi' - y')F_{y'}]_{x=x_0^0} = 0 \end{aligned} \qquad (5.12)$$

transversallik shartlariga ega bo'lamiz.

Agar

$$y = y_0 = \varphi(x) = const, \quad y = y_0 = \psi(x) = const$$

ko'rinishdagi chiziqlar qaralsa, $y'(x) = 0$, $\psi'(x) = 0$ bo'ladi va (5.12) shartlar soddalashadi:

$$[F + y'F_{y'}]_{x=x_1^0} = 0, \quad [F + y'F_{y'}]|_{x=x_0^0} = 0. \qquad (5.13)$$

4) Agar (5.2) shartlar qatnashmasa, $h(x_1)$, δx_1, $h(x_0)$, δx_0 variasiyalar ixtiyoriy bo'ladi. U holda (5.10) dan

$$
\begin{aligned}
&F_{y'}\,|_{x=x_1^0}=0,\;\; [F-y'F_{y'}]_{x=x_1^0}=0\\
&F_{y'}\,|_{x=x_0^0}=0,\;\; [F-y'F_{y'}]_{x=x_0^0}=0
\end{aligned}
\qquad (5.14)
$$

kelib chiqadi.

5) Agar (5.2) shartlar $y(x_0)=y_0$, $y(x_1)=y_1$ ko'rinishda yozilgan bo'lsa, ya'ni chegaralari qo'zg'olmas masala qaralsa, $\delta x_0=\delta x_1=h(x_0)=h(x_1)=0$ bo'lgani uchun, (5.10) transversallik shartlari bajariladi, Eyler tenglamasining umumiy yechimidagi ixtiyoriy o'zgarmaslar esa, chegaraviy shartlardan aniqlanadi.

1-misol. Chap uchi $A(0;0)$ nuqtada mahkamlangan, o'ng uchi esa $x=x_1=2$ to'g'ri chiziqda yotuvchi silliq chiziqlar ichida (5.5-chizma)

$$J(y)=\int_0^2[2xy+yy'+y'^2]dx$$

funksionalga ekstremum beruvchi chizik topilsin.

Yechilishi. Eyler tenglamasini tuzamiz:

$$F=2xy+yy'+y'^2,\quad F_y=2x+y',\quad F_{y'}=y+2y',\quad \frac{d}{dx}F_{y'}=y'+2y''$$

bo'lgani uchun, Eyler tenglamasi

$$2x+y'-y'-2y''=0 \qquad \text{yoki} \qquad y''=x$$

ko'rinishga ega bo'ladi. Uni ketma-ket ikki marta integrallab, umumiy yechimini topamiz:

$$y'=\frac{x^2}{2}+c,$$

$$y=\frac{x^3}{6}+c_1x+c_2.$$

Joiz chiziqlarning chap uchi maxkamlangan, o'ng uchi esa $x=2$ vertikal chiziq bo'ylab siljigani uchun, chap uch uchun chagaraviy shart va o'ng uchi uchun (5.11) transversallik shartini yozamiz.

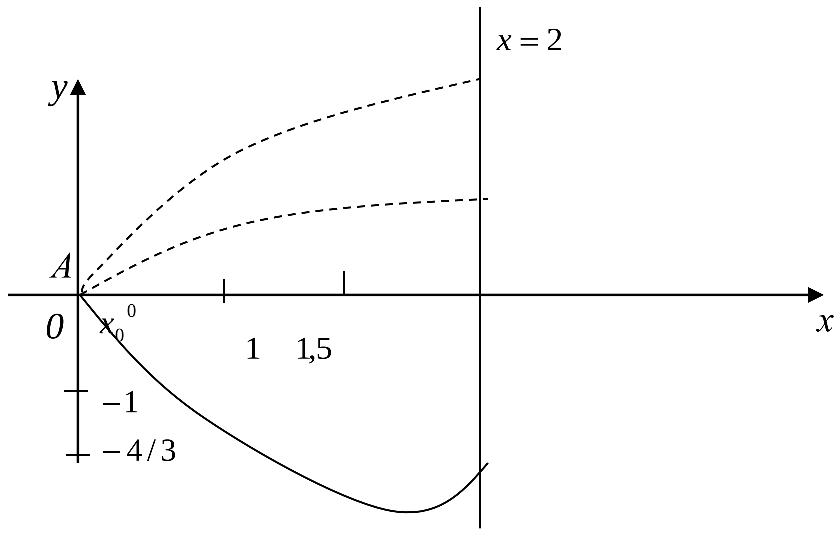
$$y(0)=0$$

$$F_{y'}\,|_{x_1=2}=(y+2y')|_{x_1=2}=y(2)+2y'(2)=0.$$

Endi C_1 va C_2 o'zgarmaslarni topamiz.

$$y(0)=C_2=0,\, y(2)+2y'(2)=\frac{8}{6}+2C_1+C_2+4+2C_1=0.$$

Bu yerdan $C_2=0$, $C_1=\frac{4}{3}$. Natijada, $y^0(x)=\frac{x^3}{6}-\frac{4x}{3}$ ekstremalga ega bo'lamiz (5,5-chizmaga q.).



5.5-chizma

8-ma’ruza. Shartli ekstremumga qo‘yilgan variatsion masalalar. Lagranj ko‘paytuvchilar qoidasi

Reja

1. Bog`lanishlari chekli bo`lgan variatsion masalalar
 1) Masalaning qo`yilishi.
 2)Masala yechimini izlash tartibi(sxemasi).
 3). Masalada shartli ekstremumning zaruriy shartini qo'llash algoritmi.
2. Differentsial bog’lanishlari bo’lgan shartli ekstremum masalalari
 1). Masalaning qo’yilishi
 2). Masala yechimini izlash tartibi (sxemasi).
 3). Masalada ekstremumning zaruriy shartini qo’llash algoritmi.

Tayanch iboralar. *Vektor funksiya,* $C^1[x_0, x_1]$ *fazo, masala yechimini izlash tartibi, Lagranj funksiyasi, Lagranj ko'paytuvchilari, Eyler tenglamalari sistemasi, umumiy yechim, chegaraviy shartlar, bog’lanishlar tenglamalari*.

1. Bog`lanishlari chekli bo`lgan variatsion masala.

**1.1. Masalaning qo`yilishi**. Quyidagi shartlarni qanoatlantiruvchi joiz $y(x)=(y_1(x),...,y_n(x))$ vektor funksiyalar to`plami M ni qaraymiz:

a) $y_i(x)$ funksiyalar $[x_0,x_1]$ kesmada aniqlangan va uzluksiz differensiallanuvchi bo'lsin, ya'ni $y_i(x)\in C^1[x_0,x_1],\ i=\overline{1,n},\ \ x_0,x_1$ - berilgan;

b) $y_i(x)$ funksiyalar

$$y_i(x_0)=y_{i0}, y_i(x_1)=y_{i1},\ i=\overline{1,n} \qquad (1)$$

chegaraviy shartlarni qanoatlantirsin, ya'ni $y_i(x)$ egri chiziqlarning har biri mahkamlangan (qo`zg`olmas) chegara nuqtalardan o`tadi, $y_{i0},y_{i1},\ i=\overline{1,n}$, berilgan sonlar;

c) $y_i(x)$ funksiyalar barcha $x\in[x_0,x_1]$ lar uchun

$$\varphi_j(x,y_1(x),...,y_n(x))=0,\ j=\overline{1,m},\ m<n, \qquad (2)$$

chekli bog`lanishlarni qanoatlantiradi., bunda $y_j(x,y_1,y_2...,y_n),\ j=\overline{1,m}$ funksiyalar barcha o`zgaruvchilari bo`yicha uzluksiz differensiallanuvchidir.

(2) tenglamalar o'zaro bog`lanmagan, ya'ni

$$rang\begin{pmatrix} \frac{\partial\varphi_1}{\partial y_1}.....\frac{\partial\varphi_1}{\partial y_n} \\ .... \\ \frac{\partial\varphi_m}{\partial y_1}....\frac{\partial\varphi_m}{\partial y_n} \end{pmatrix}=m,$$

hamda (2) bog`lanishlar (1) chegaraviy shartlarga muvofiqlashtirilgan.

Oxirgi jumlaning ma'nosi shundan iboratki, chegaraviy nuqtalar (2) tenglamalarni $x=x_0$ va $x=x_1$ bo`lganda qanoatlantirishi shart.

M to'plamda

$$J[y_1,y_2,...,y_n]=\int_{x_0}^{x_1}F(x,y_1,...,y_n,y_1',y_2',...,y_n')dx \qquad (3)$$

funksional berilgan, bu yerda $F(x,y_1,...,y_n,y_1',y_2',...,y_n')$ funksiya o'zining barcha argumentlari bo'yicha ikkinchi tartibgacha uzluksiz xususiy hosilalarga ega, deb faraz qilinadi.

M to'plamga tegishli joiz $y(x)$ vektor funksiyalar ichida shunday $y^*(x)=(y_1^*(x),...,y_n^*(x))^1$ vektor funksiyani topish kerakki, unda (3) funksional ekstremumga erishsin, ya'ni

$$J[y_1^*(x),y_2^*(x),...,y_n^*(x)]=\underset{y(x)\in M}{extr}\int_{x_0}^{x_1}F(x,y_1,...,y_n,y_1',y_2',...,y_n')dx \qquad (4)$$

bo'lsin.

Qo'yilgan masala, funksionallarning shartli ekstremumini topish haqidagi masalalar sirasiga kiradi.

1.2. Masala yechimini izlash tartibi (sxemasi). Sxema ekstremumning zaruriy sharti, birinchi variatsiyaning ekstremum beruvchi funksiyada nolga tengligiga, ya'ni $\delta J = 0$ shartga tayanadi.

Ma'lumki, (4) masala, bir o`zgaruvchili bir necha funksiyalarga bog`liq funksionalning ekstremumi haqidagi masaladan (4-ma'ruzaga q.) faqat (2) chekli bog`lanishlar mavjudligi bilan farq qiladi. Shuning uchun, funksionalning birinchi variatsiyasi uchun,

$$\delta J = \int_{x_0}^{x_1} \sum_{i=1}^{n} [F_{y_i} - \frac{d}{dx} F_{y_i'}] \delta y_i(x) dx \qquad (5)$$

ifodadan foydalanamiz.

Modomiki, $y_i(x)$ funksiyalar (2) chekli bog`lanishlarni qanoatlantirishlari shart ekan,

$$\delta\varphi_i = \sum_{i=1}^{n} [\frac{\partial \varphi_j}{\partial y_i}]\Big|_{y^*(x)} \delta y_i(x) = 0, \; j = \overline{1,m}, \qquad (6)$$

bu yerda $y^*(x)$ - funksional ekstremumga erishadigan egri chiziq bo`lib, δy_j variatsiya x ning $[x_0, x_1]$ oraliqda tanlangan qiymatida hisoblangan.

Shunday qilib, $\delta y_i(x)$ variatsiyalarning faqat $n-m$ tasini ixtiyoriy deb hisoblash mumkin (masalan, $\delta y_{m+1}(x),...,\delta y_n(x)$ larni), qolganlari esa, (6) shartlardan aniqlanadi.
Endi (6) tenglamalarning har birini qandaydir $\lambda_j(x)$ funksiyaga hadma-had ko`paytirib va $x_0$ dan $x_1$ gacha bo`lgan chegaralarda integrallab,

$$\int_{x_0}^{x_1} \lambda_j(x) \sum_{i=1}^{n} \frac{\partial \varphi_j}{\partial y_i} \delta y_i(x) dx = 0, \quad j = \overline{1,m} \qquad (7)$$

munosabatlarni olamiz.

(5) va (7) munosabatlarni hadma-had qo`shsak,

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} [F_{y_i} + \sum_{j=1}^{m} \lambda_j(x) \frac{\partial \varphi_j}{\partial y_i} - \frac{d}{dx} F_{y_i'}] \delta y_i(x) dx = 0 \qquad (8)$$

hosil bo`ladi.

Agar

$$F^*(x, y, y') = F(x, y, y') + \sum_{j=1}^{m} \lambda_j(x) \varphi_j(x, y) \qquad (9)$$

belgilashni kiritsak, bunda $F^*(x, y, y')$ - *Lagranj funksiyasi,* $\lambda_j(x), j = \overline{1,m}$ - *Lagranj ko'paytuvchilari* deb ataladi, oxirgi tenglamani,

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} [F_{y_i}^* - \frac{d}{dx} F_{y_i'}^*] \delta y_i(x) dx = 0 \qquad (10)$$

Ko`rinishda yozish mumkin.

m ta $\lambda_1(x),...,\lambda_m(x)$ ko`paytuvchilarni shunday tanlaylikki, ular $y^*(x)$ egri chiziq bilan birga m ta

$$F_{y_i}^* - \frac{d}{dx} F_{y_i'}^* = 0, \quad i = \overline{1,m} \qquad (11)$$

Eyler tenglamalari sistemasini qanoatlantirsin.

Bunday qilishning imkoniyati bor, chunki (11) sistema, (9) belgilashni hisobga olganda,

$$F_{y_i}+\sum_{j=1}^{m}\lambda_j(x)\frac{\partial\varphi_j}{\partial y_i}-\frac{d}{dx}F_{y_i'}=0,\quad i=\overline{1,m}$$

ko`rinishni oladi.

Ravshanki, (11) sistema $\lambda_j(x)$ larga nisbatan chiziqli va uning determinanti noldan farqli (masalaning qo'yilishidagi c) bandga ko'ra), demak, u

$$\lambda_1(x),...,\lambda_m(x)$$

yechimga ega.

$\lambda_1(x),...,\lambda_m(x)$ ko'paytuvchilar yuqoridagidek tanlanganda, (10) shart, quyidagi,

$$\int_{x_0}^{x_1}\sum_{i=m+1}^{n}[F_{y_i}^{\ *}-\frac{d}{dx}F_{y_i'}^{\ *}]\delta y_i(x)dx=0 \qquad (12)$$

ko'rinishni oladi, bunda $\delta y_{m+1}(x),...,\delta y_n(x)$ variatsiyalar o'zaro bog'lanmagan. U holda, variatsion hisobning asosiy masalasiga asosan (uni qo'llash uchun variatsiyalarning navbat bilan bittasini ixtiyoriy deb, qolganlarini nolga teng deb olish mumkin),

$$F_{y_i}^{\ *}-\frac{d}{dx}F_{y_i'}^{\ *}=0,\quad i=m+1,..n \qquad (13)$$

munosabatlarga ega bo'lamiz.

Nihoyat, (11) va (13) larni hisobga olib, $y^*(x)$ egri chiziq va Lagranj ko'paytuvchilari

$$F_{y_i}^{\ *}-\frac{d}{dx}F_{y_i'}^{\ *}=0,\quad i=\overline{1,n} \qquad (14)$$

Eyler tenglamalari sistemasini qanoatlantirishi zarur, degan xulosa qilish mumkin.

Shunday qilib, $2n$ ta (1) chegaraviy shartlarni hisobga olgan holda, $n+m$ ta (14) va (2) tenglamalardan $y^*(x)=(y_1^*(x),...,y_n^*(x))^1$ vektor funksiya va $\lambda_1(x),...,\lambda_m(x)$ Lagranj ko'paytuvchilari topiladi.

Bayon qilingan natijani quyidagicha ifodalaymiz.

1-t e o r e m a((4) masalada ekstremumning zaruriy sharti). Agar (1) cheegaraviy shartlarni va (2) chekli bog'lanishlarni qanoatlantiruvchi $y^*(x)=(y_1^*(x),...,y_n^*(x))^1$ vector funksiyada, bunda $y_i(x)\in C^1[x_0,x_1],\ i=\overline{1,n},$ (3) funksional ekstremumga erishsa, $y_1^*(x),...,y_n^*(x)$ funksiyalar.

$$J^*[y_1,...,y_n]=\int_{x_0}^{x_1}F^*(x,y_1,...,y_n,y_1',...,y_n')dx=\int_{x_0}^{x_1}[F(x,y_1,...,y_n,y_1',...,y_n')+\sum_{j=1}^{m}\lambda_j(x)\varphi_j(x,y_1,...,y_n)]dx$$

funksional uchun tuzilgan

$$F_{y_i}^{\ *}-\frac{d}{dx}F_{y_i'}^{\ *}=0,\quad i=\overline{1,n}$$

Eyler tenglamalari sistemasini qanoatlantiradi.

Izohlar.

1. 1- teoremaga asosan, shartli ekstremumga qo`yilagan (4) masalani yechish bog`lanishlar qatnashmaganda $J^*[y_1,...,y_n]$ funksionalning ekstremumini tekshirishga keltiriladi.
2. Bog`lanishlardan qutilishning keltirilgan usuli, funksiyalarning shartli ekstremumini topishdagi Lagranj ko`paytuvchilari usuliga o`xshashdir.
3. (2) bog`lanishlar mexanikada *golonom* bog`lanishlar deyiladi.
4. Umumiy holda

$$\overline{F}^*(x,y,y) = F(x,y,y) + \sum_{j=1}^{m} \lambda_j(x)\varphi_j(x,y)$$

umumlashgan Lagranj funksiyasidan foydalaniladi. Bunda $\lambda_0(x) \equiv 0$ va $\lambda_0(x) \neq 0$ hollar alohida qaraladi.

1.3. (4) masalada shartli ekstremumning zaruriy shartini qo`llash algoritmi.

1. $F^{**}(x,y,y) = F(x,y,y) + \sum_{j=1}^{m} \lambda_j(x)\varphi_j(x,y)$

Lagranj funksiyasini tuzish, $\lambda_1(x),...,\lambda_m(x)$- Lagranj ko`paytuvchilari.

2. (14) Eyler tenglamalari sistemasi va (2) bog`lanishlar shartlarini yozish:

$$F_{y_i}^* - \frac{d}{dx} F_{y_i'}^* = 0, \quad i = \overline{1,n}$$

$$\varphi_j(x, y_1,..., y_n) = 0, \ j = \overline{1,m}.$$

3.Eyler tenglamalari sistemasining

$$y_i(x) = y_i(x, c_1, c_2,..., c_{2n}), \ i = \overline{1,n} \ ,$$

umumiy yechimini va $\lambda_1(x),...,\lambda_m(x)$ Lagranj ko`paytuvchilari uchun ifodalarni topish.

4. $c_1, c_2,..., c_{2n}$ o`zgarmaslarni

$$y_{i0} = y_i(x_0, c_1, c_2,..., c_{2n}), \ i = \overline{1,n}$$

$$y_{i1} = y_i(x_0, c_1, c_2,..., c_{2n}), \ i = \overline{1,n}$$

chegaraviy shartlardan topish va $y^*(x) = (y_1^*(x),..., y_n^*(x))^1$ ekstremal uchun ifoda yozish (ekstremalni yozish).

1-misol. Ushbu

$$J[y_1, y_2] = \int_0^{\pi/2} [y_1^2 + y_2^2 - y_1'^2 - y_2'^2] dx$$

Funksionalning

$$y_1(0) = 1. y_2(0) = -1, y_1(\frac{\pi}{2}) = 1, y_2(\frac{\pi}{2}) = 1$$

chegaraviy shartlarni va

$$y_1 - y_2 - 2\cos x = 0$$

bog`lanishlarni qanoatlantiradigan ekstremalini toping.

Y e ch i l i sh i. 1. Lagranj funksiyasini tuzamiz. Modomiki,

$$F = y_1^2 + y_2^2 - y_1'^2 - y_2'^2, y_1(x,y) = y_1 - y_2 - 2\cos x, \ m = 1$$

ekan,

$$F^{*}=F+\lambda_1(x)y_1(x,y)=y_1^2+y_2^2-y_1'^2-y_2'^2+\lambda_1(x)[y_1-y_2-2\cos x]$$

bo`ladi.

2.Eyler tenglamalari sistemasi va bog`lanishlar tenglamasini yozamiz. Buning uchun,

$$F_{y_1}^{*}=2y_1+\lambda_1(x),\ F_{y_1'}^{*}=-2y_1',\ \frac{d}{dx}F_{y_1'}^{*}=-2y_1'',$$

$$\overline{F_{y_1}^{*}}=2y_1-\lambda_1(x),\ F_{y_2'}^{*}=-2y_1',\ \frac{d}{dx}F_{y_2'}^{*}=-2y_1'',$$

ifodalardan foydalansak, quyidagi

$$F_{y_i}^{*}-\frac{d}{dx}F_{y_i'}^{*}=2y_1+\lambda_1(x)+2y_1''=0$$

$$F_{y_{2i}}^{*}-\frac{d}{dx}F_{y_2'}^{*}=2y_2+\lambda_1(x)+2y_2''=0$$

$$y_1-y_2-2\cos x=0$$

munosabatlarni hosil qilamiz.

3. Sistemaning umumiy yechimini topamiz. Sistemaning birinchi ikkita tenglamasini hadma- had qo`shib,

$$2(y_1''+y_2'')+2(y_1+y_2)=0$$

ekanligini olamiz. Yangi, $z=y_1+y_2$ belgilash kiritib,

$$z''+z=0$$

tenglamani hosil qilamiz. Uning xarakteristik tenglamasi

$$k^2+1=0$$

Bo`lib, u $k_{1,2}=\pm i$ ildizlarga ega ekanligini hisobga olsak,

$$z(x)=c_1\cos x+c_2\sin x=y_1+y_2$$

Bo`ladi.

Ikkinchidan, hosil qilingan sistemaning uchinchi tenglamasidan

$$2\cos x=y_1-y_2$$

Bo`lishi kelib chiqadi. Oxirgi tenglamalarni qo`shib,

$$2y_1=c_1\cos x+c_2\sin x+2\cos x \text{ yoki } y_1(x)=\frac{c_1}{2}\cos x+\frac{c_2}{2}\sin x+\cos x$$

ekanligini olamiz. U holda

$$y_2(x)=y_1(x)-2\cos x,$$

$$\lambda_1(x)=2y_2(x)+2y_2''(x)$$

Bo`lishi kelib chiqadi.

4. c_1 va c_2 ixtiyoriy o`zgarmaslarni chegaraviy shartlardan topamiz:

$$y_1(0)=\frac{c_1}{2}+1=1,$$

$$y_1(\frac{\pi}{2})=\frac{c_2}{2}=1$$

$$y_2^*(x) = y_1^*(x) - 2\cos t = \sin x - \cos x,$$
$$\lambda_1(x) = 2\sin x - 2\cos x - 2\sin x + 2\cos x = 0$$

bu yerdan $c_1 = 0,\ c_2 = 2$ va $y_1^*(x) = \sin x + \cos x,$

Shuni e’tirof etish kerakki, masalada chegaraviy shartlar va bog`lanishlar tenglamalari muvofiqlashtirilgan, chunki

$$y_1(0) - y_2(0) - 2\cos 0 = 0,$$
$$y_1(\frac{\pi}{2}) - y_2(\frac{\pi}{2}) - 2\cos\frac{\pi}{2} = 0\,.$$

Bu faktni masalani yechishdan oldin tekshirish lozim. Shunday qilib, masalada

$$y_1^*(x) = \sin x + \cos x,\ y_2^*(x) = \sin x - \cos x$$

ekstremal topildi.

2. Differensial bog’lanishli variatsion masalalar.

2.1. Masalaning qo’yilishi. Quyidagi shartlarni qanoatlantiruvchi joiz $y(x) = (y_1(x),...,y_n(x))'$ vector- funksiyalar to`plami m ni qaraymiz:

a) $y_i(x)$ funksiyalar $[x_0, x_1]$ kesmada aniqlangan va uzluksiz differensiallanuvchi bo`lsin, ya’ni $y_i(x) \in C^1[x_0, x_1],\ i = \overline{1,n},\ x_0, x_1$ lar berilgan;

b) $y_i(x)$ funksiyalar,

$$y_i(x_0) = y_{i0},\ \ y_i(x_1) = y_{i1},\ i = \overline{1,n} \qquad (15)$$

chegaraviy shartlarni qanoatlantiradi, bunda $y_{i0}, y_{i1},\ i = \overline{1,n}$ sonlar berilgan, ya’ni egri chiziqlarning har biri ikkita mahkamlangan(qo`zg`olmas) chegara nuqtalardan o`tadi;

c) $y_i(x)$ funksiyalar barcha $x \in [x_0, x_1]$ lar uchun

$$\varphi_j(x, y_1,..., y_n, y_1',..., y_n') = 0,\ \ j = \overline{1,m};\ m < n \qquad (16)$$

differensial bog`lanishlarni qanoatlantiradi, bunda, $\varphi_j(x, y_1,..., y_n, y_1',..., y_n')$ funksiyalar barcha o`zgaruvchilari bo`yicha uzluksiz differensiallanuvchidir.

Bundan tashqari, (16) tenglamalar o`zaro bog`lanmagan, ya’ni

$$rang\begin{Bmatrix} \dfrac{\partial\varphi_1}{\partial y_1'} \ldots\ldots \dfrac{\partial\varphi_1}{\partial y_n'} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \dfrac{\partial\varphi_m}{\partial y_1'} \ldots\ldots \dfrac{\partial\varphi_m}{\partial y_n'} \end{Bmatrix} = m$$

M to’plamda

$$J[y_1, y_2,..., y_n] = \int_{x_0}^{x_1} F(x, y_1,..., y_n, y_1', y_2',..., y_n')dx \qquad (17)$$

funksional berilgan, bu yerda $F(x, y_1,..., y_n, y_1', y_2',..., y_n')$ funksiya o'zining barcha argumentlari bo'yicha ikkinchi tartibigacha (birinchi va ikkinchi tartibli) uzluksiz xususiy hosilalarga ega deb faraz qilinadi.

M to'plamga tegishli joiz $y(x)$ vector funksiyalar ichida shunday $y^*(x) = (y_1^*(x),..., y_n^*(x))^1$ vector funksiyani topish kerakki, unda (17) funksional ekstremumga erishsin, ya'ni

$$J[y_1^*(x), y_2^*(x),..., y_n^*(x)] = \underset{y(x)\in M}{extr} \int_{x_0}^{x_1} F(x, y_1,..., y_n, y_1', y_2',..., y_n')dx \qquad (18)$$

bo'lsin.

Bu masala - *Lagranj masalasi* deb ataladi.

2.2. Masala yechimini izlash tartibi (sxemasi).

Sxema ekstremumning zaruriy sharti, birinchi variatsiyaning ekstremum beruvchi funksiyada nolga tengligiga, ya'ni $\delta J = 0$ shartga tayanadi. (18) masalaning (4) masaladan asosiy farqi shundaki, undagi (16) bog`lanishlar tenglamalarida hosilalar qatnashmoqda.

Bu yerda ham, xuddi (4) masaladagi kabi, funksionalning birinchi variatsiyasi ifodasi

$$\delta J = \int_{x_0}^{x_1} \sum_{i=1}^{n} [F_{y_i} - \frac{d}{dx} F_{y_i'}]\delta y_i(x)dx \qquad (19)$$

ko`rinishda bo`ladi, bunda (16) differrensial bog`lanishlar qatnashganligidan, $\delta y_i(x)$ variatsiyalar ixtiyoriy bo`la olmaydi. Shu sababli, bu bosqichda variatsion hisobning asosiy lemmasini qo`llash mumkin emas.

$\delta y_i(x)$ variatsiyalar orasidagi t

$$\delta\varphi_i = \sum_{i=1}^{n} [\frac{\partial\varphi_j}{\partial y_i}]\Big|_{y^*(x)} \delta y_i(x) + \sum_{i=1}^{n} [\frac{\partial\varphi_j}{\partial y_i'}]\Big| \delta y_i'(x) = 0,\ j = \overline{1,m}, \qquad (20)$$

bu yerda xususiy hosilalar (17) funksional ekstremumga erishadigan $y^*(x)$ egri chiziqda hisoblanadi.

(20) tenglamalarning har birini hadma-had hozircha noma'lum bo'lgan $\lambda_j(x)$ ko'paytuvchiga ko'paytiramiz va x_0 dan x_1 gacha bo'lgan oraliqda integrallab,

$$\int_{x_0}^{x_1} \lambda_j(x) \sum_{i=1}^{n} \frac{\partial\varphi_j}{\partial y_i} \delta y_i(x)dx + \int_{x_0}^{x_1} \lambda_j(x) \sum_{i=1}^{n} \frac{\partial\varphi_j}{\partial y_i'} \delta y_i'(x)dx = 0, \quad j = \overline{1,m} \qquad (21)$$

munosabatlarni hosil qilamiz.

(21) munosabatlardagi ikkinchi integralning har bir qo'shiluvchisini bo'laklab integrallab va $\delta y_i(0) = \delta y_i(x_1) = 0,\ i = \overline{1,m}$ ekanligini hisobga olib (chunki chegaralar qo'zg'almas),

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} \{\lambda_j(x) \frac{\partial\varphi_j}{\partial y_i} - \frac{d}{dx}[\lambda_j(x) \frac{\partial\varphi_j}{\partial y_i'}]\}\delta y_i(x)dx = 0, \quad j = \overline{1,m} \qquad (22)$$

bo'lishini ko'ramiz.

Endi (22) va δJ ning (19) ifodasidagi $\delta J = 0$ shartlarni qo'shib,

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} \{F_{y_i} + \sum_{j=1}^{m} \lambda_j(x) \frac{\partial\varphi_j}{\partial y_i} - \frac{d}{dx}[F_{y_i'} + \sum_{j=1}^{m} \lambda_j(x) \frac{\partial\varphi_j}{\partial y_i'}]\}\delta y_i(x)dx = 0, \qquad (23)$$

tenglamani hosil qilamiz. Agar

$$F^{**}(x,y,y') = F(x,y,y') + \sum_{j=1}^{m} \lambda_j(x)\varphi_j(x,y,y') \qquad (24)$$

belgilashni kiritsak, bunda $F^{*}(x,y,y')$ - Lagranj funktsiyasi deyiladi, (23) tenglama,

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} [F_{y_i} - \frac{d}{dx} F_{y_i'}^{*}] \delta y_i(x) dx = 0, \qquad (25)$$

ko'rinishda yoziladi.

m ta $\lambda_1(x),...,\lambda_m(x)$ ko'paytuvchilarni shunday tanlaymizki, ular $y^{*}(x)$ egri chiziq bilan birga m ta

$$F_{y_i}^{*} - \frac{d}{dx} F_{y_i'}^{*} = 0, \quad i = \overline{1,m} \qquad (26)$$

Eyler tenglamasini qanoatlantirsin.

Agar tenglamalarni kengaytirilgan holda yozsak, ular $\lambda_1(x),...,\lambda_m(x)$ larga nisbatan chiziqli differrensial tenglamalar sistemasidan iborat bo’ladi va masalaning qo'yilishidagi c) bandga asosan, yechimga ega.

Lagranj ko'paytuvchilari yuqoridagi usul bilan tanlanganda (25) shart

$$\int_{x_0}^{x_1} \sum_{i=m+1}^{n} [F_{y_i}^{*} - \frac{d}{dx} F_{y_i'}^{*}] \delta y_i(x) dx = 0 \qquad (27)$$

ko'rinishni oladi, bunda $\delta y_{m+1}(x),...,\delta y_n(x)$ variatsiyalar (o'zaro) bog'lanmagan.

Oxirgi tenglikda $\delta y_{m+1}(x),...,\delta y_n(x)$ variatsiyalardan bittasini ixtiyoriy, qolganlarini nolga teng deb olib, hamda variatsion hisobning asosiy lemmasini tadbiq qilib,

$$F_{y_i}^{*} - \frac{d}{dx} F_{y_i'}^{*} = 0, \quad i = \overline{m+1,n} \qquad (28)$$

tenglamalar sistemasini hosil qilamiz.

(26) va (28) larni hisobga olib, $y^{*}(x)$ egri chiziq va Lagranj ko'paytuvchilari

$$F_{y_i}^{*} - \frac{d}{dx} F_{y_i'}^{*} = 0, \quad i = \overline{1,n} \qquad (29)$$

Eyler tenglamalari sitemasini qanoatlantirishi zarur degan xulosa qilishimiz mumkin.

Shunday qilib, $n+m$ ta (29) va (16) tenglamalar va ta (15) chegaraviy shartlardan $y*(x) = (y_1*(x),...,y_n*(x))^1$

vector- funksiya va $\lambda_1(x),...,\lambda_m(x)$ Lagranj ko’paytuvchilari topiladi.

2-te o r e m a ((18) masalada ekstremumning zaruriy shartlari). Agar (15) chegaraviy shartlar va (16) differensial bog’lanishlarni qanoatlantiruvchi $y*(x) = (y_1*(x),...,y_n*(x))^1$ vector funksiyada (17) funksional ekstremumga erishsa, $y_1*(x),...,y_n*(x)$ funksiyalar,

$$J^{*}[y_1,...,y_n] = \int_{x_0}^{x_1} F^{*}(x,y_1,...,y_n,y_1',...,y_n')dx =$$

$$= \int_{x_0}^{x_1} [F(x,y_1,...,y_n,y_1',...,y_n') + \sum_{j=1}^{m} \lambda_j(x) y_j(x,y_1,...,y_n,y_1',...,y_n')]dx$$

funksional uchun tuzilgan,

$$F_{y_i}^{\ *} - \frac{d}{dx}F_{y_i^{'}}^{\ *} = 0, \quad i = \overline{1,n}$$

Eyler tenglamalari sistemasini qanoatlantiradi.

Izohlar: 1. 2-teoremaga asosan, shartli ekstremumga qo'yilgan (18) masalani yechish - bog'lanishlar qatnashmaganda $J^*[y_1,...,y_n]$ funksionalning ekstremumini tekshirishga keltiriladi.

2. Mexanikada (16) ko'rinishdagi bog'lanishlar *golonom bo'lmagan* bog'lanishlar deyiladi.

3. Umumiy holda umumlashgan Lagranj funksiyasidan foydalaniladi.

2.3. (18) masalada ekstremumning zaruriy shartini qo'llash algoritmi.

1. $F^*(x,y,y') = F(x,y,y') + \sum_{j=1}^{m}\lambda_j(x)y_j(x,y,y')$

Lagranj funksiyasini tuzish, bunda $\lambda_1(x),...,\lambda_m(x)$ - Lagranj ko'paytuvchilari.

2. (29) Eyler tenglamalari sistemasi va (16) bog'lanishlar tenglamalarini yozish:

$$F_{y_i}^{\ *} - \frac{d}{dx}F_{y_i^{'}}^{\ *} = 0, \quad i = \overline{1,n}$$

$$\varphi_j(x, y_1,...,y_n, y_1^{'},...,y_n^{'}) = 0, \ \ j = \overline{1,m}$$

3. Eyler tenglamalari sistemasining $y_i(x) = y_i(x, c_1, c_2,...,c_{2n}), \ i = \overline{1,n}$, umumiy yechimini hamda $\lambda_1(x),...,\lambda_m(x)$ Lagranj ko`paytuvchilari uchun ifodalarni topish.

4. $c_1, c_2,...,c_{2n}$ o`zgarmaslarni

$$y_{i0} = y_i(x_0, c_1, c_2,...,c_{2n}), \ \ i = \overline{1,n},$$

$$y_{i1} = y_i(x_0, c_1, c_2,...,c_{2n}), \ \ i = \overline{1,n}$$

chegaraviy shartlardan topish va $y^*(x) = (y_1^*(x),...,y_n^*(x))^1$ ekstremal uchun ifoda (ekstremalni) yozish.

2-misol . Ushbu

$$J[y_1, y_2] = \int_0^1 [y_1^2 + y_2^2 - y_1^{'} - y_2^{'2}]dx$$

funksionalning

$$y_1(0) = 2, y_2(0) = 0, y_1(1) = 2ch1, y_2(1) = 2sh1$$

chegaraviy shartlarni va

$$y_1' - y_2 = 0$$

differensial boglanishni qanoatlantruvchi ekstremalini toping.

E ch i l i sh i. 1.Lagranj funksiyasini tuzamiz: $F(x,y,y') = y_1^{'2} + y_2^{'2}, Y_1(x,y,y') = y_1' - y_2, \ m = 1$

bo'lganligidan,

$$F^*(x,y,y') = y_1^{'2} + y_2^{'2} + \lambda_1(x)\left[y_1^{'} - y_2\right].$$

2.Eyler tenglamalari sestemasini va boglanishlar tenglamasini yozamiz;

$$F_{y_1}{}^* = 0,, F_{y'_1}{}^* = 2y_1'' + \lambda_1(x), \frac{d}{dx} F_{y'_1}{}^* = 2y''_1 + \lambda_1(x)$$

$$F^*{}_{y_2} = -\lambda_1(x), F_{y'_2}{}^* = 2y_2'', \frac{d}{dx} F^*{}_{y_2^1} = 2y_2''$$

ekanligini hisobga olsak,

$$\begin{cases} F_{y_1}{}^* - \dfrac{d}{dx} F_{y'_1}{}^* = 0 \Rightarrow \begin{cases} -2y_1'' - \lambda_1'(x) = 0 \\ -\lambda_1(x) - 2y_2'' = 0 \end{cases} \\ F_{y_2}{}^* - \dfrac{d}{dx} F_{y_2^1}{}^* = 0 \Rightarrow \end{cases}$$

$$y_1' - y_2 = o$$

bo’ladi.

3. Hosil qilingan sistemaning umumiy echimini topamiz. Sistemaning daslabki ikkita tenglamalaridan,

$$\lambda_1(x) = -2y_2'', \lambda_1'(x) = -2y''', 2y_2'' = -\lambda'_1(x) = 2y_2'''$$

ekanligini olamiz, uchinchi tenglamadan esa,

$$y_1' = y_2, y_1'' = y_2'$$

bo’lishi kelib chiqadi. U holda, $2y_1'' = 2y_2' = 2y_2'''$ yoki $y_2''' - y_2' = 0$ bo’ladi. Oxirgi tenglamaning $\lambda^3 - \lambda = 0,, \lambda(x^2 - 1) = 0$ xarakteristik tenglamasi $\lambda_1 = 1, x_2 = -1, \lambda_3 = 0$ ildizlarga ega, shuning uchun,

$$y_2(x) = c_1 e^x + c_2 e^{-x} + c_3,$$

$$y_1(x) = \int y_2(x) dx = c_1 e^x - c_2 e^{-x} + c_3 x + c_4,$$

$$\lambda_1(x) = -2y_2''(x).$$

4. $c_1, c_2, c_3, c_4,$ o’zgarmaslarni chegaraviy shartlardan topamiz;

$$\begin{cases} y_1(0) = c_1 - c_2 + c_4 = 2 \\ y_2(0) = c_1 + c_2 + c_3 = 0, \\ y_1(1) = c_1 e - c_2 e^{-1} + c_3 + c_4 = ch1.ch1 = \dfrac{e + e^{-1}}{2}. \\ y_2(1) = c_1 e + c_2 e^{-2} + c_3 = 2sh1, sh1 = \dfrac{e - e^{-1}}{2} \end{cases}$$

Bu erdan $c_1 = 1, c_2 = -1; c_3 = c_4 = 0$ bo’lishi kelib chiqadi.

Natijada, $y^*(x) = (y^*{}_1(x), y^*{}_2(x))^T$ ekstremal, $y^*{}_1(x) = e^x + e^{-x}; y^*{}_2(x) = e^x - e^{-x}$ ko’rinishda bo’ladi va $\lambda_1(x) = -2y''{}_2(x) = -2e^x + 2e^{-x}$.

9-ma'ruza. Izoperimetrik masalalar

Reja

1. Lagranj funksiyasi. Lagranj ko'paytuvchilari qoidasi.
2. Ikkinchi tartibli zaruriy shartlar va yetarli shartlar.

Tayanch iboralar. Izoperimetrik masala, izoperimetrik shart, funksiya, $C^1[x_0, x_1]$ fazo, kuchsiz va kuchli ekstremumlar, Lagranj funksiyasi, Lagranj ko'paytuvchilari, Eyler tenglamasi, Lejandr va Yakobi shartlari.

$Q \subset R^3$ – ochiq to'plam, $S = \{(x,y) : (x,y,z) \in Q\}$, $F_i(x,y,z)$, $i = \overline{0,m}$, funksiyalar Q da uzluksiz, $P_0(x_0, y_0)$, $P_1(x_1, y_1)$ – S to'plamning belgilangan nuqtalari, $x_0 < x_1$ bo'lsin.
Quyidagi:

$$J_0[y] = \int_{x_0}^{x_1} F_0(x, y, y')dx \to \min(\max), \qquad (1)$$

$$J_i[y] = \int_{x_0}^{x_1} F_i(x, y, y')dx = a_i, \; i = \overline{1,m}, \; (a_i = const), \qquad (2)$$

$$y(x_0) = y_0, \; y(x_1) = y_1, \; (x, y(x), y'(x)) \in Q, \; x \in [x_0, x_1], \; y(x) \in C^{(1)}[x_0, x_1]. \qquad (3)$$

ekstremal masalani qaraymiz.
Bu masalaga izoperimetrik masala deyiladi. (2) shartlarga esa, izoperimetrik shartlar (bog`lanishlar) deyiladi. (2), (3) shartlarni qanoatlantiruvchi $y(x) \in C^1[x_0, x_1]$ funksiyalar izoperimetrik masalada joyiz funksiyalar (chiziqlar)dan iborat.
Kuchsiz va kuchli ekstremumlar ta'rifi variatsion hisobning asosiy masalasidagiga o'xshash beriladi: agar $y^*=y^*(x)$ joyiz funksiyaning biror $V_0(y^*, \varepsilon)$ nolinchi tartibli atrofiga tegishli barcha joyiz funksiyalar uchun

$$J_0[y^*] \le J_0[y] \quad (J_0[y^*] \ge J_0[y]) \qquad (4)$$

bajarilsa, $y_0(x)$ - izoperimetrik masalada kuchli lokal minimal (maksimal) dir; kuchsiz minimal (maksimal) uchun esa, (4) munosabat $y^*=y^*(x)$ joyiz chiziqning biror $V_1(y^*, \varepsilon)$ birinchi tartibli atrofida yotuvchi barcha $y=y(x)$ joyiz chiziqlar uchun bajariladi.
Izoperimetrik masala shartli ekstremum uchun qo'yilgan variatsion masaladir. Bu masala uchun ekstremum zaruriy shartlarini bayon qilishda,

$$L(x, y, y', \lambda) = \sum_{i=0}^{m} \lambda_i F_i(x, y, y'), \quad \lambda = (\lambda_0, \lambda_1, ..., \lambda_m) \qquad (5)$$

Lagranj funksiyasidan foydalanamiz. $\lambda_0, \lambda_1, ..., \lambda_m$ sonlarga Lagranj ko'paytuvchilari deyiladi.

1-teorema. Faraz qilaylik, $F_i(x, y, y') \in C^{(1)}(Q)$, $i = \overline{0,m}$, bo'lsin. Agar $y^* = y^*(x) \in C[x_0, x_1]$ – (1)-(3) masalada kuchsiz ekstremal bo'lsa, bir vaqtda nolga teng bo'lmagan shunday $\lambda^*_0, \lambda^*_1, ..., \lambda^*_m$ sonlar topiladiki, $y^*(x)$ funksiya, $L(x, y, y', \lambda^*)$ $(\lambda^* = (\lambda^*_0, \lambda^*_1, ..., \lambda^*_m))$ Lagranj funksiyasi uchun tuzilgan,

$$L_y - \frac{d}{dx} L_{y'} = 0 \qquad (6)$$

Eyler tenglamasini qanoatlantiradi**.** Agar

$$G_i^*(x) = F_{iy}(x, y^*(x), y^{*'}(x)) - \frac{d}{dx} F_{iy'}(x, y^*(x), y^{*'}(x)), \; i = \overline{1,m}, \qquad (7)$$

funksiyalar chiziqli bog'langan bo'lsalar, $y^*=y^*(x)$ bo'ladi.

Isboti. ***Berilgan***

$$J_i[y]=\int_{x_0}^{x_1} F_i(x,y,y')dx,\ \ i=\overline{0,m},$$

funksionallarning $y^*=y^*(x)$ nuqtadagi variatsiyalarini hisoblaymiz.
Ta’rifga ko’ra,

$$\delta J_i[y^*,h]=\frac{\partial}{\partial\alpha}J_i[y^*+\alpha h]_{\alpha=0}=\int_{x_0}^{x_1}[F^*_{iy}(x)h(x)+F^*_{iy'}(x)h'(x)]dx, \tag{8}$$

$i=0,1,...,m,$

bu yerda $F^*_{iy}(x)=F_{iy}(x,y^*(x),y^{*'}(x)),\quad F^*_{iy'}(x)=F_{iy'}(x,y^*(x),y^{*'}(x)).$

$C^{(1)}[x_0,x_1]$ fazoning $C^{(1)}[x_0,x_1]=\{h(x)\in C^{(1)}[x_0,x_1]: h(x_0)=h(x_1)=0\}$ qism fazosini R^{m+1} fazoga akslantiruvchi

$Ah=(\delta J_0[y^*,h],\delta J_1[y^*,h],...,\delta J_m[y^*,h])$

chiziqli akslantirishni qaraymiz.
Bu yerda ikki hol bo’lishi mumkin:
A akslantirish $C_0^{(1)}[x_0,x_1]$ ni R^{m+1} ga to’la akslantiradi, ya’ni A akslantirishning obrazi $\operatorname{Im}A=R^{m+1}$ (regulyar hol).
A akslantirish $C_0^{(1)}[x_0,x_1]$ ni R^{m+1} ning qismiga akslantiradi (aynan bo’lgan hol).
Dastlab teorema tasdig’ining aynan bo’lgan holda to’g’riligini ko’rsatamiz.
Ma’lumki, chiziqli akslantirishda chiziqli fazoning obrazi chiziqli qism fazodan iborat bo’ladi. Demak, aynan bo’lgan holda, A akslantirishning obrazi $\operatorname{Im}A=R^{m+1}$ fazoning hos qism fazosi bo’ladi. U holda funksional analizdan ma’lum faktlarga ko’ra[a , b], bir vaqtda nolga teng bo’lmagan shunday $\lambda^*_0,\lambda^*_1,...,\lambda^*_m$ sonlar topiladiki,

$$\sum_{i=0}^{m}\lambda^*_i z_i=0,\quad \forall z=(z_0,z_1,...,z_m)\in\operatorname{Im}A$$

tenglik bajariladi. Endi A operatorning aniqlanishi va $\delta J_1[y^*,h]$ variatsiya uchun (8) ifodalarni hiasobga olib,

$$\int_{x_0}^{x_1}\left(\sum_{i=0}^{m}\lambda^*_i(F^*_{iy}(x)h(x)+F^*_{iy'}(x)h'(x))\right)dx=0,\qquad \forall h(x)\in C^1[x_0,x_1]$$

tenglikka ega bo’lamiz. Dyubua-Reymon lemmasiga ko’ra, bu yerdan

$$\sum_{i=0}^{m}\lambda^*_i F^*_{iy}(x)-\frac{d}{dx}\left(\sum_{i=0}^{m}\lambda^*_i F^*_{iy'}(x)\right)\qquad \forall x\in[x_0,x_1] \tag{9}$$

munosabatni olamiz, ya’ni $y^*(x)$ kuchsiz ekstremal (6) tenglamani qanoatlantiradi.
Hosil qilingan (9) munosabatni (7) belgilashlarda

$$\sum_{i=0}^{m}\lambda^*_i G^*_i(x)=0\qquad \forall x\in[x_0,x_1]$$

ko’rinishda yozish mumkin. Bu yerdan esa, $G^0_i(x)$ funksiyalar chiziqli bog’lanmagan holda, $\lambda^*_0\neq 0$ ekanligi kelib chiqadi.
Endi regulyar holni qaraymiz va bunday hol bo’lishi mumkin emasligini ko’rsatamiz.
Shunday $h_j(x)\in C_0^{(1)}[x_0,x_1],\ \ j=\overline{0,m}$ fun ksiyalarni olamizki, $Ah_j=e_j$ bo’lsin, bu yerda $e_0=(1,0,...,0),\ e_2=(0,1,0,...,0),\ \ e_m=(0,0,...,1)-$ R^{m+1} dagi kanonik bazis. R^{m+1} dagi nol nuqtaning atrofini yana R^{m+1} ga F akslantirishni qaraymiz:

$$\Phi(\beta)=(\varphi_0(\beta_0,\beta_1,...,\beta_m),\ \varphi_0(\beta_0,\beta_1,...,\beta_m),...,\varphi_m(\beta_0,\beta_1,...,\beta_m)\ ),$$

bu yerda

$$\varphi_i(\beta_0,\beta_1,...,\beta_m)=J_i\left[y^*+\sum_{j=0}^{m}\beta_j h_j\right],\quad i=0,1,...,n.$$

Bu φ_i funksiyalar uzluksiz differensiallanuvchi va bunda

$$\frac{\partial\varphi_i(0)}{\partial\beta_j}=\delta J_i\left[y^*,h\right]=\delta_{ij}=\begin{cases}1,\ i=j\\0,\ i\neq j\end{cases},\ 0\le i,\ \ j\le m,$$

$$\Phi(0)=(a_0,a_1,...,a_m)=z^*,\ \ a_0=J_0\left[y^*\right].$$

Teskari funksiyaning mavjudligi haqidagi teoremaga ko'ra [a,b], shunday Φ^1 silliq akslantirish va *k>0* o'zgarmas mavjud bo'ladiki, yetarli kichik z-z* uchun

$$\left|\Phi^1(z)\right|\le k\left|z-z^*\right|$$

tengsizlik bajariladi. Hususiy holda, modul bo'yicha yetarli kichik ε son uchun shunday $\beta(\varepsilon)=(\beta_0(\varepsilon),\beta_1(\varepsilon),...,\beta_m(\varepsilon))$ vektor topiladiki, $\beta(\varepsilon)=\Phi^1(a_0+\varepsilon,a_1,...,a_m)$ tenglik bajariladi, ya'ni

$$\varphi_0(,\beta(\varepsilon))=a_0+\varepsilon\ \Leftrightarrow\ J_0\left[y^*+\sum_{j=0}^{m}\beta_j(\varepsilon)h_j\right]=J_0\left[y^*\right]+\varepsilon,$$

$$\varphi_i(,\beta(\varepsilon))=a_i\ \Leftrightarrow\ J_i\left[y^*+\sum_{j=0}^{m}\beta_j(\varepsilon)h_j\right]=a_{i,}\quad i=1,...,m$$

va bunda

$$\left|\beta(\varepsilon)\right|=\Phi^1(a_0+\varepsilon,a_1,...,a_m)\le k\left|\varepsilon\right|.$$

Shunday qilib, $y^*(x)$ joyiz funksiyaning ixtiyoriy $V_1(y^*,\varepsilon)$ birinchi tartibli atrofida shunday $y(x)=y^*(x)+\sum_{j=0}^{m}\beta_j(\varepsilon)h_j(x)$ joyiz funksiya mavjudki, uning uchun

$$J_0[y]-J_0\left[y^*\right]=\varepsilon$$

bajariladi. Bu yerda ε ning moduli bo'yicha yetarki kichik har xil ishorali qiymat qabul qilishini hisobga olsak, y^* ning lokal ekstremal ekanligiga zid xulosaga kelamiz. Olingan qarama-qarshilik A akslantirish uchun regular hol bo'lmasligini ko'rsatadi. Teorema isbotlandi.

Isbotlangan teoremaga, izoperimetrik masala uchun Lagranj ko'paytuvchilari qoidasi deyiladi. Agar $y^*(x)$ ekstremalga $\lambda_0^*\neq 0$ Lagranj ko'paytuvchisi mos kelsa, $\lambda_0=1$ deb olish mumkin.

8-ta'rif. $L(x,y,y',\lambda^*)$ $(\lambda^*=(\lambda^*{}_0,\lambda^*{}_1,...,\lambda^*{}_m)\neq 0)$ Lagranj funksiyasi uchun tuzilgan (6) Eyler tenglamasini qanoatlantiruvchi $y^*(x)$ joyiz funksiyaga (1)-(3) masalaning shartli- stasionar funksiyasi deyiladi.

Ma'ruzamiz so'ngida izoperimetrik masala uchun ikkinchi tartibli zaruriy va yetarli shartlar haqida qisqagina to'xtalib o'tamiz. (to'liqroq ma'lumot uchun masalan [6] ga qarang).

Faraz qilaylik, $F_i(x,y,y')\in C^2(Q),\ \ i=\overline{0,m},\ \ y^*(x)\in C^2[x_0,x_1]$ joyiz stasionar funksiya, $\lambda^*=(\lambda^*{}_0,\lambda^*{}_1,...,\lambda^*{}_m)$ unga mos Lagranj vektori, $\lambda_0^*\neq 0$ bo'lsin. U vaqtda

$$J[y]=\int_{x_0}^{x_1}L(x,y,y',\lambda^*)dx=\int_{x_0}^{x_1}\left(F_0+\sum_{i=0}^{m}\lambda_i^*F_i\right)dx$$

funksional y^* nuqtada

$$\delta^2J\left[y^*,h\right]=\int_{x_0}^{x_1}\left[A^*(x)h'^2+C^*(x)hh'+B^*(x)h^2\right]dx$$

ko’rinishdagi ikkinchi variatsiyaga ega, bu yerda
$A^*(x)=L_{y'y'}(x,y^*(x),y^{*'}(x),\lambda^*)$, $C^*(x)=L_{y'y}(x,y^*(x),y^{*'}(x),\lambda^*)$,
$B^*(x)=L_{yy}(x,y^*(x),y^{*'}(x),\lambda^*)$.
Quyidagi

$$\delta^2 J[y^*,h]=\int_{x_0}^{x_1}\left[A^*(x)h'^2+2C^*(x)hh'+B^*(x)h^2\right]dx\to\min(\max),$$

$$\delta J_i[y^*,h]=\int_{x_0}^{x_1}\left[F_{iy}(x,y^*(x),y^{*'}(x))-\frac{d}{dx}F_{iy'}(x,y^*(x),y^{*'}(x))\right]h(x)dx=0,\ i=\overline{1,m}$$

$$h(x_0)=0,\quad h(x_1)=0$$

izoperimetrik masalani qaraymiz. $h(x)=0$ funksiya bu masalaning yechimidir. Shu masala uchun Lagranj ko’paytuvchilari qoidasini qo’llab,

$$-\frac{d}{dx}\left(A^*(x)h'+C^*(x)h\right)+C^*(x)h'+B^*(x)+\sum_{i=1}^{m}\mu_i G_i^*(x)=0 \qquad (10)$$

tenglamaga ega bo’lamiz, bu yerda $G_i^*(x)$ funksiya (7) tenglik bilan aniqlanadi. (10) tenglamaga $y^*(x)$ stasioanar funksiyaga mos keluvchi *Yakobi tenglamasi* deyiladi.

9-ta’rif. Agar (10) Yakobi tenglamasi

$$\int_{x_0}^{x_1}G_i^*(x)h(x)dx=0,\quad i=\overline{1,m},\quad h(x_0)=h(\xi)=0$$

shartlarni qanoatlantiruvchi trivial (aynan nol) bo’lmagan yechimga ega bo’lsa, ξ nuqta x_0 nuqtaga qo’shma nuqta deyiladi.

Agar $G_i^*(x)$, $i=\overline{1,m}$ funksiya $[x_0,\xi_1]$, $x_0\le\xi_0\le\xi_1\le x_1$, kesmalarda chiziqli bog’lanmagan bo’lsalar, qo’shma nuqtalarni quyidagicha aniqlash mumkin: faraz qilaylik, $h_0(x)$ bir jinsli (ya’ni $\mu_i=0,\ i=\overline{1,m}$) Yakobi tenglamasining $h_0(x_0)=0,\ h_0'(x_0)=1$ shartlarini qanoatlantiruvchi yechimi bo’lsin; $h_j(x)$– bir jinsli bo’lmagan ($\mu_j=1,\ \mu_i=0,\ i\ne j$) Yakobi tenglamasining $h_i(x_0)=0,\ h_j'(x_0)=0,\ j=\overline{1,m}$ shartlarini qanoatlantruvchi yechimi bo’lsin; u vaqtda ξ nuqta faqat va faqat

$$H(\xi)=\begin{pmatrix} h_0(\xi)\ \dots\dots\dots\dots\dots\ h_m(\xi) \\ \int_{x_0}^{\xi}h_0G_1^*dx\ \dots\dots\ \int_{x_0}^{\xi}h_mG_1^*dx \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \int_{x_0}^{\xi}h_0G_m^*dx\dots\dots\ \int_{x_0}^{\xi}h_mG_m^*dx \end{pmatrix}$$

matrisa maxsus bo’lgan holdagina $(\det H(\xi)=0)$ x_0 nuqtaga qo’shma nuqta bo’ladi.

2-teorema. Faraz qilaylik, $F_i(x,y,y')\in C^{(3)}(Q)$, $i=\overline{0,m}$, $y^*(x)\in C^2[x_0,x_1]$ kuchsiz minimal (maksimal), $\lambda^*=(\lambda^*_0,\lambda^*_1,...,\lambda^*_m)$ – unga mos Lagranj vektori bo’lsin va $G_i(x)$, $i=\overline{1,m}$ funksiyalar ihtiyoriy $\xi\in(x_0,x_1)$ uchun $[x_0,\xi]$, $[\xi,x_1]$ kesmalarda chiziqli bog’lanmagan bo’lsin. U holda quyidagilar bajariladi:

a) Lejandr sharti: $L_{y'y'}(x,y^*(x),y^{*'}(x))\ge 0\ \ (\le 0)$
b) Yakobi sharti: (x_0,x_1) intervalda x_0 nuqtaga qo’shma nuqta mavjud emas.

3-teorema. Faraz qilaylik, quyidagilar bajarilsin:

1) $Q = S \times R,\ S \subset R^2$ – ochiq to'plam;

2) $L = F_0 + \sum_{i=1}^{m} \lambda_i F_j, \quad L \in C^{(4)}(Q);$

3) $L_{y'y'}(x, y(x), y'(x), \lambda) \geq 0 \ \ (\leq 0), \quad \forall (x, y, y') \in S \times R;$

4) $y^*(x) \in C^2[x_0, x_1]$ shartli stasionar funksiya;

5) kuchaytirilgan Lejandr sharti: $L_{y'y'}(x, y^*(x), y^{*'}(x), \lambda^*) > 0 \ (< 0),$

6) kuchaytirilgan Yakobi sharti: $(x_0, x_1]$ oraliqda x_0 nuqtaga qo'shma nuqta mavjud emas.

U vaqtda *y*(x)* - izoperimetrik masalada kuchli lokal minimal (maksimal) bo'ladi.

4-teorema. Faraz qilaylik, (1)-(3) masalada J_0 – kvadratik funksioanal, ya'ni

$$J_i[y] = \int_{x_0}^{x_1} (a_i(x) y' + b_i(x) y) dx, \quad i = \overline{1, m}$$

bo'lsin, bu yerda $A_0(x), a(x), ..., a_m(x)$ funksiyalar $[x_0, x_1]$ kesmada uzluksiz differensiallanuvchi, $B_0(x), b_1(x), ..., b_m(x)$ funksiyalr esa uzluksiz. Bundan tashqari $A_0(x) > 0 \ (< 0), \quad \forall x \in [x_0, x_1]$ shart bajarilsin. U holda, agar Yakobi sharti bajarilmasa izoperimetrik masalada $\inf J_0[y] = -\infty, \ (\sup J_0[y] = +\infty)$ bo'ladi. Agar kuchaytirilgan Yakobi sharti bajarilsa, shartli stasionar funksiya mavjud, yagona va unda J_0 funksional global minimum (maksimum) ga yerishadi.

Misol.

$$\int_0^0 (y'^2 - y^2) dx \to \min, \quad \int_0^1 y dx = 0, \quad y(0) = y(1) = 0.$$

Lagranj ko'paytuvchilari qoidasini qo'llaymiz.

$$F_0 = y'^2 - y^2, \ F_1 = y, \ C^*{}_1 = F_{1y} - \frac{d}{dx} F_{1y} = 1.$$

Demak, $L = \lambda_0 F_0 + \lambda_1 F_1$ Lagranj funksiyasida $\lambda_0 = 1$ deb olish mumkin. Qulaylik uchun $\lambda = \lambda_1$ deb belgilab, $L = F_0 + \lambda F_1 = y'^2 - y^2 + \lambda y$ Lagranj funksiyasi uchun Eyler tenglamasini yozamiz:

$$L_y - \frac{d}{dx} L_{y'} = 0 \Leftrightarrow -2y + \lambda - \frac{d}{dx}(2y') = 0 \Rightarrow y'' + y - \frac{\lambda}{2} = 0.$$

Tuzilgan tenglamaning umumiy yechimi

$$y(x) = c_1 \sin x + c_2 \cos x + \frac{\lambda}{2}.$$

bo'ladi. Chegaraviy va izoperimetrik shartlardan foydalanib, $c_1,\ c_2,\ \lambda$ o'zgarmaslarni topamiz. $y(0) = 0 \Rightarrow \frac{\lambda}{2} = -c_2 \ \Rightarrow \ y(x) = c_1 \sin x + c_2 (\cos x \cdot 1).$

$$\left.\begin{array}{l} y(1) = 0 \\ \int_1^0 y dx = 0 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} c_1 \sin 1 + c_2(\cos 1 - 1) = 0 \\ c_1(1 - \cos 1) + c_2(\sin 1 - 1) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 0, \\ c_2 = 0. \end{array}$$

Demak, *y(x)*=0* yagona shartli - stasionar funksiyadir.

Kuchaytirilgan Lejandr sharti bajariladi, ya'ni $L_{y'y'} = 2 > 0$. Yakobi tenglamasini tuzamiz.

$$A^* = L_{y'y} = 2,\ B^* = L_{yy} = -2,\ C^* = L_{yy'} = 0,\ G^* = G^*{}_1 = 1,$$

$$-\frac{d}{dx}\left(A^*h' + C^*h\right) + C^*h' + B^*h + \mu_1 G^* = 0 \Rightarrow -2h - \frac{d}{dx}\left(2h'\right) + \mu = 0 \Rightarrow h'' + h - \frac{\mu}{2} = 0 \Rightarrow$$

$$\Rightarrow h'' + h - \overline{\mu} = 0,\ \overline{\mu} = \frac{\mu}{2}.$$

Tuzilgan Yakobi tenglamasiga mos $h''+h=0$ bir jinsli tenglama $h(0)=0, h'(0)=1$ shartlarni qanoatlantiruvchi $h_0(x)=\sin x$ yechimga ega. Bir jinsli bo'lmagan $h''+h+1=0$ tenglama esa, $h(0)=h'(0)=0$ shartlarni qanoatlantiruvchi $h_1(x)=\cos x - 1$ yechimga ega. $H(\xi)$ matrisani tuzamiz.

$$H(\xi) = \begin{pmatrix} h(\xi) & h_1(\xi) \\ \int\limits_0^\xi h_0 dx & \int\limits_0^\xi h_1 dx \end{pmatrix} = \begin{pmatrix} \sin\xi & \cos\xi - 1 \\ 1-\cos\xi & \sin\xi - \xi \end{pmatrix}$$

Demak, $x_0 = 0$ nuqtaga qo'shma nuqtalar quyidagi tenglamaning yechimidir:

$$\det H(\xi) = \begin{vmatrix} \sin\xi & \cos\xi - 1 \\ 1-\cos\xi & \sin\xi - \xi \end{vmatrix} = 2 - 2\cos\xi - \xi\sin\xi = 0 \Leftrightarrow 2(1-\cos\xi) - \xi\sin\xi = 0$$

$$\Rightarrow 2\sin^2\frac{\xi}{2} - \xi\sin\frac{\xi}{2}\cos\frac{\xi}{2} = 0 \Rightarrow \sin\frac{\xi}{2}\left(2\sin\frac{\xi}{2} - 2\cos\frac{\xi}{2}\right) = 0 \Rightarrow \sin\frac{\xi}{2} = 0,\quad tg\frac{\xi}{2} = \frac{\xi}{2}.$$

$tg\frac{\xi}{2} = \frac{\xi}{2}$ tenglamaning ildizi $\overline{\xi} > 1$ bo'ladi, chunki agar $\overline{\xi} < 1$ bo'lganida edi, $tg\frac{1}{2} > \frac{1}{2}$ bo'lar edi. Ammo $\left[0, \frac{\pi}{2}\right]$ oraliqda $tgx<x$, demak $tg\frac{1}{2} < \frac{1}{2}$; $\overline{\xi} \neq 1$ ekanligi ham ravshan.

$\sin\frac{\xi}{2} = 0$ tenglamaning eng kichik musbat ildizi esa, $\xi^* = 2\pi$ bo'ladi. Shunday qilib, $[0,1]$ da $x_0 = 0$ nuqtaga qo'shma nuqta mavjud emas, ya'ni kuchaytirilgan Yakobi sharti bajariladi. 13-teoremaga ko'ra, $y^*(x) = 0$ – berilgan masalaning global yechimidir.

Mustaqil ishlash uchun savollar.

1. Izoperimetrik masala. Lagranj ko'paytuvchilari qoidasi.
2. Izoperimetrik masalada Lejandr va Yakobi shartlari.
3. Ekstremumning yetarli shartlari.

10-ma'ruza. Optimal boshqaruv masalasi. Optimal boshqaruv masalasining umumiy qo'yilishi, asosiy muammolar

Reja.

1.Optimal boshqaruv masalasiga sodda misol.

2.Optimal boshqaruv masalasining umumiy qo'yilishi.

3.Optimal boshqaruv masalasining asosiy tiplari.

4.Optimallikning zaruriy sharti (maksimum prinsipi).

Tayanch iboralar: Boshqariluvchi obyekt, boshqaruv, tezkor harakat bo'yicha boshqarish, joyiz boshqarish, joyiz trayektoriya, optimal boshqarish, optimal trayektoriya, terminal boshqaruv masalasi, qo'shma sistema , Gamilton-Pontryagin funksiyasi, maksimum sharti, maksimum prinsipi va uning chegaraviy masalasi.

XX asrning ikkinchi yarmiga kelib hozirgi zamon fan va texnikasi masalalari bilan bog'liq holda variasion hisobning yangi tarmog'I - optimal boshqaruv nazariyasi vujudga kelib intensiv rivojlanmoqda [1,2,3].

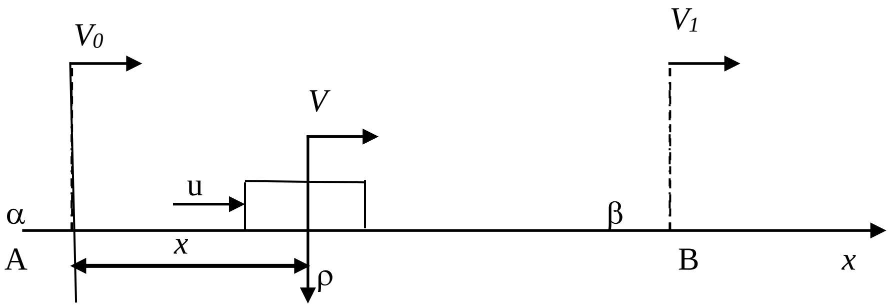
1.Optimal boshqaruv masalasining qo'yilishi. Avvalo optimal boshqaruv amaliy masalalaridan birini keltiramiz:

v_0 boshlang'ich tezlikka ega bo'lgan birlik massali material nuqtani modul bo'yicha birdan oshmaydigan kuch ta'sirida gorizontal to'g'ri chiziq bo'ylab *A* nuqtadan B nuqtaga shunday ko'chirish talab qilinadiki, bunda material nuqta B nuqtaga v_1 tezlik bilan eng qisqa vaqtda yetib kelsin.

Qo'yilgan masala tezkor harakat bo'yicha optimal boshqaruv masalasidan iborat. Uning matematik modelini tuzamiz.

Ox o'qda $A(\alpha)$ va $B(\beta)$ nuqtalarni olaylik. Material nuqta $t=t_0$ boshlang'ich vaqtda *A* nuqtada, $t=t_1(t_1>t_0)$ vaqtda esa B nuqtada bo'lsin.

$T= t_1-t_0$ material nuqtaning ko'chish vaqtidan iborat.



$x=x(t)$-material nuqtaning *t* vaqtda bosib o'tgan yo'li, $u=u(t)$ material nuqtaga *t* vaqt momentida ta'sir etayotgan kuch miqdori bo'lsin.

U vaqtda $\dot{x} = \frac{dx}{dt} = v$ - material nuqtaning tezligi, $\ddot{x} = \frac{d^2x}{dt^2} = a$ material nuqtaning tezlanishi bo'ladi.

Nyutonning ikkinchi qonuniga ko'ra $m\alpha=u$ tenglik o'rinli, bu yerda m –material nuqtaning massasi $m=1, a=\ddot{x}$ ekanligini hisobga olsak,

$$\ddot{x} = u \tag{1}$$

tenglamaga ega bo'lamiz. Masalaning qo'yilishiga ko'ra,

$$\left.\begin{aligned} x(t_0) = \alpha, \quad \dot{x}(t_0) = v_0 \\ x(t_1) = \beta, \quad \dot{x}(t_1) = v_1 \end{aligned}\right\} \tag{2}$$

shartlar kelib chiqadi. Bundan tashqari, $u(t)$ kuchga

$$|u(t) \le t, \ t \in [t_0, t_1]| \tag{3}$$

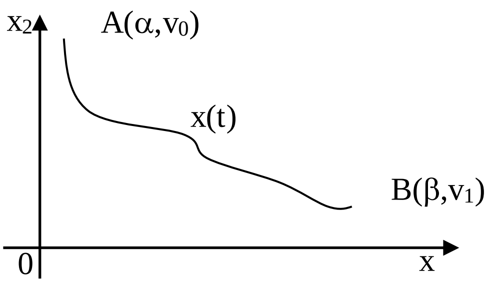
boshqarish funksiyasi (qisqacha, boshqarish) deyiladi. Odatda u, bo'lakli-uzluksiz funksiyalar sinfidan deb qaraladi. Bunday funksiyalar joyiz boshqarishlar sinfini tashkil etadi.

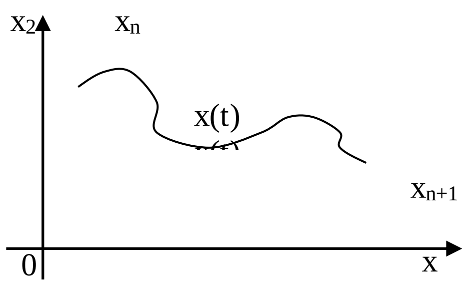
Shunday qilib, qo'yilgan masalaning matematik modeli quyidagicha: shunday $\left|u^*(t),\ t\in[t_0,t_1^*]\right|$ joyiz boshqarishni topish talab qilinadiki, (1) tenglamaning unga mos keluvchi $x^*(t)$ yechimi (2) shartlarni qanoatlantirsin va bunda ko'chish vaqti $T=t_1^*-t_0$ minimal bo'lsin.

$x_1=x,\ x_2=\dot{x}$ o'zgaruvchilarni kiritib, bu masalani

$$\left.\begin{aligned}&T(u)=t_1-t_0\to\min,\\&\dot{x}_1=x_2,\dot{x}_2=u,\\&x_1(t_0)=u,x_1(t_1)=\beta,\\&x_2(t_0)=v_0,x_2(t_1)=v_1,\\&|u|\le 1\end{aligned}\right\}\qquad(4)$$

ko'rinishda yozish mumkin.





(4) masala geometrik tilda $\{x_1,x_2\}$ tekislikda shunday $x^*(t)=\{x_1^*(t),x_2^*(t)\}$ trayektoriyani qurishni bildiradiki, u eng qisqa $T^*=t_1^*-t_0$ vaqtda $A=\{\alpha,v_0\}$ nuqtadan $B=\{\beta,v_1\}$ nuqtaga ko'chib o'tadi.

Endi optimal boshqaruv masalasining umumiy qo'yilishiga o'tamiz. [3,5,6].

Biror boshqariluvchi obyekt (jarayon)

$$\dot{x}_i=f_i(x_1,...,x_n,u_1,...,u_m,t),\ i=\overline{1,n}\qquad(5)$$

differensial tenglamalar sistemasi bilan berilgan bo'lsin, bu yerda t-vaqt, $x_1,\ldots,x_n$-obyektning faza koordinatalari, $u_1,\ldots,\ u_m$ -boshqarish parametrlari. Obyektning holati vektori $x=(x_1,\ldots,x_n)$, boshqarish vektori $u=(u_1,\ldots,u_m)$ va $f=(f_1,\ldots,f_n)$ vektor yordamida (5) sistemani

$$\dot{x}=f(x,u,t)\qquad(6)$$

vektorli differensial tenglama ko'rinishida yozamiz.

(6) boshqariluvchi obyektning faza koordinatalari $x=x(t)$ ko'rinishdagi t vaqtning biror $[t_0,t_1]$ oraliqdagi funksiyasi sifatida aniqlanishi uchun boshlang'ich t_0 vaqtda boshlang'ich $x(t_0)=x^0$ shartni va boshqarish parametrlarini t vaqtning $u=u(t)$ funksiyasi ko'rinishida aniqlash kerak.

U vaqtda $x=x(t)$ faza koordinatalari

$$\left.\begin{aligned} \dot{x}(t) &= f\{x(t),u(t),t\},\ t_0 \le t \le t_1 \\ x(t_0) &= x^0 \end{aligned}\right\} \qquad (7)$$

Koshi masalasining yechimi sifatida aniqlanadi. *u(t)* boshqarish ma'lum uzluksizlik shartlarini qanoatlantirishi zarur. Ko'pgina amaliy masalalarda boshqarishlar sifatida bo'lakli–uzluksiz funksiyalar olinadi. Ba'zi amaliy masalalarda *u(t)* ning uzluksizligi, ba'zan esa *u(t)* ning bo'lakli-silliqligi talab qilinadi. Nazariy tadqiqotlarda boshqarishlarning kengroq sinflari, masalan chegaralangan o'lchovli funksiyalar fazosi yoki $L_p^m[t_0,t_1]$ fazolar qaraladi.

Biz quyidagi asosan bo'lakli-uzluksiz boshqarishlar sinfidan foydalanamiz.

Shunday qilib, biror $x^0 \in R^n$ nuqta va $u=u(t)$ bo'lakli uzluksiz boshqarish berilgan bo'lsin. U vaqtda (7) Koshi masalasining yechimi $x=x(t)$ deb,

$$x(t) = \int_{t_0}^{t} f(x(\tau),u(\tau),\tau)d\tau + x^0, \quad t_0 \le t \le t_1 \qquad (8)$$

tenglamaning uzluksiz yechimini tushunamiz.

Bu yechimni $x(t,x^0,u)$ deb belgilaymiz. $x(t_0,x^0,u)$- obyekt trayektoriyasining chap uchi, $x(t_1,x^0,u)$- trayektoriyaning o'ng uchi deyiladi.

Agar $f_i(x,u,t), i=\overline{1,n}$ funksiyalar barcha $x \in R^n, u \in R^m, t \in [t_0,t_1]$ bo'yicha o'zining $f_{ixj}(x,u,t)$, xususiy hosilalari bilan uzluksiz bo'lsa, (8) tenglamaning $x(t_0)=x^0$ shartni qanoatlantiruvchi yechimi mavjud va yagonadir.

Biror $V \subset R^m$ to'plam berilgan bo'lsin. Shu V to'plamdan qiymatlar qabul qiluvchi $u=u(t),\ t \in [t_0,t_1]$ bo'lakli-uzluksiz boshqaruvlarni joyiz boshqaruvlar deb ataymiz va bunday boshqarishlar to'plamini U deb belgilaymiz.

Boshqaruv masalalarida boshqarish parametrlari bilan bir qatorda obyektning faza koordinitalariga ham cheklashlar qo'yiladi. Bunday cheklashlar

$$x(t) = x(t,x^0,u) \in G(t), \quad t_0 \le t \le t_1 \qquad (9)$$

ko'rinishda yoziladi, bu yerda $G(t) \subset R^n$ (9) ko'rinishdagi cheklashlarga faza cheklashlari deyiladi.

Trayektoriyaning chap va o'ng uchlari qanoatlantirishi zarur bo'lgan shartlar haqida ham to'xtalib o'tamiz. Faraz qilaylik, $S_0(t_0) \subset R^n$ va $S_1(t_1) \subset R^n$ to'plamlar berilgan bo'lsin. U vaqtda trayektoriyaning uchiga qo'yilgan shartlar

$$x(t_0) \in S_0(t_0), \quad x(t_1) \in S_1(t_1), \qquad (10)$$

kabi yoziladi. $S_0(t_0)$ va $S_1(t_1)$ to'plamlar, odatda

$$S_0(t_0) = \{y : y \in G_0(t_0), h_i(y,t_0) \le 0,\ i=1,...,m_0, h_i(y,t_0)=0, i=m_0+1,...,\xi_0\}, \qquad (11)$$

$$S_1(t_1) = \{x : x \in G(t), g_i(x,t_1) \le 0,\ i=1,...,m_1, g_i(x,t_1)=0, i=m_2+1,...,\xi_1\}, \qquad (12)$$

shaklda beriladi, bu yerda $h_i(y,t_0)$, $g_i(x,t_1)$ ma'lum funksiyalar.

$\theta_0 \subset R^1, \theta_1 \subset R^1$ to'plamlar berilgan, $\inf \theta_0 < \sup \theta_1$ bo'lsin. $t_0 \in \theta_0, t_1 \in \theta_1$, $u = u(t)$ - joyiz boshqarish, $x(t, x^0, u)$ unga mos joyiz trayektoriya bo'lsin. Har bir shunday joyiz (x^0, u, x, t_0, t_1) da aniqlangan

$$J(x^0, u, x, t_0, t_1) = \int_{t_0}^{t_1} f_0(x(t), u(t)t)dt + g_0(x^0, x(t_1), t_0, t_1) \qquad (13)$$

funksionalni qaraymiz.

$$J_* = \inf J(x^0, u, x, t_0, t_1)$$

deb belgilaymiz, bu yerda quyi chegara barcha joyiz (x^0, u, x, t_0, t_1) bo'yicha olinadi.

Agar $J(x_*^0, u_*, x_*, t_0^*, t_1^*) = J_*$ bo'lsa, joyiz $(x_*^0, u_*, x_*, t_0^*, t_1^*)$ ga optimal boshqaruv masalasining yechimi, $u_* = u_*(t)$ optimal boshqarish, $x_* = x_*(t)$ optimal trayektoriya deyiladi.

Qo'yilgan optimal boshqaruv masalasini

$$J = \int_{t_0}^{t_1} f^0(x(t), u(t)t)dt + g_0(x^0, x(t_1), t_0, t_1) \to \inf \qquad (14)$$

$$\left.\begin{array}{l} \dot{x}(t) = f(x(t), u(t), t), t_0 \le t \le t_1 \\ x(t) \in G(t),\ t_0 \le t \le t_1 \\ x(t_0) = x^0, x(t_0) \in S_0(t_0),\ x(t_1) \in S_1(t_1) \\ t_0 \in \theta_0, t_1 \in \theta_1, u(t) \in V,\ t_0 \le t \le t_1 \end{array}\right\} \qquad (15)$$

ko'rinishda belgilaymiz.

Agar $G(t) \equiv R^n$ bo'lsa, (14),(15) masala faza koordinitalariga cheklashlar qo'yilmagan optimal boshqaruv masalasi deyiladi. Agar $S_0(t)$ $(S_1(t))$ to'plam vaqtga bog'liq bo'lmasa va yagona nuqtadan iborat bo'lsa, (14),(15) masalada trayektoriyalarning chap uchi (o'ng uchi) mahkamlangan deyiladi.

Agar $S_0(t)$ (yoki $S_1(t)$),$t_0 \le t \le t_1$ to'plam R^n fazo bilan ustma-ust tushsa, optimal boshqaruv masalasida trayektoriyalarning chap (o'ng) uchi bo'sh (erkin) deyiladi. Agar $S_0(t), S_1(t), t_0 \le t \le t_1$, R^n da biror sirt yoki chiziqdan iborat bo'lsa, optimal boshqaruv masalasida trayektoriyalar chap (o'ng) uchi qo'zg'oluvchan deyiladi.

(14),(15) masaladan optimal boshqaruv masalasining asosiy tiplariga ega bo'lamiz:

a) Tezkor harakat bo'yicha optimal boshqaruv masalasi. Agar $f_0 \equiv 1$, $g_0 \equiv 1$, t_0-berilgan (ma'lum) , t_1-noma'lum (izlanayotgan) bo'lsa, tez harakat bo'yicha optimal boshqarish masalasiga ega bo'lamiz. Bu masalada kriteriy $J=T(u)=t_1(u)-t_0$ bo'ladi.

b) Terminal boshqaruv masalasi. Bu masala (14), (15) masaladan, $f_0 \equiv 0, t_0, t_1$lar belgilangan, $S_1(t) \equiv R^n, t_0 \le t \le t_1$, bo'lgan holda olinadi. Terminal boshqaruv masalasida kriteriy $J=g_0(x(t_1))$ ko'rinishda bo'ladi.

2. Pontryaginning maksimum prinsipi. Maksimum prinsipi –optimal boshqaruv masalalarida optimallikning asosiy zaruriy sharti hisoblanadi. Bu natija XX asrning 50-yillari ikkinchi yarmida akademik L.S.Pontryagin boshchiligidagi sovet matematiklari tomonidan olingan.

Quyidagi:

$$J(u,x)=\int_{t_0}^{t_1} f_0(x(t),u(t)t)dt+g_0(x^0,x(t_1))\to \inf \qquad (16)$$

$$\left.\begin{aligned}&\dot{x}(t)=f(x(t),u(t),t),t_0\le t\le t_1\\&x(t_0)=x^0,g_i(x(t_1))\le 0,\ i=1,...,k,\\&g_i(x(t_1))=0,i=k+1,...,s\\&u=u(t)\in V,\end{aligned}\right\} \qquad (17)$$

optimal boshqaruv masalasini qaraymiz.

Bu masalada t_0,t_1 vaqt momentlari belgilangan (o'zgarmas), x^0-berilgan boshlang'ich nuqta. (16), (17) masala (14), (15) masaladan $G(t)\equiv R^n, S_0(t_0)=\{x_0\}, S_1(t_1)=\{y\in R^n : g_i(y)\le 0, i=\overline{1,k}, g_i(y)\le 0, i=\overline{k+1,S},\}$ bo'lgan holda kelib chiqadi.

Faraz qilamizki, $f(x,u,t)=(f_1(x,u,t),....,f_n(x,u,t))$ vektor - funksiyaning $f_i(x,u,t)$ komponentalari va $f_0(x,u,t),\ g_0(x)$ funksiyalar x bo'yicha uzluksiz xususiy hosilalarga ega bo'lsin.

Maksimum prinsipini bayon qilish uchun Gamilton-Pontryagin funksiyasi deb ataluvchi,

$$\begin{aligned}&H(x,u,t,\psi,a_0)=-a_0f_0(x,u,t)+\psi_1f_1(x,u,t)+....+\psi_nf_n(x,u,t)=\\&=-a_0f_0(x,u,t)+\psi^Tf(x,u,t)\end{aligned} \qquad (18)$$

funksiyani qaraymiz, bu yerda $\psi=(\psi_1,...,\psi_n),\ a_0=const$

$u=u(t)$ joyiz boshqarish, $x(t)=x(t,u,x^0)$ unga mos joyiz trayektoriya, $[t_0,t_1]$ oraliqda aniqlangan bo'lsin. $(u(t),x(t)),(t_0\le t\le t_1)$ juftlikka mos ravishda $\psi=\psi(t)=(\psi_1(t),...,\psi_n(t)),$ o'zgaruvchilarga nisbatan

$$\begin{aligned}&\dot{\psi}_1(t)=\frac{\partial H(x,u,t,\psi(t),a_0)}{\partial x_1}\Bigg|_{u=u(t)}^{x=x(t)}=\\&=a_0f_{0x_i}(x(t),u(t),t)-\sum_{j=1}^{n}\psi_j(t)f_{jx_i}(x(t),u(t),t),(t_0\le t\le t_1)\end{aligned} \qquad (19)$$

sistemani qaraymiz. Unga qo'shma sistema deyiladi. (19) qo'shma sistemani vektor shaklda

$$\dot{\psi}(t)=-H_x(x(t),u(t),t,\psi(t),a_0),\ t_0\le t\le t_1 \qquad (20)$$

kabi yozish mumkin, bu yerda $H_x=(H_{x_1},...,H_{x_n})$

Agar (6) sistema x,u ga nisbatan chiziqli, ya'ni

$$\dot{x}(t)=A(t)x(t)+B(t)u(t)+f(t),(t_0\le t\le t_1)$$

ko'rinishda bo'lsa, $H(x,u,t,\psi,a_0)=-a_0f_0(x,u,t)+\psi'(A(t)+B(t)u+f(t))$ va (20) qo'shma sistema

$$\dot{\psi}(t)=a_0f_{0x}(x(t),u(t),t)-A'(t)\psi(t),(t_0\le t\le t_1)$$

kabi bo'ladi, bu yerda '- transponirlash belgisi.

(20) qo'shma sistema –chiziqli differensial tenglamalar sistemasidan iborat bo'lib, u $\psi(t_0)=\psi_0$ boshlang'ich shartni qanoatlantiruvchi yagona yechimga ega.

1-teorema. Agar $(x(t),u(t)),\ t_0 \le t \le t_1$ (16)-(17) masalaning yechimi bo'lsa, shunday $a_0, a_1, ..., a_n$ sonlar va $\psi(t)=(\psi_1(t),...,\psi_n(t)), t_0 \le t \le t_1$ vektor-funksiya mavjud bo'ladiki, quyidagilar bajariladi:

1) $a=(a_0 a_1,...,a_n) \ne 0,\ a_0 \ge 0,...,a_k \ge 0;$

2) $\psi(t)$ funksiya - (20) qo'shma sistemaning *(x(t),u(t))* ga mos keluvchi yechimidan iborat;

3) *u(t)* optimal boshqarishning barcha $t \in [t_0,t_1]$ uzluksizlik nuqtalarida $H(x(t),u,t,\psi(t),a_0)$ funksiya $u=(u_1,\dots,u_m)$ o'zgaruvchi bo'yicha *V* to'plamda aniq yuqori chegarasiga *u=u(t)* bo'lganda erishadi, ya'ni

$$\sup_{u\in V} H(x(t),u,t,\psi(t),a_0)=H(x(t),u(t),t,\psi(t),a_0),(t_0 \le t \le t_1) \qquad (21)$$

4)
$$\psi_1(t_1)=-\sum_{j=0}^{s} a_j g_{jx_i}(x(t_1), i=1,2,....,n \qquad (22)$$

$$a_j g_j(x(t_1))=0,\ \ j=1,2,...,k \qquad (23)$$

(22) shartlarga transversallik shartlari deyiladi. (21) maksimum shartlari 1-teoremada markaziy o'rinni egallaydi. Shuning uchun uni va quyida keltiriladigan 2-teoremani maksimum prinsipi deb atash qabul qilingan.

Endi boshlang'ich yoki oxirgi vaqt momentlari belgilangan quyidagi :

$$J(u,x,t_0,t_1)=\int_{t_0}^{t_1} f_0(x(t),u(t)t)dt+g_0(x(t_1),t_0,t_1)\to \inf \qquad (24)$$

$$\left.\begin{aligned}
&\dot{x}(t)=f(x(t),u(t),t),t_0 \le t \le t_1\\
&x(t_0)=x^0, g_i(x(t_1),t_0,t_1)\le 0,\ i=1,...,k,\\
&g_i(x(t_1),t_0,t_1)=0, i=k+1,...,s\\
&u=u(t)\in V, t_0 \le t \le t_1
\end{aligned}\right\} \qquad (25)$$

optimal boshqaruv masalasini qaraymiz.

Bu yerda $f=(f_0 f_1,..., f_n), f_0, g_0$ funksiyalarni o'z aniqlanish sohalarida $f_{jx_j}, g_{jx_j}, g_{jt_0}, g_{jt_1}$ xususiy hosilalari bilan birga uzluksiz deb faraz qilamiz.

2-teorema. Agar $(x(t),u(t),t_0,t_1)$ -(24)(25) masalaning yechimi bo'lsa, shunday $a_0 a_1,...,a_s$ sonlar va $\psi(t)=(\psi_1(t),...,\psi_n(t)), t_0 \le t \le t_1$ vektor-funksiya mavjud bo'ladiki, ular 1-teoremaning 1)-3) shartlarini va quyidagi transversallik shartlarini qanoatlantiradi:

$$\psi(t_1)=-\sum_{j=0}^{s} a_j g_{jx_i}(x(t_1),t_0,t_1) \qquad (26)$$

$$\max_{u\in V} H(x(t_0),u,t_0,\psi(t_0),a_0)=-\sum_{j=0}^{s} a_j g_{jt_{0i}}(x(t_1),t_0,t_1) \qquad (27)$$

(agar t_0 belgilangan bo'lsa, (27) shart qatnashmaydi);

$$\max_{u\in V} H(x(t_1),u,t_1,\psi(t_1),a_0)=-\sum_{j=0}^{s} a_j g_{jt_{1i}}(x(t_1),t_0,t_1) \qquad (28)$$

(agar t_1 belgilangan bo'lsa, (28) shart qatnashmaydi);

$$a_j g_j(x(t_1),t_0,t_1)=0, \quad j=1,2,...,k \tag{29}$$

3. Maksimum prinsipining chegaraviy masalasi. Maksimum prinsipidan amaliyotda qanday foydalanish mumkinligini ko'rib o'tamiz.

$H(x,u,t,\psi,a_0)$ funksiyani $u=(u_0 u_1,...,u_n)$, o'zgaruvchining funksiyasi deb qaraymiz va har bir belgilangan (x,t,ψ,a_0) da

$$H(x,u,t,\psi,a_0)\to \sup, u\in V \tag{30}$$

maksimallashtirish masalasini yechamiz.

$$u=u(x,t,\psi,a_0)\in V \tag{31}$$

shu masalaning yechimi bo'lsin, ya'ni

$$H(x,u(x,t,\psi,a_0),t,\psi,a_0)=\sup_{u\in V} H(x,u,t,\psi,a_0) \tag{32}$$

tenglik bajariilsin. Agar optimal boshqaruv masalasi yechimga ega bo'lsa, maksimum shartiga ko'ra (31) funksiya aniqlangan bo'ladi. Ko'p hollarda (31) funksiyani oshkor ko'rinishda yozish mumkin bo'ladi. Masalan, agar

$$f_j(x,u,t)= f_j^0(x,t)+\sum_{i=1}^{m} f_j^0(x,t)u_i, \; j=1,2,...,n$$

$$V=\{ \mathrm{u}=(\mathrm{u}_1,...,\mathrm{u}_\mathrm{m})\in \mathrm{R}^\mathrm{m}, \alpha_\mathrm{i}\le u_i\le \beta_i, \quad i=1,2,...,m\}$$

($\alpha_\mathrm{i}, \beta_i$, -berilgan sonlar) bo'lsa,

$$H(x,u,t,\psi,a_0)=-a_0 f_0^0(x,t)+\sum_{j=1}^{n}\psi_j f_j^0(x,t)+\sum_{j=1}^{n}\varphi_i(x,t,\psi,a_0)u_i$$

bo'ladi, bu yerda

$$\varphi_i(x,t,\psi,a_0)=-a_0 f_0^0(x,t)+\sum_{j=1}^{n}\psi_j f_j^0(x,t), \quad i=1,2,...m$$

U vaqtda (30) masala yechimi $u(x,t,\psi,a_0)$ ning koordinatalari

$$u=u_i(x,t,\psi,a_0)=\begin{cases}\beta_i, \varphi_i(x,t,\psi,a_0)>0\\ \alpha_i, \; \varphi_i(x,t,\psi,a_0)<0, \quad i=1,...,m\end{cases}$$

ko'rinishda bo'lishi ravshan. Xususiy holda, agar $\alpha_i=-1, \beta_i=+1$ bo'lsa, $u_i=sign\varphi_i(x,t,\psi,a_0), i=1,2,...,m$ bo'ladi.

Agar V to'plam

$$V=\left\{u\in R^m : |u|=(\sum_{i=1}^{m}u_i^2)^{\frac{1}{2}}\le r\right\}$$

ko'rinishda bo'lsa, (31) funksiyani oshkor shaklda

$$u(x,t,\psi,a_0)=\frac{\varphi(x,t,\psi,a_0)}{\|\varphi_i(x,t,\psi,a_0)\|}r$$

kabi yozish mumkin, bu yerda $\varphi=(\varphi_1,...,\varphi_n)$,

Faraz qilaylik, bizga (31) funksiya ma'lum bo'lsin. U vaqtda x,ψ o'zgaruvchilarga nisbatan quyidagi $2n$ ta differensial tenglamalar sistemasini qaraymiz:

$$\left.\begin{aligned}&\dot{x} = f(x, u(x,t,\psi,a_0),t)\\&\dot{\psi} = -H_x(x,u(x,t,\psi,a_0),t,\psi,a_0), t_0 \le t \le t_1\end{aligned}\right\} \tag{33}$$

Differensial tenglamalar kursidan yaxshi ma'lumki, (33) tenglamalar sistemasining umumiy yechimi *2n* ta ixtiyoriy parametrlarga (masalan, $x(t_0) = (x_1(t_0),...,x_n(t_0)),\ \psi(t_0) = (\psi_1(t_0),...,\psi_n(t_0))$ boshlang'ich shartlarga) bog'liq bo'ladi.

Bundan tashqari, maksimum prinsipidagi $a_0, a_1,..., a_s$ parametrlar ham noma'lum bo'lganligidan,ularni aniqlash uchun yana *s+1* ta shart kerak bo'ladi. Shunday qilib, noma'lum *2n+s+1* ta parametrlarni aniqlash uchun *2n+s+1* ta shart zarur. Ularni maksimum prinsipidan, masalan, 1-teoremadagi (22),(23) shartlar hamda

$$g_j(x(t_j)) = 0, \quad j = k+1,...,s \tag{34}$$

shartlarni olamiz. Bu shartlar jami *2n+s* ta tenglamalarni beradi. Yetishmayotgan yana bitta tenglamani olish uchun $H(x,u,t,\psi,a_0)$ funksiyaning $\psi_1,\psi_2,...,\psi_n,a_0$ o'zgaruvchilarga nisbatan chiziqli va bir jinsli ekanligini, ya'ni $H(x,u,t,a\psi,aa_0) = aH(x,u,t,\psi,a_0), \forall a \in R^1$ ekanligini hisobga olamiz. U vaqtda (32) shartdan

$$u(x,t,a\psi,aa_0) = u(x,t,\psi,a_0),\ \forall a > 0 \tag{35}$$

ekanligi kelib chiqadi. Demak, maksimum prinsipida $a_1, a_2,..., a_s,\ \psi_1,...,\psi_n$ o'zgaruvchilar musbat ko'paytuvchi aniqligida topiladi. Demak,

$$\|a\|^2 = \sum_{i=0}^{S} a_i^2 = 1 \tag{36}$$

deb olish mumkin. Agar $a_0>0$ ekanligi ma'lum bo'lsa, (36) shart o'rniga $a_0=1$ deb olish ham mumkin. (22),(23),(34),(36) tenglamalar sistemasini yechganda

$$a_0 \ge 0, a_1 \ge 0,..., a_k \ge 0, \quad g_i(x(t_1),t_0,t_1) \le 0, \quad i = 1,...,k \tag{37}$$

shartlarning bajarilishi hisobga olinadi. Shunday qilib, maksimum prinsipi asosida (32) maksimum shartidan, (33) tenglamalar sistemasi va (22),(23),(34),(36),(37) shartlardan iborat maxsus chegaraviy masalaga ega bo'ldik. Bu masalaga maksimum prinsipining chegaraviy masalasi deyiladi.

Agar *x(t)* , $\psi(t), a_1, a_2,..., a_s$, ,- maksimum prinsipining chegaraviy masalasi yechimidan iborat bo'lsa, ularni (31) ga qo'yib,

$$u(t) = u(x(t),t,\psi(t),a_0) \qquad t_0 \le t \le t_1 \tag{38}$$

funksiyani hosil qilamiz. Agar bu funksiya $[t_0,t_1]$ oraliqda bo'lakli-uzluksiz bo'lsa, u optimalikka shubhali boshqarish bo'ladi. Agar optimal boshqarish masalasining yechimi mavjud va maksimum prinsipi chegaraviy masalasi yagona yechimga ega bo'lsa, (38) bo'lakli-uzluksiz funksiya optimal boshqaruvdan iborat bo'ladi.

1-misol. $$J(u) = \int_{t_0}^{t_1} (u^2(t) + x^2(t))dt \to \inf$$

$$\left.\begin{aligned}&\dot{x}(t) = u(t), 0 \le t \le t_1\\&x(0) = x(t_1) = 0\end{aligned}\right\}$$

Bu yerda $t_1>0$ vaqt momenti berilgan, boshqarishlar to'plami $V=R^1$.Bu masala sodda bo'lib, uning yechimi $(u(t)\equiv 0, x(t)\equiv 0),(0\le t\le t_1)$ juftlikdan iborat. Shu yechimni maksimum prinsipi (1-teorema) dan foydalanib topish mumkin. Haqiqatan ham, Gamilton-Pontryagin funksiyasi $H(x,u,t,\psi,a_0)=-a_0(u^2+x^2)+\psi u$ yordamida qo'shma sistemani yozamiz:

$$\dot\psi(t)=-H_x=2a_0x$$

Agar $a_0=0$ bo'lsa, $H=\psi u$ funksiya $V=R^1$ to'plamda yuqori chegarasiga faqat $\psi=0$ bo'lganda erishadi. Ammo $a_0=\psi=0$ shart maksimum prinsipiga ziddir. Demak, $a_0>0$. U vaqtda $a_0=1$ deb hisoblash mumkin. Bu holda $H=u^2-x^2+\psi u$ funksiya u bo'yicha R^1 da yuqori chegarasiga $u=\frac{\psi}{2}$ nuqtada erishadi. U vaqtda maksimum prinsipining chegaraviy masalasi

$$\dot x=\frac{\psi}{2},\ \dot\psi=2x,\ 0\le t\le t_1, x(0)=x(t_1)=0$$

ko'rinishda yoziladi. Bu masalaning yagona yechimi $(x(t)\equiv 0,\psi(t)\equiv 0),(0\le t\le t_1)$ bo'ladi. U vaqtda $(u(t)=0,\psi(1)/2\equiv 0),(0\le t\le t_1)$ -bu bizga ma'lum optimal boshqarishdir.

2-misol. $$J(u)=\int_{t_0}^{t_1}(u^2(t)-x^2(t))dt\to\inf$$

$$\left.\begin{array}{l}\dot x(t)=u(t),0\le t\le t_1\\ x(0)=x(t_1)=0,\ t_1>0\end{array}\right\}$$

Gamilton-Pontryagin funksiyasi

$$H=-a_0(u^2-x^2)+\psi u$$

ko'rinishda bo'ladi. Qo'shma sistemani tuzamiz:

$$\dot\psi(t)=-H_x=-2a_0x$$

Agar $a_0=0$ bo'lsa, $H=\psi u$ funksiya u bo'yicha aniq yuqori chegarasiga $V=R^1$ to'plamda faqat $\psi=0$ bo'lganda erishadi. Bu esa, maksimum prinsipiga ziddir. Demak, $a_0>0$ ya'ni $a_0=1$ deb olish mumkin. U vaqtda $H=u^2-x^2+\psi u$ funksiyaning $u\in V=R^1$ bo'yicha aniq yuqori chegarasiga $u=\frac{\psi}{2}$ nuqtada erishiladi. Maksimum prinsipining chegaraviy masalasi

$$\dot x=\frac{\psi}{2},\ \dot\psi=-2x,\ 0\le t\le t_1, x(0)=x(t_1)=0$$

bo'ladi. Bu yerdagi differensial tenglamalar sistemasining umumiy yechimi

$$x(t)=c_1\sin t+c_2\cos t,\ \ \psi(t)=2c_1\cos t+2c_2\sin t$$

ko'rinishda topiladi, bu yerda c_1,c_2 -ixtiyoriy o'zgarmaslar. $x(0)=0$ shartni hisobga olib $c_2=0$ ekanligini topamiz. U vaqtda $x(t)=c_1\sin t, \psi(t)=2c_1\cos t, x(t_1)=0$ shartdan $c_1\sin t_1=0$ tenglikni olamiz. $t_1\neq\pi k(k=1,2,\ldots)$ bo'lganda, bu yerdan, $c_1=0$ bo'lishi kelib chiqadi va maksimum prinsipi chegaraviy masalasi yagona $(x(t)\equiv 0,\psi(t)\equiv 0),(0\le t\le t_1)$

yechimga ega, optimallikka shubhali boshqaruv esa, $u=\frac{\psi}{2}=0$ bo'ladi. Agar $t_1=\pi k(k=1,2,\ldots)$ bo'lsa, maksimum prinsipi chegaraviy masalasi cheksiz ko'p yechimga ega: $x(t)=c_1\sin t,\ \psi(t)=2c_1\cos t,$, bu yerda s_1-ixtiyoriy o'zgarmas. U vaqtda optimallikka shubhali boshqarish ham cheksiz ko'p bo'ladi:

$$(u(t)=c_1\cos t \quad (0\le t\le t_1)\,.$$

Topilgan boshqarish optimal bo'ladimi? Bu savolga javob t_1 ning qiymatiga bog'liq. $t_1>\pi$ va $0<t_1<\pi$ bo'lgan hollarni qaraymiz.

1) $t_1>\pi$ bo'lsin. U vaqtda $\inf J(u)=-\infty$ ekanligini ko'rsatamiz. Buning uchun

$u_m=u_m(t)=\frac{m\pi}{t_1}\cos\frac{\pi t}{t_1}$ boshqarishlar ketma-ketligini va ularga mos

$x_m=x_m(t)=m\sin\frac{\pi t}{t_1}\ (0\le t\le t_1, m=1,2,...)$ trayektoriyalar ketma-ketligini qaraymiz. U vaqtda

$$J(u_m)=\int_{t_0}^{t_1}(u_m^2(t)-x_m^2(t))dt=\frac{1}{2}t_1m^2(\frac{\pi^2}{t^2}-1)\to-\infty, m\to\infty$$

Demak, $t_1>\pi$ bo'lganda qaralayotgan optimal boshqaruv masalasi yechimga ega emas. Maksimum prinsipi chegaraviy masalasi esa yuqorida ko'rsatildiki, $t_1\neq\pi k$ bo'lganda yagona yechimga ega, $t_1=\pi k$ bo'lganda esa, cheksiz ko'p yechimga ega.

2) $0<t_1<\pi$ bo'lsin. Shunday bo'lakli-uzluksiz *v(t)* funksiyalarni qaraymizki, $\dot{x}(t)=v(t)\ (0\le t\le t_1, x(0)=x(t_1)=0$ masala yechimga ega bo'lsin. U vaqtda shu $x=v(t)$ funksiyalar uchun quyidagi munosabatga ega bo'lamiz:

$$J(v)=\int_{t_0}^{t_1}(v^2-x^2)dt=\int_{t_0}^{t_1}(v^2+x^2ctg^2t-x^2\sin^2 t)dt=\int_{t_0}^{t_1}(v^2+x^2ctg^2t-2x\dot{x}ctgt)dt=$$

$$=\int_{t_0}^{t_1}(v(t)-x(t)ctgt)^2dt\ge 0$$

$t_1<\pi$ bo'lganda $u(t)\equiv 0$ va $t_1=\pi$ bo'lganda $u(t)=c_1\cos t \quad (c_1=const)$ funksiya uchun $J(u)=0$ bo'ladi. Demak, $t_1<\pi$ bo'lganda qaralayotgan optimal boshqaruv masalasi yagona $u(t)\equiv 0(0<t < t_1)$ yechimga ega, $t_1=\pi$ bo'lganda esa cheksiz ko'p $u(t)=c_1\cos t\ ,(0\le t\le t_1,\ \ c_1$ - ixtiyoriy o'zgarmas) yechimga ega.

Mustaqil ishlash uchun savollar.

1. Optimal boshqaruvning amaliy masalalariga misollar.
2. Optimal boshqaruv umumiy masalasining qo'yilishi. Optimal boshqaruv masalasining asosiy tiplari.
3. Pontryaginning maksimum prinsipi va uning qo'llanilishi.
4. Maksimum prinsipining chegaraviy masalasi va uni yechish orqali optimal boshqarishni topish.

11-ma'ruza. Terminal boshqaruv masalasida Pontryaginning maksimum prinsipi

REJA

1.Terminal boshqaruv masalasining qo'yilishi. Maksimum prinsipi.
2. Funksional orttirmasi uchun formula.
3. " Ignasimon" variasiya. Trayektoriya bahosini aniqlash.
4. Maksimum prinsipining isboti.
5. Ekstremal boshqarishlar.
6. Chiziqli terminal boshqaruv masalasi.

Tayanch iboralar: terminal funksional (kriteriy), joyiz boshqarish, joyiz trayektoriya , funksional ottirmasi, funksional orttirmasi uchun formula, boshqarishning "ignasimon" variatsiyasi, trayektoriya orttirmasi, maksimum sharti, ekstremal boshqarish, qo'shma sistema, optimal boshqarish, optimal trayektoriya, chiziqli terminal boshqaruv masalasi, optimallikning zaruriy va yetarli sharti.

1. **Terminal boshqaruv masalasi. Maksimum prinsipi.**

Boshqarish obyekti

$$\dot{x} = f(x,u,t), \quad t \in [t_0, t_1] \qquad (1)$$

vektorli differensial tenglama bilan berilgan bo'lsin, bu yerda $x = (x_1,...,x_n)$, $u = (u_1,...,u_m)$, $f = (f_1,..., f_n)$, $f_i(x,u,t)$ funksiyalarni $f_{ixj}(x,u,t)$ hususiy hosilalari bilan birga uzluksiz deb hisoblaymiz. Joyiz boshqarishlar $[t_0, t_1]$ oraliqda aniqlangan bo'lakli uzluksiz va $V \subset R^m$ to'plamdan qiymatlar qabul qiluvchi *u=u(t) m*– vektor funksiyalardan iborat. (1) tenglamaning har bir *u=u(t)* joyiz boshqarishga mos $x = x(t)$ joyiz trayektoriyasi

$$x(t_0) = x^0 \qquad (2)$$

shartni qanoatlantiradi. Qaralayotgan obyektni boshqarish

$$J(u) = \varphi(x(t_1)) \qquad (3)$$

terminal kriteriy orqali sifat jihatidan baholanadi, bu yerda $\varphi(x) - R^n$ da uzluksiz differensiallanuvchi funksiya. Shunday $u^*(t)$ boshqarishni topish kerakki, $J(u^*) = \inf_{u \in U} J(u)$ bo'lsin, bu yerda *U*– barcha joyiz boshqarishlar to'plami. Shunday qilib, quyidagi

$$\left.\begin{aligned} &J(u) = \varphi(x(t_1)) \to \inf \\ &\dot{x} = f(x,u,t), \quad t \in [t_0, t_1] \\ &x(t_0) = x^0, \quad u = u(t) \in V \end{aligned}\right\} \qquad (4)$$

terminal boshqaruv masalasini qaraymiz. Bu masalada trayektoriyalarning chap uchi mahkamlangan ((2) shartga q.), o'ng uchi esa, erkin $\left(x(t_1) \in R^n\right)$.

(4) masala avvalgi ma'ruzamizda qaralgan optimal boshqaruv umumiy masalasining xususiy holi bo'lib, Pontryaginning maksimum prinsipi bu masala uchun quyidagicha bo'ladi.

1-teorema. Agar $u^*(t),\ t\in[t_0,t_1]$ – optimal boshqarish, $x^*(t),\ t\in[t_0,t_1]$ optimal trayektoriya bo'lsa,,

$$H\left(x^*(t),\psi^*(t),u^*(t),t\right)=\max_{u\in V} H\left(x^*(t),\psi^*(t),u,t\right),\quad t\in[t_0,t_1] \qquad (5)$$

maksimum sharti bajariladi, bu yerda

$$H(x,\psi,u,t)=\psi' f(x,u,t)=\sum_{j=1}^{n}\psi_j f_j(x,u,t),$$

$\psi'(t),\ t\in[t_0,t_1]$ funksiya

$$\dot\psi=-\frac{\partial H\left(x^*(t),\psi,u^*(t),t\right)}{\partial x} \qquad (6)$$

$$\psi(t_1)=-\frac{\partial\varphi\left(x^*(t_1)\right)}{\partial x} \qquad (7)$$

qo'shma sistemaning yechimidir.

2. Funksional orttirmasi uchun formula. Teoremaning isbotiga o'tishdan oldin, funksionalning orttirmasi uchun formula keltirib chiqaramiz.

$u=u(t),\ \ \tilde u=u(t)+\Delta u(t),\ \ t\in[t_0,t_1]$ joyiz boshqarishlar, $x=x(t)$, $\tilde x=x(t)+\Delta x(t)$, $t\in[t_0,t_1]$, ularga mos joyiz trayektoriyalar bo'lsin.

$$\Delta J(u)=J(\tilde u)-J(u)$$

ayirmaga (3) funksionalning $u=u(t)$ boshqarish bo'yicha orttirmasi deyiladi.

(3) funksionalning aniqlanishidan va $\varphi(x)$ funksiyaning differensiallanuvchiligidan,

$$\Delta J(u)=\varphi(\tilde x(t_1))-\varphi(x(t_1))=\frac{\partial\varphi'(x(t_1))}{\partial x}\Delta x(t_1)+o(\|\Delta x(t_1)\|) \qquad (8)$$

kelib chiqadi.

$\psi=\psi(t)$ ixtiyoriy differensiallanuvchi funksiya uchun o'rinli bo'lgan quyidagi ayniyatni qaraymiz:

$$\psi'(t_1)\Delta x(t_1)-\psi'(t_0)\Delta x(t_0)=\int_{t_0}^{t_1}\dot\psi'(t)\Delta x(t)dt+\int_{t_0}^{t_1}\psi'(t)\Delta\dot x(t)dt \qquad (9)$$

Trayektoriyalarning chap uchi mahkamlangan, ya'ni $x(t_0)=\tilde x(t_0)=x^0$ bo'lgani uchun, $\Delta x(t_0)=0$.

$$\psi(t_1)=-\frac{\partial\varphi(x(t_1))}{\partial x} \qquad (10)$$

deb olamiz. (9) va (10) larni hisobga olgan holda, (8) dan

$$\Delta J(u)=-\int_{t_0}^{t_1}\dot\psi'(t)\Delta x(t)dt-\int_{t_0}^{t_1}\psi'(t)\Delta\dot x(t)dt+o(\|\Delta x(t_1)\|) \qquad (11)$$

tenglikni olamiz. (11) dagi ikkinchi integralni qaraymiz. Tushunarliki, $\Delta x(t_0)=\tilde x(t_0)-x(t)$ funksiya, quyidagi,

$$\Delta\dot x(t)=f(x(t)+\Delta x(t),\ u(t)+\Delta u(t),t)-f(x(t),u(t),t)$$

differensial tenglamani qanoatlantiradi. Shuning uchun,

$$H(x,\psi,u,t)=\psi' f(x,t,u) \qquad (12)$$

Gamilton–Pontryagin funksiyasi yordamida quyidagini yozamiz:

$$\int_{t_0}^{t_1}\psi'(t)\Delta\dot{x}(t)dt=\int_{t_0}^{t_1}\left(H(x+\Delta x,\psi,u+\Delta u,t)-H(x,\psi,u,t)\right)dt=$$

$$=\int_{t_0}^{t_1}\left(H(x,\psi,u+\Delta u,t)-H(x,\psi,u,t)\right)dt=\int_{t_0}^{t_1}\frac{\partial H(x,\psi,u+\Delta u,t)}{\partial x}\Delta x(t)dt+ \qquad (13)$$

$$+\int_{t_0}^{t_1}o\left(\|\Delta x(t)\|\right)dt$$

Faraz qilaylik, $\psi=\psi(t)$ funksiya

$$\dot{\psi}(t)=\int_{t_0}^{t_1}\frac{\partial H(x(t),\psi(t),u(t),t)}{\partial x}dt \qquad (14)$$

qo'shma sistemaning (10) boshlang'ich shartni qanoatlantiruvchi yechimi bo'lsin. (14) differensial tenglama chiziqli, ya'ni

$$\dot{\psi}(t)=\int_{t_0}^{t_1}\frac{\partial f(x(t),u(t),t)}{\partial x}\psi(t)$$

ko'rinishda bo'lgani uchun, $\psi(t)$ funksiya $[t_0,t_1]$ oraliqda bir qiymatli aniqlangandir.

(13) va (14) ni (10) ga keltirib qo'yib, funksional orttirmasini quyidagicha yozamiz:

$$\Delta J(u)=\int_{t_0}^{t_1}\Delta_{\tilde{u}}H(x(t),\psi(t),u(t),t)dt+\eta \qquad (15)$$

bu yerda $\Delta_{\tilde{u}}\ H(x,\psi,u,t)=H(x,\psi,\tilde{u},t)-H(x,\psi,u,t)$,

$$\eta=\eta_1+\eta_2+\eta_3,\ \eta_1=\left(\|\Delta x(t_1)\|\right), \qquad \eta_2=-\int_{t_0}^{t_1}o_1\left(\|\Delta x(t)\|\right)dt,$$

$$\eta_3=-\int_{t_0}^{t_1}\frac{\partial H_{\tilde{u}}'(x(t),\psi(t),u(t),t)}{\partial x}\Delta x(t)dt \qquad (16)$$

Hosil qilingan (15) formulaning ba'zi xususiy hollarini qarab chiqamiz.

a) Agar $\varphi(x)$ qavariq funksiya bo'lsa, $\eta_1\geq 0$ bo'ladi; agar $\varphi(x)$ chiziqli funksiya bo'lsa, $\eta_1=0$ bo'ladi.

Haqiqatan ham, differensiallanuvchi $\varphi(x)$ funksiya

$$\varphi(x+\Delta x)-\varphi 9x\geq\frac{\partial\varphi'(x)}{\partial x}\Delta x$$

tengsizlikni qanoatlantirganligi uchun, == bajariladi. == chiziqli bo'lgan holda esa == bo'lishi ravshan.

b) Agar (1) sistema x bo'yicha chiziqli bo'lsa, $\eta_2=0$ bo'ladi. Haqiqatan ham, agar (1) sistema

$$\dot{x}=A(u,t)x+b(u,t)$$

ko'rinishda bo'lsa, $H(x,\psi,u,t)=\psi'\left[A(u,t)+b(u,t)\right]$ funksiya x bo'yicha chiziqli va shuning uchun $o_1\left(\|\Delta\dot{x}\|\right)=0$ bo'ladi.

v) Agar (1) sistema , x va u o'zgaruvchilar ajralgan, ya'ni,

$$\dot{x} = f(x,t) + b(u,t)$$

ko'rinishda bo'lsa, $\eta_3 = 0$ bo'ladi. Haqiqatan ham, bu holda $\Delta_{\tilde{u}}\ H(x,\psi,u,t) = \psi'[b(\tilde{u},t) - b(u,t)]$. Shuning uchun, $\partial\Delta_{\tilde{u}}H/\partial x = 0$.

Agar

$$J(u) = \varphi(x(t_1)) \to \inf,$$
$$\dot{x} = A(t)x + b(u,t),\ \ t \in [t_0, t_1]$$
$$x(t_0) = x^0,\ \ u(t) \in V.$$

masalada $\varphi(x)$ qavariq funksiya bo'lsa, yuqoridagi a), b), v) natijalarga ko'ra, (15) formula,

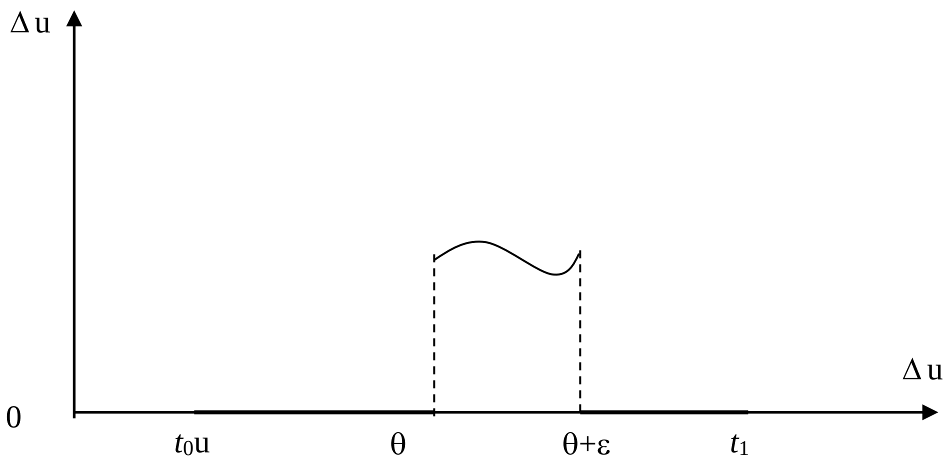
$$\Delta J(u) \ge -\int_{t_0}^{t_1} \Delta_{\tilde{u}}H(x(t),\psi(t),u(t),t)dt = \int_{t_0}^{t_1} \psi'(t)\left[b(u(t)+\Delta u(t),t) - b(u(t),t)\right]dt \quad (17)$$

ko'rinishni oladi.

3**. "Ignasimon variatsiya".** Berilgan $u = u(t)$ boshqaruvni o'zgartirib,

$$\tilde{u}(t) = \begin{cases} u(t), & t \notin [\theta, \theta+\varepsilon), \\ v, & t \in [\theta, \theta+\varepsilon) \end{cases} \quad (18)$$

boshqaruvni qaraymiz, bu yerda $\theta \in [t_0, t_1)$, $\varepsilon > 0$, $\theta + \varepsilon < t_1$, $v \in V$ (1-rasmga q.) $\tilde{u}(t)$– joyiz boshqaruvdir.



1-chizma.

$u(t)$ boshqaruvning $\Delta_{\theta v\varepsilon}u(t) = \tilde{u}(t) - u(t)$ ko'rinishdagi variatsiyasiga "ignasimon" variatsiya (Maksheyn variatsiyasi) deyiladi. Bu variatsiya θ, ε, v parametrlarga bog'liq bo'lib,

$$\Delta_{\theta v\varepsilon}u(t) = \begin{cases} 0, & t \notin [\theta, \theta+\varepsilon), \\ v - u(t), & t \in [\theta, \theta+\varepsilon) \end{cases} \quad (19)$$

ko'rinishga ega.

4. Trayektoriyaning "ignasimon" variatsiyaga mos orttirmasi. Agar $u = u(t)$ boshqaruvga (19) ko'rinishdagi $\Delta_{\theta v\varepsilon} u(t)$ "ignasimon" variatsiya berilsa, $x = x(t)$ trayektoriyaning $\Delta_{\theta v\varepsilon} x(t)$ orttirmasi qanoatlantiradigan

$$\Delta\dot{x} = f\left(x+\Delta x,\tilde{u},t\right) - f\left(x,u,t\right), \ \Delta x\left(t_0\right) = 0$$

tenglama, quyidagi ikkita tenglamaga ajraladi:

$$\begin{aligned} &\Delta\dot{x} = f\left(x+\Delta x,u,t\right) - f\left(x,u,t\right), \ \Delta x\left(t_0\right) = 0 \\ &t_0 \le t \le \theta, \quad \theta+\varepsilon \le t \le t_1 \end{aligned} \qquad (20)$$

$$\Delta\dot{x} = f\left(x+\Delta x,v,t\right) - f\left(x,u,t\right), \ \theta \le t \le \theta+\varepsilon \qquad (21)$$

$\Delta x(t_0) = 0$ bo'lgani uchun $\Delta_{\theta v\varepsilon} x(t_0) \equiv 0, \ t_0 \le t \le \theta$ funksiya (50) tenglamaning $[t_0, \theta]$ oraliqdan yagona yechimidir. Shunday qilib, (21) tenglama uchun boshlang'ich shart $\Delta_{\theta v\varepsilon} x(\theta) = 0,$ bo'ladi. Shuning uchun (21) tenglama integral tenglama shaklida quyidagicha yoziladi:

$$\Delta x(t) = \int_0^t \left[f\left(x+\Delta x,v,\tau\right) - f\left(x,u,\tau\right) \right] d\tau \qquad (22)$$

$f(x,u,t)$ funksiyaning $x(t)$ trayektoriya biror $\Delta -$ atrofidagi Lipshis o'zgarmasi $L = L(\Delta)$ bo'lsin, ya'ni

$$\left\| f\left(x+\Delta x,v,t\right) - f\left(x,u,t\right) \right\| \le L \left\| \Delta x \right\| \qquad (23)$$

tengsizlik bajarilsin. U vaqtda (22) tenglamadan

$$\begin{aligned} &\left\| \Delta x(t) \right\| \le \int_0^1 \left\| f\left(x+\Delta x,v,\tau\right) - f\left(x,u,\tau\right) \right\| d\tau \le \\ &\le \int_0^1 \left\| \Delta_v f\left(x,u,\tau\right) \right\| d\tau + L \int_0^1 \left\| \Delta x\left(\tau\right) d\tau \right\| \end{aligned} \qquad (24)$$

kelib chiqadi, bu yerda $\Delta_v f(x,u,\tau) = f(x,v,\tau) - f(x,u,\tau)$.

(1) tenglamaning o'ng tomoniga qo'yilgan shartlardan va $x(t)$ trayektoriyaning uzluksizligidan kelib chiqadiki, yetarlicha kichik $\varepsilon > 0$ uchun L o'zgarmas ε ga bog'liq bo'lmaydi.

Quyidagi

$$C = \int_0^1 \left\| \Delta_v f\left(x,u,\tau\right) \right\| d\tau, \quad w = \left\| \Delta x \right\| \qquad (25)$$

belgilashlardan foydalansak, (24) tengsizlik ko'rinishda yoziladi. Bu yerdan,

$$\begin{aligned} &w(t) \le c + L\int_0^t \left[c + L\int_0^\tau w(s)\,ds \right] d\tau = c + Lc(t-\theta) + L^2 \int_0^t (t-\tau)\, w(\tau) d\tau \le \\ &\le c + Lc(t-\theta) + c\frac{L^2 (t-\theta)^2}{2} + L^3 \int_0^1 (t-\tau) \int_0^s w(s)\, d\tau \le \ldots \le \\ &\le c\left[1 + \frac{L(t-\theta)}{1!} + \frac{L^k (t-\theta)^2}{2!} + \ldots + \frac{L^k (t-\theta)^k}{k!} + \ldots \right] = ce^{L(t-\theta)} \end{aligned} \qquad (26)$$

kelib chiqadi. (24) belgilashni hisobga olib, (25) tengsizlikdan

$$\|\Delta x(t)\| \le \left(\int_{\theta}^{t} \|\Delta_v f(x,u,\tau)\| d\tau \right) e^{L(t-\theta)}$$

tengsizlikni olamiz. Bu yerdan

$$\|\Delta x(t)\| \le \left(\int_{0}^{\theta+\varepsilon} \|\Delta_v f(x,u,t)\| dt \right) e^{L\varepsilon}, \quad t \in [\theta, \theta+\varepsilon] \qquad (27)$$

kelib chiqadi. $\Delta_v f(x(t), u(t), t)$ funksiya faqat v va t ga bog'liq va t bo'yicha bo'lakli uzluksiz bo'lgani uchun,

$$k(v) = \max_{t \in [t_0, t_1]} \|\Delta_v f(x(t), u(t), t)\| \qquad (28)$$

mavjud. Shunday qilib, (26) va (27) dan

$$\|\Delta_{\theta v \varepsilon}\| \le k_1 \varepsilon, \quad \theta \le t \le \theta + \varepsilon \qquad (29)$$

kelib chiqadi, bu yerda $k_1 = k(v) e^{L\varepsilon}$.

Yendi $\Delta_{\theta v \varepsilon} x(t)$ funksiyani $[\theta+\varepsilon, t_1]$ oraliqda qaraymiz. Bu yerda u (20) tenglamani va (29) ga ko'ra ,

$$\|\Delta_{\theta v \varepsilon}(\theta+\varepsilon)\| \le k_1 \varepsilon \qquad (30)$$

boshlang'ich shartni qanoatlantiradi. Soddalik uchun,(22) dagi L Lipshis o'zgarmasidan foydalanib va (19) ni integral shaklda yozib, quyidagiga yega bo'lamiz:

$$\|\Delta x(t)\| \le \|x(\theta+\varepsilon)\| + \int_{\theta+\varepsilon}^{t} \|f(x+\Delta x, u, \tau) - f(x,u,\tau)\| d\tau \le k_1 \varepsilon + L \int_{\theta+\varepsilon}^{t} \|\Delta x(\tau)\| d\tau$$

Bu ham (24) ga o'xshash tengsizlikdir. Shuning uchun, bu yerda ham (26) tipidagi tengsizlik o'rinli. Demak,

$$\|\Delta_{\theta v \varepsilon} x(t)\| \le k_1 \varepsilon e^{L(t-\theta-\varepsilon)}.$$

Bu yerdan

$$\|\Delta_{\theta v \varepsilon} x(t)\| \le k_2 \varepsilon, \quad \theta + \varepsilon \le t \le t_1 \qquad (31)$$

tengsizlikni olamiz, bunda $k_2 = k(V)^{L(t-\theta)}$ o'zgarmas ε ga bog'liq emas.

$k = k(V)^{L(t_1 - t_0)}$ bo'lsin. $k_1 < k,\ k_2 < k$ ekanligi ravshan. U vaqtda $\Delta_{\theta v \varepsilon} x(t) \equiv 0 \quad t \in [t_0, \theta]$, (29), (31) munosabatlarni birlashtirib, $[t_0, t_1]$ oraliqda $x(t)$ trayektoriyaning "ignasimon" variatsiyasiga mos orttirmasi,

$$\|\Delta_{\theta v \varepsilon} x(t)\| \le k\varepsilon \qquad (32)$$

kabi baholanishini ko'ramiz, bu yerda k o'zgarmas ε ga bog'liq emas.

5. Maksimum prinsipining isboti. Endi 1-teorema isbotini keltiramiz. $u^*(t) \quad t \in [t_0, t_1]$ optimal boshqaruvning (18) ko'rinishdagi ignasimon variatsiyasidan foydalanamiz, ya'ni

$$\tilde{u}(t) = \begin{cases} u^*(t), & t \notin [\theta, \theta+\varepsilon), \\ v, & t \in [\theta, \theta+\varepsilon) \end{cases}$$

ko'rinishdagi joyiz boshqaruvlarni qaraymiz, bu yerda $\theta \in [t_0, t_1]$, $\theta + \varepsilon < t_1$, $\varepsilon > 0$, $v \in V$. U vaqtda,

$$J(\tilde{u}) - J(u^*) \geq 0 \qquad (33)$$

funksionalning orttirmasi uchun hosil qilingan (14) formulaga ko'ra, (33) munosabat,

$$0 \leq \Delta J(u^*) = -\int_e^{\theta+\varepsilon} \Delta_v H(x^*(t), \psi^*(t), u^*(t), t)\,dt + \eta_{\theta v\varepsilon} \qquad (34)$$

ko'rinishida yoziladi, bu yerda

$$\eta_{\theta v\tau} = 0\left(\left\|\Delta_{\theta v\varepsilon} x^*(t_1)\right\|\right) - \int_0^t 0_1\left(\left\|\Delta_{\theta v\varepsilon} x^*(t)\right\|\right)dt - \\ -\int_0^{\theta+\varepsilon} \frac{\partial \Delta_v H'(x^*, \psi^*, u^*, t)}{dx} \Delta_{\theta v\varepsilon} x^*(t)\,dt \qquad (35)$$

(32) munosabatni hisobga olsak, (35) dan $\eta_{\theta v\varepsilon} = o_2(\varepsilon)$ ekanligi kelib chiqadi. $\Delta_V H(x^*(t), \psi^*(t), u^*(t), t)$ funksiya $t = \theta$ nuqtada o'ngdan uzluksiz bo'lganligi uchun,

$$\int_0^{\theta+\varepsilon} \Delta_v H'(x^*(t), \psi^*(t), u^*(t), t)\,dt = \Delta_v H(x^*(\theta), \psi^*(\theta), u^*(\theta), \theta)\varepsilon + o_3(\varepsilon) \qquad (36)$$

tenglik bajariadi. Shunday qilib, (36) ni hisobga olgan holda, (35) dan

$$-\Delta_v H(x^*(\theta), \psi^*(\theta), u^*(\theta), \theta)\varepsilon + o_4(\varepsilon) \geq 0 \qquad (37)$$

munosabatni olamiz $(o_4(\varepsilon) = o_2(\varepsilon) + o_3(\varepsilon))$. (37) ni $\varepsilon > 0$ songa bo'lib va $\varepsilon \to 0$ da limitga o'tib,

$$\Delta_v H(x^*(\theta), \psi^*(\theta), u^*(\theta), \theta) \leq 0$$

tengsizlikni olamiz, ya'ni

$$H(x^*(\theta), \psi^*(\theta), v, \theta) \leq H(x^*(\theta), \psi^*(\theta), u^*(\theta), \theta), \qquad \forall v \in V. \qquad (38)$$

Bu munosabat (4) tenglikning $\theta \in [t_0, t_1]$ nuqtada o'rinli ekanligini bildiradi. $u^*(t)$ boshqarishni $\theta = t_1$ nuqtada uzluksiz deb hisoblash mumkin. U vaqtda (38) munosabatda $\theta \to t_1$ da limitga o'tsak, uning $\theta = t_1$ nuqtada ham o'rinli ekanligini ko'ramiz. Teorema isbotlandi.

6. Ekstremal boshqaruvlar. Agar $x = x(t)$, $\psi = \psi(t)$, $u = u(t)$, funksiyalar $[t_0, t_1]$ oraliqda asosiy va qo'shma sistemani hamda maksimum shartini qanoatlantirsa, ya'ni

$$\dot{x}(t) = \frac{\partial H(x(t), \psi(t), u(t), t)}{\partial \psi}, \quad x(t_0) = x^0, \qquad (38)$$

$$\dot{\psi}(t) = -\frac{\partial H(x(t), \psi(t), u(t), t)}{\partial x}, \quad \psi(t_1) = -\frac{\partial \varphi(t_1)}{\partial x}, \qquad (39)$$

$$H(x(t), \psi(t), u(t), t) = \max_{v \in V} H(x(t), \psi(t), v, t), \quad t \in [t_0, t_1] \qquad (40)$$

munosabatlar bajarilsa, $u(t)$ joyiz boshqaruvga ekstremal boshqaruv deyiladi.

Maksimum prinsipi optimallikning zaruriy shartidan iborat, ya’ni ekstremal boshqaruvlar orasida optimal bo’lmaganlari ham topiladi.

1-misol.

$$\left.\begin{array}{l} x_2(1)\to \min, \\ \dot{x}_1 = u,\ \dot{x}_2 = -x_1^2,\ \ x_1(0) = x_2(0) = 0,\ \ t\in[0,1], \\ |u|\le 1. \end{array}\right\} \qquad (41)$$

Bu masala (4) ko’rinishdagi terminal boshqarish masalasidir: $f=(f_1,f_2)$, $f_1 = u,\ \ f_2 = -x_1^{\,2},\ \ x^0 = (0{,}0),\ t_0 = 0,\ t_1 = 1,\ \ \varphi(x) = x_2,\ \ x = (x_1, x_2)$.

Gamilton-Pontryagin funksiyasini tuzamiz:

$$H(x,\psi,u) = \psi_1 u - \psi_2 x_1^{\,2}.$$

Qo’shma sistemani tuzamiz:

$$\left.\begin{array}{l} \dot{\psi}_1 = -\dfrac{\partial H}{\partial x_1} = 2\psi_2 x_1, \\ \dot{\psi}_2 = -\dfrac{\partial H}{\partial x_2} = 0 \end{array}\right\}$$

Bu sistemaning

$$\psi_1(1) = -\frac{\partial\varphi(x(1))}{\partial x_1} = 0, \qquad \psi_2(1) = -\frac{\partial\varphi(x(1))}{\partial x_2} = -1$$

shartlarni qanoatlantiruvchi yechimi,

$$\psi_2(t) = -1,\ \ t\in[0,1],\ \ \psi_1(t) = -2\int_1^t x_2(\tau)\,d\tau$$

bo’ladi. Qaralayotgan masala uchun (4) maksimum sharti,

$$\psi_1(t)u(t) = \max_{|u|\le 1}\psi_1(t)u$$

ko’rinishga keladi. Demak, har bir

$$u(t) = sign\psi_1(t),\ \ t\in[0,1] \qquad (42)$$

joyiz boshqaruv – ekstremal boshqaruv bo’ladi.

$$u(t)\equiv 1,\ t\in\left[0,\frac{1}{3}\right),\ u(t) = -1,\ \ t\in\left[\frac{1}{3},1\right] \qquad (43)$$

ko'rinishdagi boshqaruv ham ekstremal boshqaruv bo'ladi, chunki unga mos keluvchi trayektoriyaning birinchi komponentasi

$$x_1(t)=t,\ \ t\in\left[0,\frac{1}{3}\right],\ \ x_1(t)=-t+\frac{2}{3},\ \ t\in\left[\frac{1}{3},1\right],$$

bo'lib,

$$\psi_1(t)=2\int_t^1 x_1(\tau)d\tau=2\int_t^{1/3} x_1(\tau)d\tau+2\int_{1/3}^1 x_1(\tau)d\tau=\frac{1}{9}-t^2\ge 0,\ \ t\in\left[0,\frac{1}{3}\right]$$

$$\psi_1(t)=2\int_t^1 x_1(\tau)d\tau=2\int_t^1\left(-\tau+\frac{2}{3}\right)d\tau=t^2-\frac{4}{3}t+\frac{1}{3}\le 0,\ \ t\in\left[\frac{1}{3},1\right],$$

ya'ni (42) shart bajariladi. (43) boshqaruv ekstremal boshqaruv bo'lsa-da, optimal boshqaruv emas. Haqiqatan ham, $\bar{u}(t)\equiv 1,\ t\in[0,1]$ joyiz boshqaruv uchun $J(\bar{u})=-\frac{1}{3}$, (43) uchun esa, $J(u)=-\frac{1}{27}$, ya'ni $J(\bar{u})<J(u)$.

Ekstremal boshqaruvlarning muhim xossalarini quyidagi teoremada keltiramiz.

2-teorema. $u(t)$ $t\in[t_0,t_1]$ – ekstremal boshqaruv bo'lsin. U vaqtda:

1) $H(x(t),\psi(t),u(t),t)$ funksiya $t\in[t_0,t_1]$ oraliqda uzluksiz;

2) $u(t)$ ning har bir $t\in[t_0,t_1]$ uzluksizlik nuqtasida

$$\frac{dH(x(t),\psi(t),u(t),t),}{dt}=\frac{\partial H(x(t),\psi(t),u(t),t)}{\partial t} \tag{44}$$

tenglik bajariladi.

Isboti. Teoremaning birinchi tasdig'i isbotini keltiramiz. (40) munosabatga ko'ra, t va $\bar{t}=t+\Delta t$ vaqt momentlari uchun,

$$H(t)=H(x(t),\psi(t),u(t),t)\ge H(x(t),\psi(t),u(\bar{t}),t)$$

$$H(\bar{t})=H(x(t),\psi(\bar{t}),u(\bar{t}),t)\ge H(x(\bar{t}),\psi(\bar{t}),u(t),t)$$

tengsizliklar bajariladi. Bu yerdan,

$$\begin{aligned}&H(x(\bar{t}),\psi(\bar{t}),u(\bar{t}),\bar{t})-H(x(t),\psi(t),u(t),t)\le H(\bar{t})-H(t)\le\\&\le H(x(\bar{t}),\psi(\bar{t}),u(\bar{t}),\bar{t})-H(x(t),\psi(t),u(\bar{t}),t)\end{aligned} \tag{45}$$

Olingan (45) tengsizlikning chap va o'ng tomonlari $\Delta t\to 0$ da nolga intiladi. Demak, $H(\bar{t})\to H(t)$, $\Delta t\to 0$. Bu esa, *H(t)* funksiyaning ixtiyoriy $t\in[t_0,t_1]$ nuqtada uzluksizligini ko'rsatadi.

Teoremaning ikkinchi tasdig'i isbotini adabiyotlardan ([6] dan) qarash mumkin.

7. Chiziqli terminal boshqarish masalasi. Chiziqli boshqarish masalasi uchun chiziqli terminal kriteriyli quyidagi masalani qaraymiz:

$$\left.\begin{aligned}
&J(u)=c'x(t_1)\to \min,\\
&\dot{x}=A(t)x+b(t,u),\quad t\in[t_0,t_1]\\
&x(t_0)=x^0,\ u=u(t)\in V,\quad t\in[t_0,t_1]
\end{aligned}\right\}\qquad (46)$$

bu yerda $x\in R^n,\ u\in R^m,\quad V\subset R^n,\quad A(t)-n\times n-$matrisa-funksiya, $b(t,u)=(b_1(t,u),...,b_m(t,u)),\ \ c\in R^n,\ \ x^0\in R^n$.

A(t) matrisaning elementlari $[t_0,t_1]$ da uzluksiz, $b_i(t,u),\ \ i=\overline{1,n}$,funksiyalar $[t_0,t_1]\times V$ da uzluksiz deb faraz qilamiz.

(46) masala uchun quyidagi teorema o'rinlidir

3-teorema. $u=u(t),\qquad t\in[t_0,\ t_1]$ bo'lakli-uzluksiz funksiyaning (46) masalada optimal boshqaruv bo'lishi uchun,

$$\max_{v\in V}\psi'(t)b(t,v)=\psi'(t)b(t,u(t)),\ \ t\in[t_0,t_1]\qquad (47)$$

shartning bajarilishi zarur va yetarlidir, bu yerda $\psi(t),t\in[t_0,t_1\ ]$ funksiya

$$\dot{\psi}=-A'(t)\psi,\quad \psi(t_1)=-c\qquad (48)$$

qo'shma sistemaning yechimidan iborat.

Isboti. $H(x,\psi,u,t)=\psi^1[A(t)x+b(t,u)]$ Gamilton-Pontryagin funksiyasi yordamida (47) shartni (40) maksimum sharti ko'rinishida yozish mumkin. (48) sistema esa, (39) ko'rinishda yoziladi. Demak, (47) shartning zaruriyligi 1-teoremadan kelib chiqadi. Shu shartning $u(t)$ optimal boshqaruv bo'lishi uchun yetarliligi (16) munosabatdan kelib chiqadi, chunki ixtiyoriy $\tilde{u}=y(t)$ joyiz boshqaruv uchun,

$$J(\dot{u})-J(u)=\int_{t_0}^{t_1}\psi'(t)\big[b(t,\tilde{u}(t))-b(t,u(t))\big]dt\geq 0$$

bajariladi. Teorema isbotlandi.

2-misol.
$$\begin{aligned}
&J(u)=x_1(1)+x_2(1)\to \min,\\
&\dot{x}_1=x_2,\dot{x}_2=x_1+u,\\
&x_1(0)=x_2(0)=0,|u|\leq 1,t\in[0,1].
\end{aligned}$$

Bu masala (46) ko'rinishdagi masaladir:

$$x=(x_1,x_2),A=\begin{pmatrix}0&1\\0&0\end{pmatrix},b(t,u)=(b_1(t,u),b_2(t,u)),b_1(t,u)=0,$$

$$b_2(t,u)=u,c=(c_1,c_2)=(1,1),V=[-1,1].$$

(47) maksimum sharti,

$$\max_{v\in V}\psi'(t)b(t,v)=\max_{v\in V}\psi_2(t)u(t)=\psi_2(t)u(t)$$

ko'rinishda bo'ladi, bu yerda $\psi(t)=(\psi_1(t),\psi_2(t))$

$$\psi_1(t)=-\psi_2,\quad \psi_2=-\psi_1(1)=-1,\ \psi_2(1)=-1$$

qo'shma sistema yechimidan iborat. Demak, optimal boshqaruv

$$u(t)=sign\psi_2(t),\ t\in[0,1]$$

ko’rinishda bo’ladi. Qo’shma sistemaning yechimi

$$\psi^*{}_1(t) = -e^{1-t}, \quad \psi^*{}_2(t) = -e^{1-t}$$

bo’ladi. Shunday qilib, optimal boshqarish

$$u^*(t) = sign\psi^*{}_2 = sign\left(-e^{1-t}\right) = -1, \quad t \in [0,1]$$

formula bilan aniqlanadi. Optimal trayektoriya esa,

$$\dot{x}_1 = x_2, \ \dot{x}_2 + u^*(t)$$

sistemaning $x_1(0) = x_2(0) = 0$ shartni qanoatlantiruvchi yechimidan iborat. Bu sistemani yechib,

$$x_1^*(t) = 1 - \frac{1}{2}e^t - \frac{1}{2}e^{-t}, \quad x_1^*(t) = -\frac{1}{2}e^t - \frac{1}{2}e^{-t}$$

optimal trayektoriyani topamiz. Funksionalning minimal qiymati

$$\min_{u \in U} J(u) = J\left(u^*\right) = x_1^*(1) + x_2^*(1) = 1 - e$$ bo’ladi.

Mustaqil ishlash uchun savollar

1. Terminal boshqarish masalasining qo’yilishi.Asosiy va qo’shma sistemalar,maksimum sharti.
2. Funksional orttirmasi uchun formulani keltirib chiqaring.
3. Boshqaruvning <<ignasimon>> variatsiyasi va unga mos trayektoriya orttirmasini baholash.
4. <<Ignasimon>> variatsiyadan foydalanib, maksimum shartini isbotlang.
5. Ekstremal boshqaruvlar va ularning xossalari.
6. Chiziqli terminal boshqarish masalasida maksimum shartining optimallik uchun zarur va yetarli ekanligini ko’rsating.

12-ma’ruza. Chiziqli sistemalarni optimal boshqarish. Boshqarishning sintezi

Reja.

1. Chiziqli boshqaruv sistemasi va uning erishish to’plami. Ekstremal prinsip.
2. Chiziqli sistemaning regulyarligi va normalligi.
3. Chiziqli tezkor masala. Optimallikning zaruriy va yetarli shartlari.
4. Pontryaginning maksimum prinsipi.
5. Chiziqli stasionar tezkor masala. Optimal boshqarishning sintezi.

Tayanch iboralar: *Chiziqli boshqaruv sistemasi, yechimlarning fundamental matrisasi, Koshi formulasi, erishish to’plami, ekstremal prinsip, regulyar chiziqli sistema,normal chiziqli sistema, optimallikning zaruriy shartlari, optimallikning yetarli shartlari, maksimum sharti, maksimum prinsipi, chiziqli statsionar tezkor masala, optimal boshqarishning sintezi.*

1. Chiziqli boshqaruv sistemasi. Ekstremal prinsip. Boshqarilayotgan obyektining harakati (jarayon),

$$\dot{x} = A(t)x + B(t)u, \quad t \ge t_0, \quad u \in V \qquad (1)$$

vektorli chiziqli differensial tenglama bilan berilgan bo'lsin, bu yerda $x = (x_1,...,x_n)$ holat vektori , $u = (u_1,...,u_m)$ boshqaruv vektori, $V \subset R^m$ boshqaruvlar to'plami, *t*-vaqt, t_0-boshlang'ich vaqt momenti.

(1) tenglamada $n \times n$ o'lchovli *A(t)* matrisa va $n \times m$ o'lchovli *B(t)* matrisa elementlarini $t \ge t_0$ nuqtalarda uzluksiz deb faraz qilamiz. *V* boshqaruvlar to'plamini R^m ning bo'sh bo'lmagan qavariq kompakt to'plami deb hisoblaymiz.

Odatdagidek, (1) boshqaruv sistemasi uchun joyiz boshqarishlar sifatida, *V* boshqarishlar to'plamidan qiymatlar qabul qiluvchi bo'lakli-uzluksiz $u = u(t),\, t \in [t_0, t_1], m$-vektor-funksiyalarni qaraymiz va bunday boshqarishlar to'plamini *U* deb belgilaymiz.

Differensial tenglamalar kursidan yaxshi ma'lumki, har bir $u = u(t),\, t \in [t_0, t_1],$ - joyiz boshqarishga va

$$x(t_0) = x^0 \qquad (2)$$

boshlang'ich shartga (1) tenglamaning $[t_0, t_1],$- oraliqda bo'lakli-silliq $x(t) = x(t, u, x^0),$-yechimi mos keladi hamda bu yechim

$$x(t) = F(t,t_0)x^0 + \int_{t_0}^{t} F(t,\tau)B(\tau)u(\tau)d\tau,\ t \in [\mathrm{t}_0, \mathrm{t}_1] \qquad (3)$$

Koshi formulasi [2] orqali ifodalanadi, bu yerda $F(t,\tau) - (1)$ tenglamaga mos

$$\dot{x} = A(t)x \qquad (4)$$

bir jinsli tenglama yechimlarining fundamental matrisasidan iborat, ya'ni $F(t,\tau) - n \times n$ - o'lchovli matrisa bo'lib, uning *i*- ustuni (4) tenglamaning $x(\tau) = e^i$ (e^i-*E* birlik matrisaning *i*-ustuni) boshlang'ich shartni qanoatlantiruvchi $x^i(t,\tau)$ yechimidan iborat.

(3) formulaga ko'ra , $x(t) = x(t, u, x^0),$ trayektoriyaning $t = t_1$ vaqt momentiga mos nuqtasi,

$$x(t_1) = F(t_1,t_0)x^0 + \int_{t_0}^{t_1} F(t_1,t)B(t)u(t)dt, \qquad (5)$$

ko'rinishda bo'ladi. Barcha $u(t) \in U$ boshqarishlarga mos (5) ko'rinishdagi nuqtalarni qaraymiz. Ular R^n da qandaydir $Q(t_1)$ to'plamni hosil qiladi. Shu to'plamni (1),(2) boshqaruv sistemasining t_1 vaqt momentidagi *erishish to'plami* deb ataymiz. Demak, ta'rifga ko'ra,

$$Q(t_1) = \{x \in R^n : x = F(t_1,t_0)x^0 + \int_{t_0}^{t_1} F(t_1,t)B(t)u()dt,\ u(t) \in U\}.$$

$Q(t_1)$ to'plam elementlarinig (5) ko'rinishda ifodalanishidan va *V* boshqarishlar to'plamining qavariq kompaktligidan, $Q(t_1)$ to'plamning ham chegaralangan qavariq to'plam ekanligi kelib chiqadi.

Endi $Q(t_1)$ to'plamning yopiqligini ta'minlovchi shartni keltiramiz. Bu shart (1) sistemaning regulyarlik xossalari orqali ifodalanadi.

Avvalo *V* boshqarishlar to'plamining *k*-o'lchamli yoqi tushunchasini keltiramiz.

Agar $S\subset V$ qism to'plam *V* to'plamga o'tkazilgan, normallari chiziqli bog'lanmagan *m-k* ta tayanch tekisliklar kesishmasiga tegishli bo'lsa, *S* to'plam,
V ning *k*-o'lchovli yoqi deyiladi. *V* to'plamning o'zini esa, *m*-o'lchovli yoqdan iborat deb hisoblaymiz.

Agar ixtiyoriy $c\in R^n, c\neq 0$ vektor va ixtiyoriy $[t_0,t_1]$ kesma uchun $c'F(t_1,t)B(t)$ funksiya *V* to'plamning har bir *k*- o'lchamli yoqiga $[t_0,t_1]$ oraliqning chekli sondan ko'p bo'lmagan nuqtalari yoki kesmalarida ortogonal bo'lsa, (1) sistema regulyarlik shartini qanoatlantiradi, deyiladi.

Quyidagi tasdiq o'rinli:

1-lemma [2]. Agar (1) sistema regulyarlik shartini qanoatlantirsa, $Q(t_1)$- yopiq, chegaralangan, qavariq to'plam bo'ladi.

Chiziqli boshqaruv sistemalarini o'rganishda, ekstremal prinsip deb ataluvchi, quyidagi tasdiqdan keng foydalaniladi.

2-lemma. Faraz qilaylik, (1) sistema regulyarlik shartini qanoatlantirsin. U vaqtda har bir $c\in R^n, c\neq 0$ vektor $t_1\geq t_0$ va son uchun $Q(t_1)$ to'plamning shunday $\bar{x}$ chegaraviy nuqtasi mavjul bo'ladiki, unda

$$c'\bar{x}=\max_{x\in Q(t_1)} c'x \qquad (6)$$

munosabat bajariladi. (6) tenglikning o'rinli bo'lishi uchun, shunday $\bar{u}(t)\in U$ boshqarish topilib,

$$\bar{x}=F(t_1,t_0)x^0+\int_{t_0}^{t_1}F(t_1,t)B(t)u(t)dt, \qquad (7)$$

$$c'F(t_1,t)B(t)\bar{u}(t)=\max_{v\in V}c'F(t_1,t)B(t)v,\ t\in[t_0,t_1] \qquad (8)$$

tengliklarning bajarilishi zarur va yetarlidir.

Isboti. 1-lemmaga ko'ra $Q(t_1)$ to'plam kompakt bo'lganligi uchun, Veyershtrass teoremasidan, uzluksiz $f(x)=c'x$ funksiyaning $Q(t_1)$ da global maksimum nuqtasi $\bar{x}$ mavjudligi kelib chiqadi, ya'ni (6) tenglik bajariladi. Bu nuqta $\bar{x}$ to'plamning chegaraviy nuqtasi bo'ladi. Haqiqatan ham, agar $\bar{x}\in \operatorname{int}Q(t_1)$ deb faraz qilsak, yetarli kichik $\varepsilon>0$ uchun $x+\varepsilon c\in Q(t_1)$ bo'ladi va $c'(x+\varepsilon c)=c'x+\varepsilon$, $\|c\|'>c'x$, ya'ni (6) ga zid munosabat bajariladi.

$Q(t_1)$ to'plamning aniqlanishiga ko'ra, $\bar{x}$ nuqta, biror $\bar{u}(t)\in U$ boshqarish yordamida, (7) formula bo'yicha hosil qilinadi. Endi (6) shartni $c'\bar{x}\geq c'x,\ \forall x\in Q(t_1)$ ko'rinishda yozib va bu yerga $\bar{x}$ nuqtaning (7) ifodasini va $x\in Q(t_1)$ nuqtaning, Koshi formulasi orqali,

$$x=F(t_1,t_0)x^0+\int_{t_0}^{t_1}F(t_1,t)B(t)u(t)dt,\ u(t)\in \mathrm{U}$$

ifodasini qo’ysak, quyidagi

$$\int_{t_0}^{t}[c'F(t_1,t)B(t)\bar{u}(t)-c'F(t_1,t)B(t)\bar{u}(t)]dt>0 \qquad (9)$$

munosabatni hosil qilamiz. Bu tengsizlik ixtiyoriy $\bar{u}(t)\in U$ boshqaruv uchun, jumladan,

$$u(t)=\begin{cases}\bar{u}(t),\ t\notin(\theta-\varepsilon,\theta],\\ v,\ t\in(\theta-\varepsilon,\theta],\end{cases} \qquad (10)$$

ko’rinishdagi boshqarish uchun ham bajariladi, bu yerda $\theta\in(t_0,t_1], v\in V,\ \varepsilon>0$ -yetarli kichik $(t_0<\theta-\varepsilon<t_1)$. Agar (10) boshqarishni (20) ga qo’yib, uni *ε>0* ga bo’lsak va *ε→0* da limitga o’tsak,

$$\lim_{\varepsilon\to 0}\int_{\theta-\varepsilon}^{\theta}[c'F(t_1,t)B(t)\bar{u}(t)-c'F(t_1,t)B(t)v]dt=$$

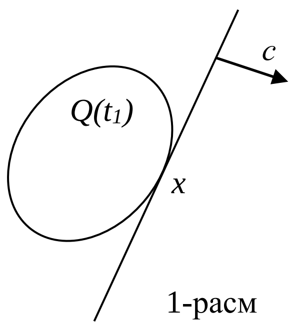
$$=c'F(t_1,\theta)B(\theta)\bar{u}(\theta)-c'F(t_1,\theta)B(\theta)\quad v\geq 0$$

tengsizlikni hosil qilamiz. Bu esa, $v\in V$ vektorning ixtiyoriyligiga ko’ra,

$$c'F(t_1,\theta)B(\theta)u(\theta)=\max_{v\in V}c'F(t_1,\theta)B(\theta)v \qquad (11)$$

tenglikning bajarilishini ko’rsatadi. Shunday qilib, (8) tenglik barcha $\theta\in(t_0,t_1],$ uchun bajarilishi ko’rsatildi. Agar *u(t)* boshqarishning bo’lakli uzluksizligini hisobga olsak, uni $t=t_0$ nuqtada chapdan uzluksiz deb hisoblashimiz mumkin. Natijada, (11) da $\theta\to t_0$ da limitga o’tsak, uning $\theta=t_0$ nuqtada ham o’rinli ekanligini ko’ramiz. Shunday qilib, (8) tenglikning barcha $t\in[t_0,t_1],$- uchun to’g’riligi ko’rsatildi. Lemma isbotlandi.

Keltirilgan ekstremal prinsip sodda geometrik ma’noga ega.Ekstremal prinsipga ko’ra, (8) shartni qanoatlantiruvchi boshqarishlar (ularni *ekstremal boshqarishlar* ham deb ataydilar) va faqat shu boshqarishlargina chiziqli sistema trayektoriyasinii erishish to’plami chegaraviy nuqtasiga o’tkazadi. (8), (7) formulalar bo’yicha *u(t)* boshqarishni va $\bar{x}\in Q(t_1)$ nuqtani hosil qiluvchi *c≠0* vektor esa, *Q(t₁)* to’plamdagi $\bar{x}$ nuqtaga o’tkazilgan tayanch tekislikning normalidan iborat (1-rasm).



1-расм

Yuqorida ta’kidlanganidek, chiziqli sistema uchun regulyarlik sharti erishish to’plamining yopiqligini ta’minlaydi. Endi erishish to’plamining yopiqligi bilan bir qatorda uning qat’iy qavariqligini ham ta’minlovchi yana bir shartni keltiramiz. U *normallik sharti* deyiladi.

(1) chiziqli sistema uchun normallik sharti, har bir *c≠0* va $t_1>t_0$ da (8) maksimum shartidan *u(t)* bo’lakli-uzluksiz funksiyaning $[t_0,t_1]$ dagi barcha uzluksizlik nuqtalarida bir qiymatli aniqlanishini ifodalaydi.

Normallik sharti, regulyarlik shartidan kuchliroq talabdir, chunki normallik shartiga ko'ra, ixtiyoriy $c\neq0$ va $t_1>t_0$ uchun $c'F(t_1,t)B(t)$ funksiya *V* to'plamning yoqlariga $[t_0,t_1]$ oraliqning faqat alohida olingan nuqtalaridagina ortogonal bo'lishi mumkin, ya'ni $[t_0,t_1]$ ning qism intervallarida ortogonallik qaralmaydi.

3-lemma. Agar (1) chiziqli sistema normal bo'lsa va *V* boshqarishlar to'plami bittadan ko'p elementli bo'lsa, ixtiyoriy (2) boshlang'ich shart va $t_1>t_0$ uchun $Q(t_1)$ erishish to'plami qat'iy qavariq bo'ladi.

Lemmaning isbotini [2] dan qarash mumkin.

2.Chiziqli tezkor masala. Optimallikning zaruriy va yetarli shartlari. (1) sistema uchun ikki nuqtali tezkor masalani qaraymiz: shunday $u^*(t)\in U$ boshqarishni topish talab qilinadiki, unga mos $x^*(t)$ trayektoriya, t_0,t_1 vaqt momentlarida berilgan x^0,x^1 nuqtalardan o'tsin, ya'ni

$$x^*(t_0)=x^0,\, x^*(t_1^*)=x^1 \qquad (x^0\neq x^1)$$

shartlar bajarilsin va t_1-t_0 o'tish vaqti minimal bo'lsin. Bunda $u^*(t)$ ga optimal boshqarish, $x^*(t)$ ga optimal trayektoriya, t_1^* ga optimal vaqt momenti (tezkor moment) deyiladi. $(u^*(t),x^*(t),t_1^*)$ esa masalaning yechimidir. Qaralayotgan masalani , qisqacha,

$$\left.\begin{aligned} & t_1-t_0\to\min \\ & \dot{x}=A(t)x+B(t)u,\ t\in[t_0,t_1] \\ & x(t_0)=x^0, x(t_1)=x^1, \\ & u(t)\in u \end{aligned}\right\} \qquad (12)$$

ko'rinishda belgilaymiz.

Quyidagi funksiyani kiritamiz:

$$\rho(t)=\min_{\|c\|=1}\left[c'F(t_1,t)x^0+\int_{t_0}^{t}\max_{u\in V}c'F(t,\tau)B(\tau)ud\tau-c'x'\right],\ t\geq t_0$$

Bu funksiya barcha $t_0>t_1$ nuqtalarda uzluksizdir.

1-teorema. Faraz qilaylik, (1) sistema regulyarlik shartini qanoatlantirsin. $(u^*(t),x^*(t),t_1^*)$ - (12) masalaning yechimi bo'lsin. U vaqtda:

1) t_1^* optimal vaqt momenti

$$\min_{\|c\|=1}\left[c'F(t,t_0)x^0+\int_{t_0}^{t}\max_{u\in V}c'F(t,\tau)B(\tau)ud\tau-c'x'\right]=0 \qquad (13)$$

tenglamaning minimal ildiziga teng;

2) $u^*(t)$ optimal boshqarish,

$$c^{*'}F(t_1^*,t_0)B(t)u^*(t)=\max_{u\in V}c'F(t_1^*,t_0)B(t)u,\ t_0\leq t\leq t_1^* \qquad (14)$$

maksimum shartini qanoatlantiradi, bu yerda $c^*\in R^n$, $c^*\neq0$ vektor $t=t_1^*$ bo'lganda (13) ning chap tomonidagi ifodaning ixtiyoriy minimum nuqtasidir;

3) $x^*(t)$ optimal trayektoriya

$$\dot{x}^*=A(t)x^*+B(t)u^*(t),\ x^*(t_0)=x^0 \qquad (15)$$

munosabatlarni qanoatlantiradi.

Isboti.1) $x^1 \in Q(t_1)$ munosabat $\rho(t_1) \geq 0$ ga teng kuchlidir. Shuning uchun t_1^* optimal vaqt momenti

$$t_1^* = \min\{t_1 : \rho(t_1) \geq 0,\ t_1 > t_0\},$$

ya'ni $\rho(t_1^*) \geq 0$ bajariladi. *ρ(t)* funksiyaning uzluksizligidan va t_1^* ning optimalligidan $\rho(t_1^*) > 0$ bo'la olmasligi kelib chiqadi. Demak, $\rho(t_1^*) = 0$ va t_1^*-(13) tenglamaning eng kichik ildizidan iborat.

2) $u^*(t)$ optimal boshqarish bo'lgani uchun, Koshi formulasiga ko'ra,

$$x^1 = F(t_1^*, t_0)x^0 + \int_{t_0}^{t_1} F(t_1^*, t)B(t)u^*(t)dt, \qquad (16)$$

bajariladi. Endi (13) tenglikda $t = t_1^*$ deb uni $c \in R^n, \|c\| = 1$ bo'yicha ixtiyoriy c^* minimum nuqtasi uchun yozamiz:

$$c^{*'} F(t_1^*, t_0)x^0 + \int_{t_0}^{t_1} \max_{v \in V} c^{*'} F(t_1^*, t)B(t)u^*(t)dt - c^{*'} x^1 = 0,$$

Bu tenglikda x^1 o'rniga uning (16) dagi ifodasini keltirib qo'yib, quyidagi

$$\int_{t_0}^{t_1} [\max_{v \in V} c^{*'} F(t_1^*, t)B(t)u - c^{*'} F(t_1^*, t)B(t)u^*(t)]dt = 0,$$

tenglikni olamiz. Bu oxirgi tenglikda integral ostidagi funksiya manfiymas va *t* bo'yicha bo'lakli-uzluksiz bo'lgani uchun undan talab qilingan (14) munosabatni hosil qilamiz.

3) $x^*(t)$ optimal trayektoriya (15) munosabat orqali bir qiymatli aniqlanadi.

Teorema isbotlandi.

2-teorema. Faraz qilaylik, (1) sistema normallik shartini qanoatlantirsin. Agar $u^*(t)$ boshqarish, $x^*(t)$ trayektoriya va t_1^* vaqt momenti 1-teoremadagi 1)-3) shartlarni qanoatlantirsa, $(u^*(t), x^*(t), t_1^*)$ -(12) tez harakat masalasining yechimi bo'ladi.

Isboti. *ρ(t)* funksiyaning ta'rifi va uning uzluksizligidan

$$t_1^* = \min\{t_1 : \rho(t_1) = 0,\ t_1 > t_0\},$$

ekanligi kelib chiqadi, ya'ni (13) tenglamaning minimal ildizi optimal vaqt momentini aniqlaydi. $x^*(t)$ optimal trayektoriya (15) tenglik orqali bir qiymatli aniqlanganligi uchun, $u^*(t)$ boshqarishning optimalligini ko'rsatsak, yetarli bo'ladi.

(13) tenglikda $t = t_1^*$ deb uni $c \in R^n, \|c\| = 1$ bo'yicha ixtiyoriy s* minimum nuqtasi uchun yozamiz:

$$0 = c^{*'} F(t_1^*, t_0)x^0 + \int_{t_0}^{t_1} \max_{v \in V} c^{*'} F(t_1^*, t)B(t)udt - c^{*'} x^1 = c^{*'} F(t_1^*, t_0)x^0 +$$
$$+ \int_{t_0}^{t_1} c^{*'} F(t_1^*, t)B(t)u^*(t)dt - c^{*'} x^1 = c^{*'} x^*(t_1) - c^{*'} x^1 \qquad (17)$$

Ekstremal prinsipga (2-lemma) va $u^*(t)$ boshqarish hamda $x^*(t)$ trayektoriyalarning aniqlanishiga ko'ra,

$$c^{*'}(x - x^*(t_1)) = 0 \qquad (18)$$

tekislik $Q(t_1)$ erishish to'plamining $x^*(t_1)$ nuqtasiga o'tkazilgan tayanch tekislik bo'ladi.

(17) tenglik ko’rsatadiki, x^1 nuqta (18) tayanch tekislikda yotadi. t_1^* -tezkor moment bo’lgani uchun $x^1 \in Q(t_1^*)$ bo’ladi. Agar $x^*(t_1^*) \neq x^1$ bo’lsa, $\left[x^1, x^*(t_1^*)\right]$ kesma *Q(t₁)* ning chegarasiga tegishli bo’lar edi. Ammo 3-lemmaga ko’ra, $Q(t_1)$ qat’iy qavariq to’plam ekanligidan, bunday bo’lishi mumkin emas. Demak, $x^*(t_1^*) = x^1$ ya’ni $u^*(t),\ t \in [t_0, t_1^*]$ optimal boshqarish bo’ladi.

Teorema isbotlandi.

Optimallikning zaruriy va yetarli shartlarini ifodalovchi 1-2-teoremalarga qo’shimcha qilib shuni aytish mumkinki, normallik sharti bajarilganda optimal boshqarish (va demak, optimal trayektoriya ham) yagona bo’ladi. Bu tasdiq (14) maksimum shartidan osongina kelib chiqadi.

3. Tezkor masala uchun Pontryaginning maksimum prinsipi.

Optimal boshqaruv nazariyasida optimallikning zaruriy sharti Pontryaginning maksimum prinsipi ko’rinishida ifodalanadi. Quyida 1-teoremada keltirilgan zaruriy shartlarni maksimum prinsipi shaklida yozish mumkinligini ko’rsatamiz.

Quyidagi

$$\psi^*(t) = F(t_1^*, t)' c^* \qquad (19)$$

funksiyani kiritamiz va (14) shartni

$$\psi^{*'}(t) B(t) u^*(t) = \max_{v \in V} \psi^{*'}(t) B(t) u,\ t_0 \le t \le t_1 \qquad (20)$$

ko’rinishda yozamiz. (19) funksiya

$$\dot{\psi} = -A'(t)\psi, \psi(t_1^*) = c^* \qquad (21)$$

qo’shma sistemaning yechimidir. Agar

$$H(x, \psi, u, t) = \psi'[A(t)x + B(t)u]$$

Gamilton-Pontryagin funksiyasidan foydalansak, $x^*(t)$ ning (15) ni, $u^*(t)$ ning (14) ni va $\psi^*(t)$ ning esa, (21) ni qanoatlantirishini,

$$\dot{x}^*(t) = H_\psi(x^*(t), \psi^*(t), u^*(t), t) \qquad (22)$$

$$\dot{\psi}^*(t) = -H_x(x^*(t), \psi^*(t), u^*(t), t) \qquad (23)$$

$$H(x^*(t), \psi^*(t), u^*(t), t) = \max_{v \in V} H(x^*(t), \psi^*(t), u, t), \quad t \in [t_0, t_1^*] \qquad (24)$$

ko’rinishda yozish mumkin. Agar bu sistemani yana bita,

$$H(x^*(t_1^*), \psi^*(t_1^*), u^*(t_1^*), t_1^*) \ge 0 \qquad (25)$$

munosabat bilan to’ldirsak, tezkor masala uchun quyidagi maksimum prinsipiga ega bo’lamiz.

3-teorema (maksimum prinsipi). Faraz qilaylik, (1) sistema uchun regulyarlik sharti bajarilsin. Agar $(u^*(t), x^*(t), t_1^*)$ (12) masalaning yechimi bo’lsa, shunday trivial (aynan nol) bo’lmagan $\psi^*(t)$ vektor funksiya topiladiki, (22)-(25) shartlar bajariladi.

Mustaqil ishlash uchun savollar.

1. Chiziqli boshqarish sistemasi erishish to’plamining xossalari (qavariqlik, kompaktlik, ekstremal prinsip).
2. Chiziqli boshqarish sistemasi uchun regulyarlik va normallik shartlari. Bu shartlar erishish to’plamining qanday xossalarini ta’minlaydi?

3. Chiziqli tez harakat masalasi uchun optimallikning zaruriy va yetarli shartlari. Pontryaginning maksimum prinsipi.

13-ma'ruza. Chiziqli tezkor masala, chiziqli tezkor masala yechimining mavjudligi. Chiziqli tezkor masalada optimallikning zaruriy shartlari

4. Chiziqli statsionar tezkor masala. Quyidagi

$$\begin{aligned} &t_1 - t_0 \to \min,\ \dot{x} = Ax + Bu, x(t_0) = x^0, x(t_1) = x^1, \\ &u = u(t) \in V, t \in [t_0, t_1], \end{aligned} \tag{26}$$

tezkor masalani qaraymiz, bu yerda A-$n\times n$- matrisa va B-$n\times m$- matrisalar o'zgarmas, $V \subset R^m$ qavariq kompakt ko'pyoqlidan iborat.

4-lemma [2]. Agar V ko'pyoqlining istalgan qirrasiga parallel bo'lgan w vektor uchun

$$rank\{Bw, ABw, ..., A^{n-1}Bw\} = n \tag{27}$$

bo'lsa, (26) sistema uchun normallik sharti bajariladi.

Yuqorida keltirilgan natijalar (26) statsionar tezkor masala uchun quyidagi teoremada aniqlashtiriladi.

4-teorema. Faraz qilaylik, (26) sistema uchun (27) shart bajarilsin. U vaqtda:

a) chiziqli statsionar tezkor masalaning $(u^*(t), x^*(t), t_1^*)$ yechimi uchun shunday $\psi^*(t)$ funksiya mavjud bo'ladiki,

$$\dot{x}^*(t) = H_\psi(x^*(t), \psi^*(t), u^*(t)), \dot{\psi}^*(t) = -H_x(x^*(t), \psi^*(t), u^*(t)) \tag{28}$$

$$H(x^*(t), \psi^*(t), u^*(t)) = \max_{v \in V} H(x^*(t), \psi^*(t), u), \quad t_0 \le t \le t_1^*] \tag{29}$$

shartlar bajariladi va $u^*(t)$ boshqarish $[t_0, t_1^*]$ da bo'lakli o'zgarmas bo'lib, uning qiymatlari V ko'pyoqlining uchlaridan iborat (bu yerda $H(x, \psi, u) = \psi'[Ax + Bu]$).

b) agar V ko'pyoqlining u^1 ichki nuqtasi uchun $Ax^1 + Bu^1 = 0$ shart bajarilsa va $u^*(t), x^*(t), \psi^*(t)$ funksiyalar (28),(29) shartlarni va $x^*(t_1^*) = x^1$ tenglikni qanoatlantirsa, $u^*(t)$- optimal boshqarish, $x^*(t)$- optimal trayektoriya. t_1^* - optimal vaqt momenti bo'ladi.

c) optimal boshqarish yagonadir.

Isboti. Teoremaning *a)* tasdig'i 3-teoremaning natijasidir. *b)* tasdiq esa, 4-lemmani hisobga olgan holda, normallik shartidan va (29) maksimum shartidan kelib chiqadi. *b)* tasdiqni (26) masalaning xususiy holi bo'lgan,

$$t_1 \to \min,\ \dot{x} = Ax + Bu, x(0) = x^0, x(t_1) = 0, \quad |u| \le 1 \tag{30}$$

masala uchun isbotlaymiz, bu yerda A-$n\times n$ matrisa, $B \in R^m$,u- skalyar boshqaruv parametri.

(30) sistema uchun (27) shart,

$$rank\{B, AB, ..., A^{n-1}B\} = n \tag{31}$$

ko'rinishda bo'ladi.

Asosiy va qo'shma sistemalardan iborat,

$$\dot{x} = A(t)x + Bu,\ \dot{\psi} = -A'(t)\psi$$

sistemani qanoatlantiruvchi ixtiyoriy $u(t), x(t), \psi(t),\ t \geq 0$ funksiyalar uchun *u(t)* ning uzluksizlik nuqtalarida quyidagi

$$\frac{d}{dt}\psi'(t)x(t) = \dot{\psi}(t)x(t) + \psi'(t)\dot{x}(t) = -\psi'(t)Ax(t) + \psi'(t)Ax(t) +$$
$$+\psi'(t)bu(t) = \psi'(t)bu(t)$$

tenglik bajariladi. Bu tenglikni $[\sigma,\tau]$ oraliqda integrallab,

$$\psi'(\tau)x(\tau) - \psi'(\sigma)x(\sigma) = \int_{\sigma}^{\tau}\psi'(t)bu(t)dt \qquad (32)$$

tenglikni hosil qilamiz.

Faraz qilaylik, $u^*(t)$- boshqarish va unga mos $x^*(t), \psi^*(t)$ funksiyalar $[0, t_1^*]$ oraliqda (28),(29) hamda $x^*(0) = x^0,\ x^*(t_1^*) = 0$ shartlarni qanoatlantirsin, ammo $u^*(t)$ optimal bo'lmasin. U holda, shunday $u^0(t), t\in[0,t^0]$ boshqarish mavjudki, unga mos $x^0(t)$ trayektoriya $x^0(0) = x^0,\ x^{-0}(t_1^0) = 0$ shartlarni qanoatlantiradi va $t_1^0 < t_1^*$ tengsizlik bajariladi. (32) munosabatdan foydalanib, quyidagiga ega bo'lamiz:

$$\begin{aligned}\psi^{*'}(t_1^0)x^*(t_1^0) &= \left[\psi^{*'}(t_1^0)x^*(t_1^0) - \psi^{*'}(0)x^*(0)\right] - \left[\psi^{*'}(t_1^0)x^0(t_1^0) - \psi^{*'}(0)x^0(0)\right] = \\ &= \int_0^{t_1^0}\left[\psi^{*'}(t)bu^*(t) - \psi^{*'}(t)bu^0(t)\right]dt\end{aligned} \qquad (33)$$

Teoremaning shartiga ko'ra, $u^0(t), t\in[0,t^0]$, boshqarish maksimum prinsipini qanoatlantiradi, ya'ni

$$\psi^{*'}(t)bu^*(t) \geq \psi^{*'}(t)bu^0(t), \quad t\in\left[0,t_1^*\right]\supset\left[0,t_1^0\right]$$

tengsizlik bajariladi. Shuning uchun, (33) dan, $\psi^{*'}(t_1^0)x^*(t_1^0) \geq 0$ tengsizlik kelib chiqadi.

Ikkinchi tomondan, maksimum shartiga ko'ra,

$$\psi^{*'}(t)bu^*(t) = \max_{|u|\leq 1}\psi^{*'}(t)bu,\ t\in[0,t_1^*]$$

bo'lgani uchun, $u^*(t) = sign\,\psi^{*'}(t)b$ ko'rinishda bo'ladi. (31) shartdan $\psi^{*'}(t)b \neq 0,\ t\in[0,t_1^*]$, bo'lishi kelib chiqadi. Shunday qilib, $t_1^0 < t_1^*$ ekanligini hisobga olsak, (32) ga asosan,

$$\psi^{*'}(t_1^0)x^*(t_1^0) = \psi^{*'}(t_1^0)x^*(t_1^0) - \psi^{*'}(t_1^0)x^*(t_1^*) = -\int_{t_1^0}^{t_1^*}\psi^{*'}(t)bu^*(t)dt = -\int_{t_1^0}^{t_1^*}|\psi^{*'}(t)b|\,dt < 0$$

munosabat bajariladi. Olingan qarama-qarshilik, $u^*(t), t\in[0,t_1^*]$ boshqarishning optimalligini ko'rsatadi. Teorema isbotlandi.

Optimal boshqarishlarni qurishda quyidagi natija ham muhim ahamiyatga ega [2,3].

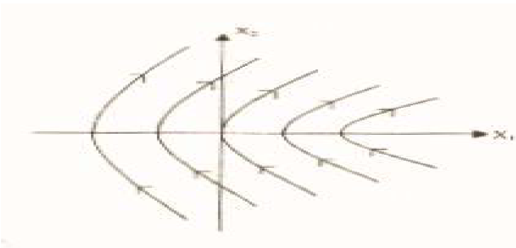
5-teorema (*n* intervallar haqida) [2]. Faraz qilaylik, (25) statsionar tezkor masala uchun (27) shart bajarilsin, $V = \left\{v\in R^m : |v_i| \leq 1,\ i = \overline{1,m}\right\}$ va *A* matrisaning barcha xos qiymatlari haqiqiy bo'lsin. U vaqtda $u^*(t) = (u_1^*(t),...,u_m^*(t))$ ekstremal boshqarish har bir $u_i^*(t)$ koordinatasining o'zgarmaslik intervallari soni *n* tadan oshmaydi.

Misol. (30) masalada $n=2, x^0=(x_1^0, x_2^0), x^1=(0,0), A=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\ b=\begin{pmatrix}0\\ 1\end{pmatrix}$ bo'lsin. Bu holda 5-teoremaning barcha shartlari bajarilgan bo'ladi, shuning uchun ekstremal boshqarish, *u=+1 va u=-1* qiymatlar qabul qiladi va o'tish nuqtasi bittadan ko'p emas, ya'ni o'zgarmaslik oraliqlari soni ikkitadan oshmaydi.

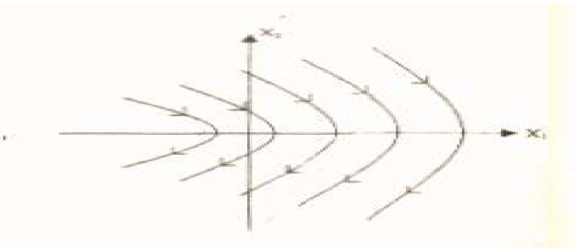
u=±1 bo'lganda $x_2=\pm 1$ nuqtaning harakat tenglamasi -=-=-= ko'rinishda bo'ladi. Bu yerdan

$$\frac{dx_1}{dx_2}=\pm x_2,\ x_1=\pm\frac{x_2^2}{2}+const \qquad (34)$$

bo'lishi kelib chiqadi. ularga mos harakat trayektoriyalari 2-3-chizmalarda keltirilgan.



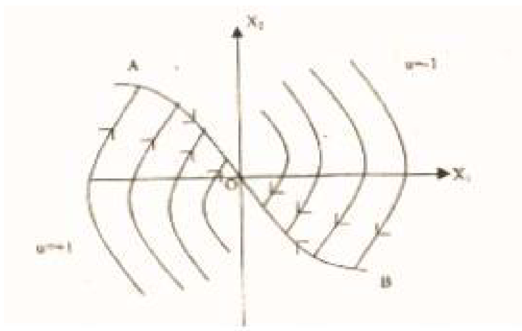
2-rasm.



3-rasm.

Koordinatalar boshiga kelib tushuvchi *AO,OV* trayektoriyalarni ajratamiz (4-5-chizmalar).

Agar boshlang'ich x^0 nuqta *AO* yoki *OV* yoy ustida yotsa, maksimum prinsipini qanoatlantiruvchi jarayon topilgan hisoblanadi. Aks holda (34) trayektoriyalar ichidan *AO* yoki *OV* orqali o'tadiganini aniqlaymiz. Natijada koordinata boshiga olib keluvchi va faza tekisligini to'ldiruvchi egri chiziqlar oilasini quramiz (6-chizma).

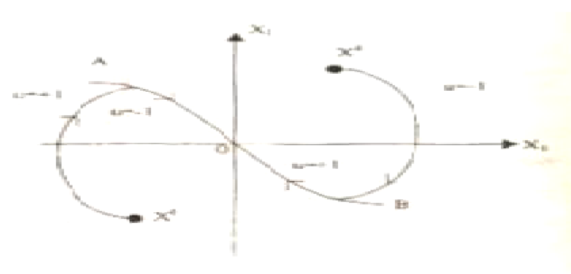


Bu chiziqlar oilasiga tegishli har bir egri chiziq (34) parabolalarning ikki bo'lagidan iborat bo'lib, ular maksimum prinsipini qanoatlantiradi. 4-teoremaga ko'ra, bu trayektoriyalar optimaldir. Koordinata boshiga yo'nalgan optimal harakat quyidagicha amalga oshadi:

$$u=\begin{cases}-1, & \text{агар } x^0 \text{ бошлангичнукта } AOB \text{ чизикданюкори}\\ +1, & \text{агар } x^0 \text{ бошлангичнукта } AOB \text{ чизикданкуйида ёки } OB \text{ ёй устида ётса.}\end{cases}$$

Boshqarishga (35) formula bo'yicha beriladigan qiymatlar $x=(x_1,x_2)$ nuqta *AOV*

4-расм.



5-расм.

chiziqqa (yoki koordinitalar boshiga) tushguncha saqlab qolinadi. $x=(x_1,x_2)$ nuqta *AOV* chiziqqa tushgan vaqt momentida boshqarishning qiymatini *-1* dan *+1* ga yoki aksincha o'zgartirib, x nuqta koordinata boshiga tushguncha saqlab qolinadi. Boshqaruvning qiymatlari o'zgaradigan *AOV* chiziq, *o'tish chizig'i* deb ataladi.

Endi $v(x^0)$ deb, (35) formula bilan aniqlanuvchi funksiyani belgilaymiz. U vaqtda optimal boshqarish $u=v(x^0)$ qoida bo'yicha hosil qilinadi. x^0 nuqta ixtiyoriy bo'lgani uchun, bu qoida optimal boshqarishni barcha $x=(x_1,x_2)$ nuqtalarda $u=v(x)$ formula bo'yicha quradi. Ana shunday qoida bo'yicha optimal boshqarishni qurishga uning *optimal sintezi* deyiladi. Demak, (35) formula qaralayotgan misolda optimal boshqarishning sintezini beradi. Optimal trayektoriya esa,

$$\dot{x}_1=x_2,\ \ \dot{x}_2=v(x)$$

sistemani integrallash orqali quriladi.

Mustaqil ishlash uchun savollar.

1. Chiziqli boshqarish sistemasi erishish to'plamining xossalari (qavariqlik, kompaktlik, ekstremal prinsip).
2. Chiziqli boshqarish sistemasi uchun regulyarlik va normallik shartlari. Bu shartlar erishish to'plamining qanday xossalarini ta'minlaydi?
3. Chiziqli tez harakat masalasi uchun optimallikning zaruriy va yetarli shartlari. Pontryaginning maksimum prinsipi.
4. Chiziqli stasionar tez harakat masalasida optimallik shartlari va ulardan foydalanib optimal boshqaruvni qurish. Optimal boshqaruvning sintezi.

14-ma'ruza. Chiziqli stasionar tezkor masala. Holatning umumiylik sharti (lemma). Chiziqli stasionar tezkor masalada yechimning mavjudligi va yagonaligi (teorema). N intervallar haqidagi teorema

Reja:

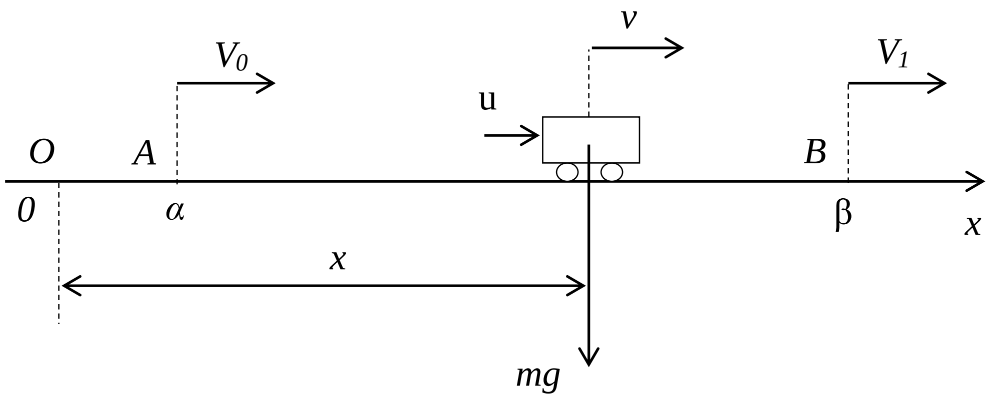
1. Optimal boshqaruv masalalari haqida.
2. Sodda tezkor masalaning qo'yilishi va uning matematik ifodasi.
3. Tezkor masala va braxistoxrona haqidagi masala orasidagi bog'lanish.
4. Tezkor masalaning umumiy umumiy qo'yilishi.
5.Tezkor masala yechimining mavjudligi.
6. Tezkor masala qaralganda kelib chiqadigan muammolar.

1.Optimal boshqaruv nazariyasi- variatsion hisob rivojlanishining hozirgi bosqichini ifodalab, u yangi texnika rivojining sohalarida amaliyot tomonidan qo'yilgan masalalar bilan bog'liq holda XX –asrning o'rtalarida paydo bo'ldi.

Optimal boshqaruv matematik nazariyasining asosida, optimallikning zaruriy shartidan iborat **Pontryaginning maksimum prinsipi** va optimallikning yetarli shartidan iborat **Bellmanning dinamik programmalashtirish** usuli yotadi.

2. Bu masala avtomatik boshqaruv bo'yicha mutaxasislar tomonidan qo'yilgan daslabki optimal boshqaruv masalasi bo'lganligi uchun, uni *optimal boshqaruvning asosiy masalasi* deb ham atashadi. Uning mohiyatini quyidagi eng sodda misolda tushuntirishga harakat qilamiz.

m massali aravachani, moduli *L* dan oshmaydigan *u* gorizontal kuch yordamida, gorizontal to'g'ri chiziq bo'ylab (ishqalish hisobga olinmagan holda), v_0 tezlikga ega bo'lgan boshlang'ich A holatdan, berilgan v_1 tezlikka ega bo'ladigan oxirgi B holatga minimal vaqtda o'tkazish talab qilinadi(1-chizma).



1-chizma.

Masalaning matematik qo'yilishini boshqarish obyektining matematik modelini tuzishdan boshlaymiz. Nyutonning ikkinchi qonuniga binoan, aravachaning harakati

$$m\ddot{x} = u \qquad (8.1)$$

tenglama orqali ifodalanadi. Bunda $\ddot{x} = \ddot{x}(t) = \frac{d^2x(t)}{dt^2}$ – aravachaning t vaqt momentidagi tezlanishi, $u = u(t)$- boshqarish obyektiga t vaqt momentida ta'sir etuvchi kuch miqdori.

Masalaning fizik qo'yilishidan, aravachaning $x(t)$ holati hamda $\dot{x}(t) = \frac{dx}{dt}$ tezligi uchun boshlang'ich $(t = 0)$ va oxirgi $(t = t^*)$ momentlarida quyidagi shartlar kelib chiqadi:

$$x(0) = \alpha, \ \ \dot{x}(0) = v_0; \quad x(t^*) = \beta; \quad \dot{x}(t^*) = v_1. \qquad (8.2)$$

Farazimizga ko'ra, aravachaga ta'sir etuvchi u kuchning qiymatlari ham chegaralangan:

$$|u(t)| \leq L, \quad t \in [0, t^*]. \qquad (8.3)$$

Optimal boshqaruv masalasining dastlabki qo'yilishlarida muhandislar tomonidan, *u* kuch o'zgarishi mumkin bo'lgan $u(t), t \geq 0$ qonunlarni bo'lakli-uzluksiz $u(t), t \in [0, t^*]$, funksiyalar orqali ifodalangan, deb faraz qilishgan.

Shunday qilib, qaralayotgan masalaning matematik modeli shunday t^{*0} moment va (8.3) bog'lanishni qanoatlantiruvchi $u^0(t), \ \ t \in [0, t^{0*}]$ bo'lakli-uzluksiz funksiyani topishdan iboratki, (8.1) tenglamaning mos $x^0(t), \ t \in [0, t^{0*}]$ yechimida (8.2) chegaraviy shartlar bajariladi va o'tish jarayoni t^{*0} vaqti minimal bo'ladi.

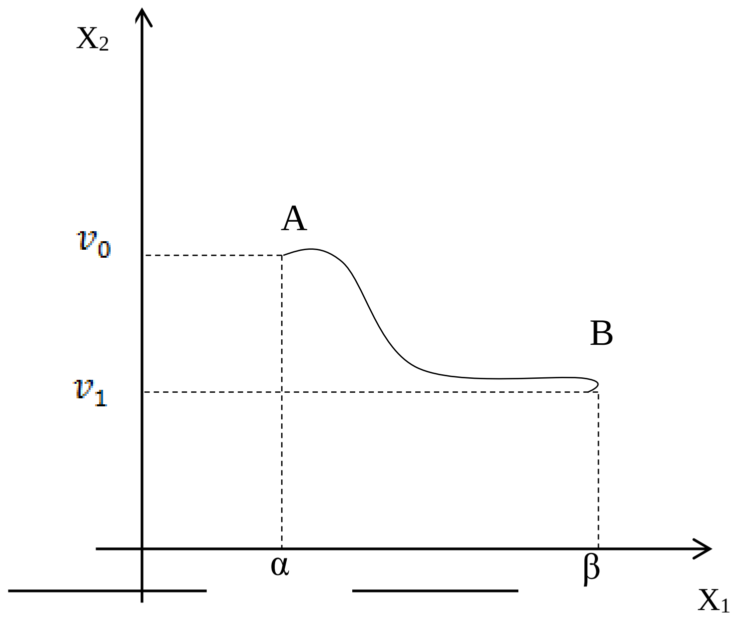
Endi tezkor masalaning variatsion hisobda ko'rib o'tilgan braxistoxrona haqidagi masala bilan taqqoslaymiz .

Buning uchun, (8.1)-(8.3) masalada $x_1 = x,\ x_2 = \dot{x}$ faza o'zgaruvchilariga (boshqaruv ob'yektining (x_1, x_2) holat o'zgaruvchilariga) o'tamiz. Unda yuqoridagi masala

$$\begin{cases} \dot{x_1} = x_2, & x_1(0) = \alpha,\ x_1(t^*) = \beta, \\ \dot{x_2} = \frac{1}{m}u; & x_2(0) = v_0,\ x_2(t^*) = v_1 \\ \quad |u(t)| \le L, \quad t \in [0, t^*]; \\ \quad t^* \to min. \end{cases} \tag{8.4}$$

ko'rinishni oladi.

(8.4) masalaning geometrik talqini quyidagicha: x_1, x_2 o'zgaruvchilarning faza tekisligida (8.4) sistemaning shunday $x^0(t) = (x_1^0(t), x_2^0(t)), t \ge 0,$ trayektoriyasini qurish kerakki, u bo'ylab harakat qilganda faza nuqtasi $A(\alpha, v_0)$ holatdan $B(\beta, v_1)$ holatga eng qisqa t^{*0} vaqtda o'tsin (2-chizma).



2-chizma.

3.Bunday talqinda qaralayotgan tezkor masala variatsion hisobdagi braxistoxrona haqidagi masalaga o'xshashdir. Tezkor masalaning braxistoxrona haqidagi masaladan muhim farqi unda (8.3) tengsizlik qatnashganligida va shuning uchun uni yechishda variatsion hisobda qo'llanilgan usullar va natijalardan foydalanish mumkin bo'lmaydi.

4.n –o'lchovli R^n faza fazosida harakati

$$\dot{x} = f(x, u) \tag{8.5}$$

oddiy differnsial tenglama orqali ifodalangan obyektni qaraymiz. Bunda

$x = x(t) = (x_1(t), \dots, x_n(t))$ –ob'yektning t vaqt momentidagi **holati;** $\dot{x} = \dot{x}(t) = \frac{dx}{dt}$ –uning tezligi $u = u(t) = (u_1(t), \dots, u_r(t)) - r$ –o'lchovli **boshqarish ta'sirining** t vaqt momentidagi qiymati.

Faraz qilaylik, r –o'lchovli R^n fazoda $U \in R^n$ to'plam berilgan bo'lsin.

1-tarif. U to'plamdan qiymatlar qabul qiluvchi bo'lakli uzluksiz $u(t),\ t \geq 0$ funksiya, ya'ni $u(t) \in U,\ t \geq 0$ munosabatni qanoatlantiruvchi funksiya, **qulay boshqarish ta'siri** deb ataladi.

Qo'shimcha ravishda har bir qulay boshqarishga (8.5) vektor tenglamaning

$x(0) = x_b$

boshlang'ich shartni qanotlantiruvchi yagona $x(t),\ t \geq 0,$ yechimi mos keladi, deb hisoblaymiz.

2-tarif. Agar berilgan $x_b, x_{ox} \in R^n$ lar uchun $u(t),\ t \geq 0,$ qulay boshqarish ta'siriga mos $x(t),\ t \geq 0$ trayektoriya qandaydir $0 < t^* = t^*(u) < \infty$ vaqt momentida

$x(0) = x_b;\quad x(t^*) = x_{ox}$

shartlarni qanoatlantirsa, bunday $u(t),\ t \geq 0,$ qulay boshqarish ta'siri **programma** deb ataladi.

Endi tezkor masalaning umumiy qo'yilishi quyidagi ko'rinishni oladi: programmalar ichida shunday $u^0(t),\ t \geq 0,$ (optimal) programmani topish kerakki, uning uchun o'tish vaqti $t^{*0} = t^0(u^0)$ minimal bo'lsin.

Bu masala yechimining mavjudligi haqida teoremani isbotsiz keltiramiz (A.F.Filippov, 1959).

1-teorema . Faraz qilaylik, X-chegaralangan to'plam; $X \subset R^n$; $f(x, U) \in C, x \in X$, $u \in V$; (5) sistemaning har xil programmalarga mos trayektoriyalari to'plami bo'sh bo'lmagan va chegaralangan $(x(t) \in X, t \geq 0)$; har bir $x \in X$ uchun

$f(x, U) = \{f(x, u), u \in U\}$

tezliklar to'plami qavariq kompaktdan iborat bo'lsin.U holda tezkor masala o'lchovi boshqarishlar ta'sirlari sinfida yechimga ega bo'ladi.

Bundan buyon qulay boshqarishlar ta'sirlari sifatida bo'lakli uzluksiz funksiyalarni qaraymiz .

O'lchovli funksiya ta'rifini keltiramiz.

3-ta'rif. Agar ixtiyoriy $\varepsilon > 0$ uchun shunday $u_\varepsilon(t), t \epsilon T,$ funksiya topilib, u berilgan $u(t), t \epsilon T,$ funksiyadan uzunliklari yig'indisi ε dan oshmaydigan $\omega \subset T$ oraliklar to'plamida farq qilsa, $u(t), t \epsilon T$,- o'lchovli funksiya deyiladi.

Shuni e'tirof etish kerakki, optimal boshqaruv nazariyasining matematik modellariga boshqaruvga bog'liq bo'lmagan ko'plab ekstremal masalalar ham keltiriladi. Bunday masalalarda optimallashtirish ob'yektlari statik (dinamik xarakterga ega bo'lmagan) bo'lishi ham mumkin .Variatsion hisobning har bir masalasini optimal boshqaruv nazariyasi masalasiga keltirish mumkin. ($F(x, y, y')$ da $y' = u$ deb olinadi) .Shu sababli, optimal boshqaruv nazariyasi klassik bo'lmagan variatsion hisob deb ham yuritiladi.

Optimal boshqaruv nazariyasida klassik variatsion hisob masalalari uchun an'anaviy bo'lgan muammolarga (mavjudlik , yagonalik, yechimlarning uzluksiz bog'liqligi , optimallikning zaruriy va yetarli shartlar va h.k) qo'shimcha ravishda, unda qaralmaydigan (stabilizatsiya, boshqariluchanlik, kuzativchanlik , optimal boshqarishning sintezi va h.k) muammolar ham tahlil qilinadi.

Optimal boshqaruv masalalari ularda ishtirok etayotgan funksionallarning ko’rinishlariga bog’liq bo’lgan holda tiplarga bo’linadi.

Agar masalaning sifat kriteriysi – funksional integral ko’rinishida berilgan bo’lsa (ya’ni integrant qatnashsa), masala **Lagranj tipidagi** masala deyiladi. Agar masalaning sifat kriteriysi tezkor masaladagi kabi faqat oxirgi vaqt momentida trayektoriyaning qiymatiga bog’liq funksiyaning qiymati orqali ifodalansa (ya’ni terminant qatnashsa), bunday masala **Mayer tipidagi masala** yoki **terminal boshqaruv masalasi** deyiladi.

Agar optimal boshqaruv masalasining sifat kriteriysi – integral funksional bilan terminatning yig’indisidan iborat bo’lsa, bunday masalaga Bolts masalasi deyiladi.

Takrorlash uchun savollar.

1. Sodda mexanik harakatni boshqarish masalasi –sodda tezkor masala qanday qo’yiladi?
2. Tezkor masalaning umumiy qo’yilishini keltiring.
3. Tezkor masalani variatsion hisobdagi braxistoxrona haqidagi masala bilan taqqoslang.
4. Optimal boshqaruv masalalari qaralganda paydo bo’ladigan muammolarni sanab bering.
5. Optimal boshqaruv masalalarining bir necha tiplarini yozing.

15-ma’ruza. Dinamik programmalashtirish usuli va uning optimal boshqaruv masalalariga qo’llanilishi

Reja:

1. Masalaning qo`yilishi
2. Masalaning yechilishi

1. Masalaning qo`yilishi. Matematik programmalashtirishning quyidagi

$$u = \sum_{i=1}^{n} f_i(x_i) \to \max, \quad \sum_{i=1}^{n} a_i x_i \le (=) b, a_i > 0, x_i \ge 0, i = \overline{1, n}, \qquad (1)$$

masalasini qaraymiz. Bu masalaning o`ziga xos hususiyati shundan iboratki, uning maqsad funktsiyasi separabeldir, ya’ni bir o`zgaruvchili $f_i(x_i), i = \overline{1, n}$, funktsiyalar yig`indisidan iborat.

Bir qator iqtisodiy masalalarni (1) masala ko`rinishida matematik modellashtirish mumkin. Shunday masalalardan biri-***resurslar taqsimoti*** haqidagi masaladir. Bizga *b* miqdordagi xomashyo (resurs) va $n$ -ta texnologik jarayonlar berilgan bo`lsin; agar xomashyoning x miqdorini i -chi texnologik jarayonda foydalansak , $f_i(x)$ miqdordagi foyda olinishi ma’lum bo`lsa, maksimal umumiy foyda olish uchun xomashyoni jarayonlar o`rtasida qanday taqsimlash kerak?

Agar x_i deb i -chi jarayon uchun ajratilgan resurs miqdorini belgilasak, $u=\sum_{i=1}^{n} f_i(x_i)$ -jami foyda, $\sum_{i=1}^{n} x_i = b, \quad x_i \geq 0, i=\overline{1,n}$, bo`ladi. Natijada resurslar taqsimoti haqidagi masala quyidagicha matematik modellashtiriladi:

$$\sum_{i=1}^{n} f_i(x_i) \to \max, \qquad \sum_{i=1}^{n} x_i = b, \qquad x_i \geq 0, i=\overline{1,n}. \qquad (2)$$

2. Masalaning yechilishi. Amerikalik olim R. Bellman tomonidan asoslangan sxemaga ko`ra , (1) masalani dinamik programmalashtirish usuli bilan echish quyidagi bosqichlarda amalga oshiriladi.

Birinchi bosqichda (1) masalani unga o`xshash masalalar oilasiga invariant kiritamiz, ya’ni

$$u=\sum_{i=1}^{n} f_i(x_i) \to \max, \sum_{i=1}^{k} a_i x_i \leq (=) y, x_i \geq 0, i=\overline{1,k}, \qquad 1 \leq k \leq n, 0 \leq y \leq b \qquad (3)$$

masalalar oilasini qaraymiz. (3) masala maqsad funktsiyasining optimal qiymatini $B_k(y)$ deb belgilaymiz:

$$B_k(y)=\max \sum_{i=1}^{n} f_i(x_i), \sum_{i=1}^{k} a_i x_i \leq (=) y, x_i \geq 0, i=\overline{1,k}. \qquad (4)$$

$B_k(y)$ - Bellman funktsiyasi deyiladi.

Ikkinchi bosqichda rekurrent

$$B_k(y)=\max_{0\leq z \leq \frac{y}{a_k}} [f_k(z)+B_{k-1}(y-a_k z)], \qquad k=\overline{2,n}, \qquad 0 \leq y \leq b \qquad (3)$$

Bellman tenglamasidan va $b_1(y)=\max_{0\leq z\leq y/a_1} f_1(z)$ (asosiy bog`lanish tenglik shaklda bo`lgan holda $b_1(y)=f_1(y/a_1)$ boshlang`ich shartdan foydalanib ketma-ket $B_2(y), B_3(y), \ldots, B_n(y)$ funktsiyalarini topamiz.

Oxirgi, uchinchi bosqichda (1) masalaning $\{x_1^0, x_2^0, \ldots, x_n^0\}$ echimini quramiz: x_n^0 ni $f_n(z)+B_{n-1}(b-a_n z), 0 \leq z \leq b/a_n$ funktsiyaga maksimum beruvchi nuqta sifatida aniqlaymiz, ya’ni $f_n(x_n^0)+B_{n-1}(b-a_n x_n^0)=\max_{0\leq z\leq b/a_n} [f_n(z)+B_{n-1}(b-a_n z),$ x_{n-1}^0 ni ham shunga o`xshash $f_{n-1}(x_{n-1}^0)+B_{n-2}(b_1-a_{n-1}x_{n-1}^0)=\max_{0\leq z\leq b/a_{n-1}} [f_{n-1}(z)+B_{n-2}(b_1-a_{n-1}z)]$ shartdan aniqlaymiz, bu erda $b_1=b-a_n x_n^0$. Shunday davom etib, $x_{n-1}^0, i=2,3,\ldots,n-2$ nuqtalarni $f_{n-i}(x_n^0)+B_{n-i-1}(b_i-a_{n-i}x_{n-i}^0)=\max_{0\leq z\leq b_i/a_{n-1}} [f_{n-i}(z)+B_{n-i-1}(b_i-a_{n-i}z)$ shartdan topamiz, bu erda

$$b_i = b-\sum_{k=0}^{i-1} a_{n-k} x_{n-k}, i=2,3,\ldots,n-2.$$

x_1^0 ni esa, $f_1(x_1^0) = \max_{0 \le z \le b_{n-1}/a_1} f_1(z)$ shartdan topamiz (asosiy bog`lanish tenglik ko`rinishda bo`lganda esa , $x_1^0 = (b - \sum_{k=0}^{i-1} a_{n-k} x_{n-k}^0) / b_1$ bo`ladi). (1) masala maqsad funktsiyasining maksimal qiymati $B_n(b)$ ga teng .

1-Misol. Resurslar taqsimoti haqidagi (2) masalada

$n = 3, b = 4,\ f_1(x) = x,\ f_2(x) = x^2 + x,\ f_3(x) = x^2 - x^3 + 24x$

bo`lsin. Demak, quyidagi masala berilgan:

$$\sum_{i=1}^{3} f_i(x_i) = x_1 + x_2^2 + x_2 + x_3^2 + 24x_3 \to \max, \quad x_1 + x_2 + x_3 = 4, \quad x_i \ge 0, i = 1,2,...$$

Unga uxshash masalalar oilasi quyidagicha bo`ladi:

$$\sum_{i=1}^{k} f_i(x_i) \to \max, \quad \sum_{i=1}^{k} x_i = y, \quad x_i \ge 0, i = \overline{1,k}, \quad k = 1,2,3, \qquad 0 \le y \le 4.$$

Bellman funktsiyasi

$$B_k(y) = \max \sum_{i=1}^{k} f_i(x_i), \quad \sum_{i=1}^{k} x_i = y, \quad x_i \ge 0, i = \overline{1,k}, \quad k = 1,2,3, \qquad 0 \le y \le 4.$$

Uchun Bellman tenglamasini yozamiz:

$$B_k(y) = \max_{0 \le z \le y} \left[f_k(z) + B_{k-1}(y - z) \right], \quad k = 2,3, \quad 0 \le y \le 4.$$

Bu tenglama uchun boshlang`ich shart $B_1(y) = f_1(y) = y$ bo`ladi. Bellman tenglamasidan ketma-ket $B_2(y)$ va $B_3(y)$ funktsiyalarni topamiz:

$$B_2(y) = \max_{0 \le z \le y} \left[f_2(z) + B_1(y - z) \right] = \max_{0 \le z \le y} \left[z^2 + z + y - z \right] = y^2 + y$$

$$B_3(y) = \max_{0 \le z \le y} \left[f_3(z) + B_2(y - z) \right] = \max_{0 \le z \le y} \left[z^2 - z^3 + 24z + (y - z)^2 + y - z \right]$$

Endi optimal taqsimoti aniqlaymiz.

$$B_3(4) = \max_{0 \le z \le 4} \left[z^2 - z^3 + 24z + (4 - z)^2 + 4 - z \right] = \max_{0 \le z \le y} \left[2z^2 - z^3 + 15z + 20 \right] = 56$$

bu erda maksimumga **z = 3** da erishiladi. Demak, $b_1 = b - x_3^0 = 1; \quad B_2(1) = \max_{o \le z \le 1}[z^2 + 1] = 2$,

bu erda maksimumga **z =1** da erishiladi. Demak, $\mathbf{x^o_2 = 1}$. U vaqtda $x_1^0 = 4 - x_3^0 - x_2^0 = 0$.

SHunday qilib, optimal taqsimot quyidagicha bo`ladi: $\mathbf{x^o_1 = 0}$, $\mathbf{x^o_2 = 1}$, $\mathbf{x^o_3 = 3}$ **:** maksimal foyda $B_3(4) = 56$ ga teng.

Mustaqil ishlash uchun savollar

1. Dinamik programmalashtirish masalasi qanday jarayon uchun qo‘yiladi?
2. Dinamik programmalashtirish masalasini necha bosqichda yechish mumkin va ularni keltiring?

16-ma’ruza. Differensial o’yinlar haqida asosiy tushunchalar

Reja:

1. Matrisaviy o’yin. O’yining quyi va yuqori baholari.
2. O’yinning egar nuqtasi. Sof strategiyalar.

Tayanch so’z va iboralar: pozitsion o’yinlar, normal o’yinlar, matrisali o’yinlar, differensial o’yinlar, strategiya, konfliktli vaziyat, optimal strategiya, sof strategiya, egar nuqta, o’yinning quyi va yuqori baholari, to’lov matritsasi, o’yinning sof bahosi.

1. Matrisaviy o’yin. O’yining quyi va yuqori baholari

Ikkita A va B o’yinchilar qatnashgan antogonistik o’yinni qaraymiz. O’yinchilar qarama-qarshi maqsadni ko’zlaydi. Biri qandaydir yutuqqa ega bo’lsa, ikkinchisi shu miqdorda yutqazadi. Demak A o’yinchining yutug’i B o’yinchi yutug’ining qarama-qarshi ishora bilan olinganiga teng bo’lgani uchun, bu o’yinda A o’yinchining yutugini tahlil qilish yetarli.

A o’yinchi (biz uni I o’yinchi deymiz) m ta $A_1, A_2, \ldots, A_m$ strategiyalariga, B o’yinchi (biz uni II o’yinchi deymiz) n ta $B_1, B_2, \ldots, B_m$ strategiyalarga ega bo’lsin. Bunday o’yinga $m\times n$ o’lchamli o’yin (ba’zan qisqacha $m\times n$-o’yin) deyiladi.

I o’yinchi o’zining mumkin bo’lgan strategiyalaridan biri A_i ni, $i=1,2,\ldots,m,$ II o’yinchi esa, I o’yinchining tanlash natijasidan bexabar holda, B_j strategiyani $(j=1,2,\ldots,n)$ tanlangan bo’lsin. Strategiyalarni tanlash natijasida I o’yinchining yutug’i $W_1(A_i, B_j)$ va II o’yinchining yutug’i $W_2(A_i, B_j)$ bo’lsa, ular $W_1(A_i, B_j) + W_2(A_i, B_j) = 0$ munosabatni qanoatlantiradi. Agar $W(A_i, B_j) = W_1(A_i, B_j)$ deb olsak, $W_2(A_i, B_j) = -W(A_i, B_j)$ bo’ladi. $W(A_i, B_j) = a_{ij}$ deb belgilaylik. Bu qiymatlarni satrlari I o’yinchining strategiyalariga, ustunlari esa II o’yinchining strategiyalariga mos keladigan jadval (1-jadval) ko’rinishida yozamiz. Bunday jadval to’lovlar matrisasi deb ataladi.

1-jadval

B / A	B_1	B_2	...	B_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
...	$\ldots$	$\ldots$	$\ldots$	$\ldots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

To’lovlar matrisasining har bir musbat a_{ij} elementi mos strategiyalar qo’llanilganda I o’yinchining yutug’i (yoki II o’yinchining yutqazig’i) miqdorini bildiradi. Matrisaning har bir manfiy a_{ij} elementi esa mos strategiyalar qo'llanilganda I o'yinchining yutqazig'i (yoki II o'yinchining yutug'i) miqdorini bildiradi. Ikkala o'yinchining ham maqsadi – o'z yutug'ini maksimallashtirishdan (yoki o'z yutqazig'ini minimallashtirishdan) iborat.

1-misol. O'yinda I va II o'yinchilar ishtirok qiladilar. O'yinchilardan har biri boshqasidan bexabar holda 1, 2 yoki 3 ta barmog'ini ko'rsatishi mumkin. Agar I va II o'yinchilar ko'rsatgan barmoqlar soni o'rtasidagi ayirma musbat bo'lsa, I o'yinchi shu sonlar ayirmasi qadar ochko yutadi va aksincha, agar ayirma manfiy bo'lsa, II o'yinchi shuncha yutadi. Agar sonlar o'rtasidagi ayirma nol bo'lsa, o'yin durang bilan tugaydi.

O'yinda har bir o'yinchining uchtadan shaxsiy yurishi bor. I o'yinchi strategiyalari: A_1 – 1 ta barmoqni ko'rsatish, A_2 – 2 ta barmoqni ko'rsatish, A_3 – 3 ta barmoqni ko'rsatish. II o'yinchining (ya'ni, I o'yinchi raqibining) strategiyalari esa, B_1 – 1 ta, B_2 – 2 ta, B_3 – 3 ta barmoqni ko'rsatishdan iborat. O'yinchilarning ular tegishli strategiyalarni qo'llaganlardagi yutuqlarini to'lov matrisasi (2-jadval) ko'rinishida yozib qo'yamiz.

2-jadval

II I	B_1	B_2	B_3
A_1	0	-1	-2
A_2	1	0	-1
A_3	2	1	0

Bu matrisa elementlari qanday hosil qilinganligini ko'ramiz. Agar I o'yinchi A_1 strategiyasini, II o'yinchi B_3 strategiyasini qo'llasa, u vaqtda I o'yinchi ikki ochko yutqazadi. Bu yutqazish to'lov matrisasida birinchi satr va uchinchi ustunlar kesishishidagi (1;3) katakka yozilgan ($a_{13} = 2$). Agar I o'yinchi A_2 strategiyasini, raqib esa B_1 strategiyani qo'llasa, u holda I o'yinchi bir ochko yutadi. To'lov matrisasida bu yutuq (2;1) katakka musbat ishora bilan yozib qo'yilgan ($a_{21} = 1$). Jadvalning boshqa elementlari ham shu tariqa hosil qilingan.

To'lov matrisasi 1-jadvalda keltirilgan $m \times n$ – o'yinni qaraymiz. Masala $A_1, A_2, \ldots, A_m$ strategiyalar orasidan I o'yinchining eng yaxshi strategiyasini, $B_1, B_2, \ldots, B_m$ strategiyalardan esa II o'yinchining eng yaxshi strategiyasini topishdan iborat. Bu masalani yechishda o'yinda ishtirok etuvchi raqiblar bir xil aql-idrokka ega va ulardan har biri o'z maqsadiga erishish uchun hamma chora-tadbirlarni ko'radi deb hisoblaymiz. Bu tamoyildan foydalanib I o'yinchining eng yaxshi strategiyasini topamiz. Buning uchun uning hamma strategiyalarini ketma-ket tahlil qilamiz. I o'yinchi o'zining A_i strategiyasini tanlaganda biz unga II o'yinchi o'zining I o'yinchi yutug'ini minimallashtiruvchi B_j strategiyasi bilan javob beradi deb hisoblashimiz kerak. Shunga ko'ra to'lov matrisasining har bir satridagi

a_{ij} sonlardan minimalini topamiz va uni $\alpha_i, i=\overline{1,m}$, bilan belgilab, to'lov matrisasining yonida qo'shimcha ustunga yozib qo'yamiz:

$$\alpha_i = \min_{j=\overline{1,n}} a_{ij}, i=1,2,\dots,m \quad . \qquad (1)$$

α_i sonlarni bilgan I o'yinchi o'zining strategiyalaridan shundayini tanlaydiki, bu unga eng katta yutuq bersin. Bu maksimal yutuqni α deb belgilaymiz, ya'ni $\alpha = \max_{i=\overline{1,m}} \alpha_i$. Shunday qilib, (1) ni hisobga olsak, $\alpha = \max_{i=\overline{1,m}} \min_{j=\overline{1,n}} a_{ij}$ hosil bo'ladi.

α soni I o'yinchining kafolatli yutug'i bo'lib, o'yinning quyi bahosi (maksimini) deb ataladi. O'yinning quyi bahosi α ni ta'minlovchi strategiya maksimin strategiya deb ataladi. Agar I o'yinchi o'zining maksimin strategiyasiga amal qilsa, II o'yinchi qanday yo'l tutishidan qa'tiy nazar, unga α dan kam bo'lmagan yutuq ta'minlanadi.

II o'yinchi o'z yutqazig'ini kamaytirishga, ya'ni I o'yinchi yutug'ini minimumga aylantirishga harakat qiladi. Shu sababli o'zining eng yaxshi strategiyasini tanlab olish uchun u to'lov matrisasi ustunlarining har biridagi maksimal sonni topishi va bu qiymatlar orasidan eng kichigini tanlab olishi kerak.

Har bir ustundagi maksimal elementni β_j deb belgilaymiz va bu elementlarni 3-jadvalning qo'shimcha satriga yozib qo'yamiz. β_j lar orasidan eng kichik qiymatlisini β deb belgilaymiz. β - o'yinning yuqori bahosi (minimaksi) bo'lib, u $\beta = \min_{j=\overline{1,n}} \max_{i=\overline{1,m}} a_{ij}$ formula bo'yicha topiladi.

II o'yinchiga β "yutuqni" ta'minlaydigan strategiya uning minimaks strategiyasi deb ataladi. Agar II o'yinchi o'zining minimaks strategiyasiga amal qilsa, har qanday holda ham uning yutqazig'i β dan oshmaydi.

3-jadval

B A	B_1	B_2	...	B_n	α_i
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}	α_1
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}	α_2
...	…	…	…	…	...
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	α_m
β_j	β_1	β_2	...	β_n	α β

O'yinning quyi va yuqori baholari uchun $\max_{1\le i\le m} \min_{1\le j\le n} a_{ij} \le \min_{1\le j\le n} \max_{1\le i\le m} a_{ij}$, tengsizlikning hamisha o'rinli, ya'ni $\alpha \le \beta$ ekanligini ko'rsatish mumkin.

2. O'yinning egar nuqtasi. Sof strategiyalar

Quyi va yuqori baholari o'zaro teng, ya'ni $\alpha = \beta$ bo'lgan o'yinlar mavjud. Bunday o'yinlar egar nuqtali o'yinlar deb ataladi. Egar nuktali o'yilarda yuqori va quyi baholarining umumiy qiymati o'yining sof bahosi, bu qiymatga erishishni ta'minlovchi A_{i^*} va B_{j^*} strategiyalar esa optimal strategiyalar deyiladi.
Optimal strategiyalarning (A_{i^*}, B_{j^*}) jufti matrisaviy o'yinning egar nuqtasi (muvozanat vaziyati) deb ataladi. To'lovlar matrisasida shu egar nuqtaga mos $a_{i^* j^*} = \gamma$ element bir vaqtning o'zida i - satrda minimal, j - ustunda maksimaldir.
2-misol. To'lov matrisasi 2-jadvalda keltirilgan o'yinning yechimini topamiz. Shu jadvalga mos α_i va β_j larning kiymatlarini topib, ular yordamida 4-jadvalni hosil qilamiz.
4- jadval

II I	B_1	B_2	B_3	α_i
A_1	0	-1	-2	-2
A_2	1	0	-1	-1
A_3	2	1	0	0
β_j	2	1	0	

O'yinning quyi bahosi
$\alpha = \max\limits_i \alpha_i = max(-2;-1;0) = 0, \ \ \alpha = \alpha_3$.
O'yinning yuqori bahosi
$\beta = \min\limits_j \beta_j = min(2;1;0) = 0, \ \ \beta = \beta_3$.
$\alpha = \beta$ bo'lgani uchun o'yin egar nuqtaga ega. O'yinning sof bahosi $\gamma = 0$. Optimal strategiyalar: I o'yinchining A_3 strategiyasi va II o'yinchining B_3 strategiyasi. Egar nuqta esa (A_3, B_3) bo'ladi.
3-misol. To'lov matrisasi 5-jadvalda keltirilgan o'yinning yechimi topilsin.
α_i va β_j larning qiymatlarini topamiz va ularni 5-jadvalga kiritamiz.
5- jadval

II I	B_1	B_2	B_3	B_4	α_i
A_1	6	5	8	5	5

A_2	7	3	2	3	2
A_3	6	5	7	5	5
β_j	7	5	8	5	

O’yinning quyi va yuqori baholarini topamiz:

$$\alpha = \max_i \alpha_i = max(5;2;5) = 5,\quad \alpha = \alpha_1 = \alpha_3 = 5$$

$$\beta = \min_j \beta_j = min(7;5;8) = 5,\quad \beta = \beta_2 = \beta_4 = 5$$

α va β ning qiymatlaridan ko’rinib turibdiki, o’yinda optimal strategiyalarning A_1B_2, A_1B_4, A_3B_2, A_3B_4 juftlariga mos to’rtta egar nuqta mavjud. O’yinning sof bahosi $\gamma = 5$.

Muammoli masala va topshiriqlar

O’yin vaziyatini sifat jihatidan aks ettiruvchi masalani qo’ying va uning matematik modelini tuzing.
1-topshiriqda qo’yilgan o’yin masalasida tomonlarni, ularning sonini, o’yin xarakterini va o’yinchilar strategiyalarini aniqlang.
O’yinda I va II o’yinchilar ishtirok qiladilar. O’yinchilardan har biri boshqasidan bexabar holda 1, 2 yoki 3 ta barmog’ini ko’rsatishi mumkin. Agar I va II o’yinchilar ko’rsatgan barmoqlar soni yig’indisi juft bo’lsa, I o’yinchi shu yig’indiga teng ochko yutadi va aksincha, agar yig’indi toq bo’lsa, II o’yinchi shuncha ochko yutadi.Shu o’yinda:
a) tomonlarning strategiyalari va ularga mos yutuqlarini aniqlang;
b) o’yinchilarning minimaks va maksimin strategiyalarini toping.
To’lovlar matrisasi H quyidagicha bo’lgan o’yinda quyi va yuqori baholarni toping va egar nuqta (muvozanat vaziyati) mavjudligini tekshiring:

$$H = \begin{pmatrix} 9 & -5 & 2 & 6 & -7 \\ -1 & 5 & 8 & -2 & 4 \\ 5 & 7 & -5 & 0 & 5 \\ 6 & 1 & -2 & 3 & 8 \end{pmatrix}.$$

Mustaqil ishlash uchun savollar

1. O’yinlar nazariyasi nima bilan shug’ullanadi. O’yinlarning turlari.
2. Nol yig’indili o’yin. Strategiya, optimal strategiya tushunchalari.
3. Matrisaviy o’yin, to’lovlar matrisasi, minimaks va maksimin strategiyalar.
4. O’yinning quyi va yuqori baholari, sof baho, sof optimal strategiyalar.

AMALIY MASHG‘ULOTLAR

1-amaliy mashg‘ulot. Variatsion hisobning predmeti, funksionalning ekstremumi. Funksionalning variatsiyasi, ekstremumi. Variatsiya terminidagi ekstremum shartlari

1. Funksional tushunchasi.

Faraz qilaylik, $y(x)$ funksiyalarning qandaydir M sinfi berilgan bo‘lsin.

Agar har bir $y(x)\in M$ funksiyaga qandaydir qonun bo‘yicha biror aniq V son mos qo‘yilgan bo‘lsa, u holda M sinfda V *funksional aniqlangan* deyiladi va u $V=V(y)$ kabi yoziladi.

$V(y)$ funksional aniqlangan, $y(x)$ funksiyalarning M sinfi funksionalning *berilish (aniqlanish) sohasi* deyiladi. $V(y)$ funksionalning qiymatlari taqqoslanadigan $y(x)$ funksiyalar (egri chiziqlar) *joiz egri chiziqlar* yoki *taqqoslanadigan egri chiziqlar* deyiladi.

$V(y)$ funksional argumenti $y(x)$ ning δy variatsiyasi yoki orttirmasi deb, tanlangan M sinfga mansub $y(x)$ va $y_0(x)$ funksiyalar orasidagi ayirmaga aytiladi:

$$\delta y = y(x) - y_0(x) \ .$$

Funksionallarga misollar qaraymiz:

1). $M=C[0,1]-$ $[0,1]$ kesmada uzluksiz bo‘lgan barcha $y(x)$ funksiyalardan iborat bo‘lsin va

$$V(y)=\int_0^1 y(x)d(x) \qquad (1)$$

bo‘lsin. U holda $V(y)$ – $y(x)$ ning funksionalidan iborat, chunki har bir $y(x)\in C[0,1]$ funksiyaga $V(y)$ ning aniq qiymati mos keladi. (1) ifodada $y(x)$ ning o‘rniga aniq funksiyalar qo‘yib, $V(y)$ ning ularga mos qiymatlarini hosil qilamiz. SHunday qilib,

agar $y(x)$=5 bo‘lsa; $V(5)=\int_0^1 5dx = 5$;

agar $y(x)$=cos π x bo‘lsa, $V(\cos \pi x)=\int_0^1 \cos \pi x dx = 0.$

2). $M=C^1[a,b]-[a,b]$ kesmada birinchi tartibli hosilalari bilan birga uzluksiz funksiyalar sinfi bo‘lsin va $V(y)=$ $y'(x_0)$, $x_0\in[a,b]$. Ravshanki, $V(y)$- ko‘rsatilgan funksiyalar sinfida aniqlangan funksiyalardan iborat; bu sinfdan aniqlangan har bir funksiyaga aniq son - bu funksiyaning tanlangan x_0 nuqtada hosilasining qiymati mos keladi.

Agar, masalan, $a=1$, $b=3$ va $x_0=2$ bo‘lsa, $y(x)=x^2$ funksiya uchun

$$V(x^2)=2x\big|_{x=2}=4;$$

$y(x)=\ln x$ funksiya uchun

$$V[\ln x]=\frac{1}{x}\Big|_{x=2}=\frac{1}{2}$$

3). $M=C^2[a,b]$ bo‘lsin. U holda

$$V[y]=\int_a^b\sqrt{1+y'^2(x)}dx \qquad (2)$$

M da aniqlangan funksiyalardan iborat bo‘ladi. (2) funksionalning geometrik ma’nosi - uchlari $A(a,y(a))$ va $B(b,y(b))$ nuqtalarda bo‘lgan $y=y(x)$ egri chiziqning uzunligini bildiradi.

2. Egri chiziqlarning yaqinligi.

1- *ta’rif.* Agar $[a,b]$ kesmada berilgan $y=y(x)$ va $y=y_1(x)$ funksiyalar uchun, oldidan berilgan $\varepsilon>0$ son va barcha $x\in[a,b]$ larda $|y(x)-y_1(x)|<\varepsilon$ tengsizlik bajarilsa, bu funksiyalar *nolinchi tartibli yaqinlik* ma’nosida yaqin funksiyalar deyiladi.

2-*ta’rif*. Agar $[a,b]$ kesmada berilgan $y=y(x)$ va $y=y_1(x)$ funksiyalar uchun, oldindan berilgan $\varepsilon>0$ son va barcha $x\in[a,b]$ larda

$$|y(x)-y_1(x)|<\varepsilon \quad \text{va} \quad |y'(x)-y'_1(x)|<\varepsilon$$

tengsizliklar bajarilsa, bu funksiyalar *birinchi tartibli yaqinlik* ma’nosida yaqin funksiyalar deyiladi.

3-*ta’rif*. Agar $[a,b]$ kesmada berilgan $y=y(x)$ va $y=y_1(x)$ funksiyalar uchun, oldindan berilgan $\varepsilon>0$ son va barcha $X=\in[a,b]$ larda

$$|y(x)-y_1(x)|<\varepsilon,\ |y'(x)-y'_1(x)|<\varepsilon,\ ...\ |y^k(x)-y^k{}_1(x)|<\varepsilon$$

tengsizliklar bajarilsa, bu funksiyalar k - *tartibli yaqinlik* ma’nosida yaqin funksiyalar deyiladi.

4-ta’rif. $[a,b]$ kesmada uzluksiz bo‘lgan $y=y(x)$ va $y=y_1(x)$ funksiyalar orasidagi *masofa* deb, $[a,b]$ kesmada $|y(x)-y_1(x)|$ miqdorning maksimumiga teng nomanfiy ρ_0 songa aytiladi:

$$\rho_0=\rho_0[y(x),\mathrm{y}_1(x)]=\max_{a\le x\le b}|\mathrm{y}(\mathrm{x})-\mathrm{y}_1(x)|$$

Faraz qilaylik, $y=y(x)$ va $y=y_1(x)$ funksiyalar $[a,b]$ kesmada n-tartibgacha uzluksiz hosilalarga ega bo‘lsin.

5-ta’rif. $y=y(x)$ va $y=y_1(x)$ funksiyalar (egri chiziqlar) orasidagi *n-tartibli masofa* deb, quyidagi

$$|y(x)-y_1(x)|,|y'(x)-y'_1(x)|,...,|y^{(n)}(x)-y_1{}^{(n)}(x)|$$

miqdorlarning $[a,b]$ kesmadagi maksimumlarining eng kattasiga aytiladi.

Bu masofa,

$$\rho_n=\rho_n[y(x),\mathrm{y}_1(x)]=\max_{1\le k\le n}\ \max_{\mathrm{a}\le\mathrm{x}\le\mathrm{b}}|\mathrm{y}^{(\mathrm{k})}(\mathrm{x})-\mathrm{y}_1{}^{(k)}(x)|$$

kabi belgilanadi.

6-ta’rif. $y = y(x)$ egri chiziqning (funksiyaning) *n*-tartibli ε- atrofi deb, shunday $y = y_1(x)$ egri chiziqlar to‘plamiga aytiladiki, ularning $y = y(x)$ egri chiziqdan *n*-tartibli masofasi ε dan kichik bo‘ladi:

$$\rho_n = \rho_n[y(x), y_1(x)] < \varepsilon.$$

7-ta’rif. $y = y(x)$ funksiyaning nolinchi tartibli ε- atrofi uning kuchli ε-atrofi deyiladi.

3. Funksionalning uzluksizligi

8-ta’rif. Agar ixtiyoriy $\varepsilon > 0$ son uchun shunday δ son mavjud bo‘lib $\delta = \delta(\varepsilon))$,

$$|y(x) - y_1(x)| < \delta, |y'(x) - y'_0(x)| < \delta, ..., |y^{(n)}(x) - y_0^{(n)}(x)| < \delta$$

shartlarni qanoatlantiruvchi barcha $y = y(x)$ joiz funksiyalar uchun

$$|V[y] - V[y_0]| < \varepsilon$$

tengsizlik bajarilsa, $y(x)$ funksiyalarning M sinfida aniqlangan $V[y]$ funksional $y = y_0(x)$ funksiyada *n-tartibli yaqinlik ma’nosida uzluksiz* deyiladi.

n-tartibli yaqinlik ma’nosida uzluksiz bo‘lmagan funksional ko‘rsatilgan yaqinlik ma’nosida uzilishga ega deyiladi.

$M - y(x)$ funksiyalarning chiziqli normalangan fazosi bo‘lsin.

9-ta’rif. Agar M fazoda aniqlangan $L(y)$ funksional:

1) bir jinsli, $L(cy) = cL(y)$, c -ixt. o‘zgarmas son;

2) additiv, $L(y_1 + y_2) = L[y_1] + L[y_2]$, $y_1(x) \in M, y_2(x) \in M$ bo‘lsa, u chiziqli funksional deyiladi.

Misollar.

1-misol. $y(x) = \dfrac{\sin n^2 x}{n}$, n-etarli katta son, va $y_1(x) \equiv 0$ funksiyalar $[0, \pi]$ kesmada nolinchi tartibli yaqinlik ma’nosida yaqin ekanligini ko‘rsating.

Yechilishi. Berilgan funksiyalar ayirmasining modulini baholaymiz:

$$|y(x) - y_1(x)| = \left|\frac{\sin n^2 x}{n}\right| \le \frac{1}{n},$$

ya’ni $[0, \pi]$ kesmaning barcha nuqtalarida bu ayirma modul bo‘yicha (etarli katta *n* lar uchun) etarli kichikdir.

Bu funksiyalar uchun birinchi tartibli yaqinlik yo‘q, chunki $|y'(x) - y_1'(x)| = n|\cos n^2 x|$, va masalan, $x = \dfrac{2\pi}{n}$ nuqtalarda $|y'(x) - y_1'(x)| = n$ bo‘lishini olamiz va demak, $|y'(x) - y_1'(x)|$ miqdor, etarli katta n larda juda katta qilinishi mumkin.

2-misol. *y(x)=* $\dfrac{\sin nx}{n^2}$, *n*-etarli katta son, va $y_1(x) \equiv 0$ egri chiziqlar $[0, \pi]$ kesmada birinchi tartibli yaqinlik ma’nosida yaqin ekanligini ko‘rsating.

Yechilishi. Modomiki,

$|y(x)-y_1(x)|=\left|\frac{\sin nx}{n^2}\right|\le\frac{1}{n^2}$ ва $|y^{|}(x)-y_1^{|}(x)|=\left|\frac{\cos nx}{n}\right|\le\frac{1}{n}$ lar etarli kichik miqdorlar ekan, berilgan egri chiziqlar birinchi tartibli yaqinlik ma'nosida yaqin egri chiziqlar bo'ladi.

3-misol. $[0,\pi]$ kesmada $y=x$ va $y=x^2$ egri chiziqlar orasidagi masofani (ρ_0) toping.

Echilishi. Tarifga ko'ra,

$$\rho_0=\max_{0\le x\le 1}\left|x^2-x\right|=\max_{0\le x\le 1}(x-x^2).$$

[0,1] kesmaning chetlarida $y=x-x^2$ funksiya nolga aylanadi. $\max\limits_{0\le x\le 1}(x-x^2)$ ni hisoblaymiz. Ravshanki, $y'=1-2x$, $x=\frac{1}{2}$ bo'lganda $y'=0$ bo'ladi, demak,

$$\rho_0=\max_{0\le x\le 1}\left|x^2-x\right|=(x-x^2)\Big|_{x=\frac{1}{2}}=\frac{1}{4}.$$

4.Funksionalning variatsiyasi.

$V[y]$ funksional $y(x)$ funksiyalarning M to'plamida berilgan bo'lsin.

10-*ta'rif*. $V[y]$ funksionalning δy argument orttirmasiga mos kelgan *orttirmasi* deb,

$$\Delta v=\Delta v[y(x)]=v[y(x)+\delta y(x)]-v[y(x)] \qquad (4)$$

($\delta y(x)=\bar{y}(x)-y(x)$, bunda $y(x)\in M,\bar{y}(x)\in M$) miqdorga aytiladi.

11-*ta'rif*. Agar $V[y]$ funksionalnnig $\Delta v=v[y(x)+\delta y(x)]-v[y(x)]$ orttirmasini,

$$\Delta v=L[y(x),\delta y]+\beta(y(x),\delta y)/\|\delta y\|,$$

ko'rinishda tasvirlash mumkin bo'lsa, bu erda $L[y(x),\delta y]-$ δy ga nisbatan chiziqli funksional va $\|\delta y\|\to 0$ da $\beta(y(x),\delta y)\to 0$, funksional orttirmasining δy ga nisbatan chiziqli qismi, ya'ni $L[y(x),\delta y]$ funksional, *V[y] funksionalning birinchi variatsiyasi* deb ataladi va u δV kabi belgilanadi. Bu holda *V[y]* funksional *y(x)* nuqtada differensiallanuvchi deyiladi.

12-*ta'rif* (funksional variatsiyasining ikkinchi ta'rifi). $V[y]$ funksionalning $y=y(x)$ nuqtadagi variatsiyasi deb, $V[y(x)+\alpha\delta y(x)]$ funksionaldan α parametr bo'yicha hosilaning $\alpha=0$ dagi qiymatiga aytiladi:

$$\delta V=\frac{d\alpha}{d}V[y(x)+\alpha\delta y(x)]/\alpha=0 \qquad (5)$$

Shuni e'tirof etish kerakki, agar funksionalning variatsiyasi uning orttirmasidagi bosh chiziqli qism sifatida mavjud bo'lsa, α parametr bo'yicha hosilaning $\alpha=0$ dagi qiymati shaklidagi variatsiya ham mavjud bo'ladi va ular o'zaro ustma-ust tushadi.

5. Funksionalning ekstremumi.

13 -*ta'rif*. Agar $V[y]$ funksionalning $y=y_0(x)$ egri chiziqqa yaqin bo'lgan ixtiyoriy egri chiziqdagi qiymati, uning $V[y_0]$ qiymatidan katta bo'lmasa, ya'ni

$$\Delta V=V[y]-V[y_0]\le 0$$

bo'lsa, $V[y]$ funksional $y=y_0(x)$ egri chiziqda *maksimumga erishadi*, deyiladi.

Agar $\Delta V \le 0$ bo‘lib, faqat $y = y_0(x)$ bo‘lganda $\Delta V = 0$ bo‘lsa, $y = y_0(x)$ da *qat’iy maksimumga erishiladi*, deyiladi.

Minimumga erishiladigan $y = y_0(x)$ egri chiziq shunga o‘xshash aniqlanadi.Bu holda, $y = y_0(x)$ egri chiziqqa yaqin barcha egri chiziqlarda $\Delta V \ge 0$ bo‘ladi.

6. Kuchli va kuchsiz ekstremumlar.

14-*ta’rif*. Agar $V[y]$ funksional aniqlangan sohadan olingan, $y = y_0(x)$ egri chiziqning nolinchi tartibli ε-atrofida yotuvchi barcha joiz $y = y(x)$ egri chiziqlar uchun $V[y] \le V[y_0]$ tengsizlik bajarilsa, $V[y]$ funksional $y = y_0(x)$ egri chiziqda *kuchli nisbiy (lokal) maksimumga erishadi*, deyiladi.

Funksionalning kuchli nisbiy (lokal) minimumi shunga o‘xshash aniqlanadi.

15-*ta’rif*. Agar $V[y]$ funksional aniqlangan sohadan olingan, $y = y_0(x)$ egri chiziqning qandaydir birinchi tartibli ε-atrofida yotuvchi barcha joiz $y = y(x)$ egri chiziqlar uchun $V[y] \le V[y_0]$ tengsizlik bajarilsa, $V[y]$ funksional $y = y_0(x)$ egri chiziqda *kuchsiz nisbiy (lokal) maksimumga erishadi*, deyiladi.

Funksionalning kuchsiz nisbiy (lokal) minimumi shunga o‘xshash aniqlanadi.

$V[y]$ funksionalning (kuchli va kuchsiz) maksimum va minumumlari uning *ekstremumlari* deyiladi.

Har qanday kuchli ekstremum bir vaqtda kuchsiz ekstremum ham bo‘ladi, lekin uning teskarisi o‘rinli emas.

$V[y]$ funksionalning, u aniqlangan barcha funksiyalardagi ekstremumi *absolyut (global) ekstremum* deb ataladi. Har qanday absolyut (global) ekstremum bir vaqtning o‘zida kuchli va kuchsiz nisbiy (lokal) ekstremum ham bo‘ladi, lekin har qanday nisbiy (lokal) ekstremum absolyut ekstremum ham bo‘lishi shart emas.

7. Ekstremumning zaruriy sharti.

Teorema. Agar $V[y]$ differensiallanuvchi funksional $y = y_0(x)$ da ekstremumga erishsa, bu yerda $y_0(x)$-funksional aniqlangan sohaning ichki nuqtasi, $y = y_0(x)$ da funksionalning (birinchi) variatsiyasi nolga teng bo‘ladi, ya’ni

$$\delta V[y_0(x)] = 0.$$

$\delta V = 0$ shartni qanoatlantiruvchi funksiyalar *statsionar funksiyalar* deyiladi.

Misollar.

1-misol. Agar $y(x) = x,\ y_1(x) = x^2$ bo‘lsa, $C[0,1]$ da aniqlangan $V[y] = \int_0^1 y(x)y'(x)dx$ funksionalning orttirmasini toping.

Yechilishi. (4) formuladan foydalanamiz:

$$\Delta V = V[x^2] - V[x] = \int_0^1 x^2\, 2x dx - \int_0^1 x 1 dx = \int_0^1 (2x^3 - x)dx = 0 .$$

2-misol. $C[a,b]$ fazoda aniqlangan $V[y] = \int_a^b y(x)dx$ funksionalning shu fazodan olingan har bir $y(x)$ nuqtada differensiallanuvchiligini ko‘rsating.

Yechilishi. (4) formuladan

$$\Delta V = V[y+\delta y]-V[y]=\int_a^b [y(x)+\delta y(x)]dx-\int_a^b y(x)dx=\int_a^b \delta y(x)dx,$$

ya'ni $$\Delta V=\int_a^b \delta y(x)dx$$

ekanligini olamiz. Bu esa, $\delta y(x)$ ga nisbatan chiziqli funksionaldir. Berilgan holda funksionalning ΔV orttirmasi $\delta y(x)$ ga nisbatan chiziqli funksionalga keltirildi.

Demak, qaralayotgan $V[y]$ funksional har bir $y(x)$ nuqtada differensiallanuvchi va uning variatsiyasi

$$\delta V=\int_a^b \delta y(x)dx \quad \text{ekan.}$$

3-misol. $C[a,b]$ da aniqlangan $V[y]=\int_a^b y^2(x)dx$ funksionalning har bir $y(x)$ nuqtada differensiallanuvchi bo'lishini ko'rsating.

Yechilishi. (4) formuladan orttirma

$$\Delta V=\int_a^b [y(x)+\delta y(x)]^2dx-\int_a^b y^2(x)dx=\int_a^b 2\, y(x)\,\delta y(x)dx+\int_a^b (\delta y(x))^2\,dx \qquad (6)$$

kabi ifodalanishi kelib chiqadi.

(6) ning o'ng tomonidagi birinchi integral har bir tanlangan $y(x)$ funksiya uchun $\delta y(x)$ ga nisbatan chiziqli funksionaldan iborat. (6) ning o'ng tomonidagi ikkinchi integralni baholaymiz:

$\int_a^b (\delta y(x))^2\,dx=\int_a^b |\,\delta y(x)|^2\,dx \le (\max\limits_{a\le x\le b}|\,\delta y(x)\,|^2)\int_a^b dx=(b-a)\,\|\,\delta y(x)\,\|^2=((b-a)\,\|\,\delta y(x)\,\|)\,\|\,\delta y(x)\,\|$ Ravshanki, $\|\,\delta y\,\|\to 0$ da $(b-a)\,\|\,\delta y(x)\,\|\to o$, demak, funksionalning ΔV orttirmasi $L[y,\delta y]$ va $\|\,\delta y\,\|$ ga nisbatan ikkinchi tartibli kichik qo'shimchaning yig'indisi shaklida bo'lishi mumkin. Tarifga ko'ra, berilgan funksional $y(x)$ nuqtada differensiallanuvchidir va uning variatsiyasi

$$\delta V=2\int_a^b y(x)\delta y(x)dx\,.$$

4-misol. $V[y]=\int_a^b y^2(x)dx$ funksionalning variatsiyasini ikkinchi tarifdan foydalanib toping.

Yechilishi. Qaralayotgan funksionalning variatsiyasi birinchi tarif bo'yicha hisoblanganda, $\delta V=2\int_a^b y(x)\delta y(x)dx$ ekanligini ko'rdik (3-misolga q). Endi funksionalning variatsiyasini ikkinchi tarifdan foydalanib topamiz. Ravshanki,

$$V[y(x)+\alpha\delta y(x)]=\int_a^b [y(x)+\alpha\delta y(x)]^2\,dx\ .$$

U holda ,

$$\frac{d}{d\alpha}V[y+2\delta y]=2\int_a^b (y+\alpha\delta y)\delta y dx$$

va demak,

$$\delta V=\frac{d}{d\alpha}V[y+\alpha\delta y]|_{\alpha=o}=2\int_a^b y(x)\delta y(x)dx.$$

Buerda funksionalning birinchi va ikkinchi tariflar manosidagi variatsiyalari ustma-ust tushadi.

5-misol. Ushbu

$$V[y]=\int_a^b f(x,y,y')dx \qquad (7)$$

funksionalning variatsiyasini toping.

Yechilishi. Variatsiyaning ikkinchi ta’rifi bo‘yicha

$$\delta V=\left[\frac{d}{d\alpha}\int_a^b f(x,y,+\alpha\delta y,y'+\alpha\delta y')dx\right]|_{\alpha=o}$$

Bu erda $\delta y'-y(x)$ argumentdan olingan $y'(x)$ hosilaning variatsiyasi. $V[y]$ funksionalning α parametr bo‘yicha hosilasini hisoblasak,

$$\int_a^b [f_y(x,y+\alpha\delta y,y'+\alpha\delta y')\delta y+f_{y'}(x,y+\alpha\delta y,y'+\alpha\delta')\delta y']dx$$

bo‘ladi. $a=0$ deb olib, (7) funksionalning variatsiyasini topamiz,

$$\delta V=\int_a^b (f_y\delta y+f_{y'}\delta y')dx \qquad (8)$$

6-misol. $V[y]=\int_0^1 (x^2+y^2)dx$ funksional $y\equiv 0$ egri chiziqda qat’iy minimumga erishishini ko‘rsating.

*Yechilishi***.** $[0,1]$ kesmada uzluksiz ixtiyoriy $y(x)$ funksiya uchun,

$$V[y]=\int_0^1 (x^2+y^2)dx-\int_0^1 x^2dx=\int_0^1 y^2dx\geq 0,$$

bo‘ladi, bunda tenglik faqat $y(x)\equiv 0$ bo‘lganda o‘rinli.

Mustaqil yechish uchun masalalar.

1– masala

Quyidagi egri chiziqlarning yaqinlik tartibini toping.

1. $[0,2\pi]$ da $y(x)=\dfrac{\cos nx}{n^2+1}$ va $y_1(x)=0$ egri chiziqlar.

2.$[0,\pi]$ da $y(x)=\dfrac{\sin x}{n}$ va $y_1(x)=0$ egri chiziqlar.

3.[0,1] da $y(x)=\sin\dfrac{x}{n}$ va $y_1(x)=0$ egri chiziqlar.

4.$[0,2\pi]$ da $y(x)=\dfrac{\sin x}{n+1}$ va $y_1(x)=0$ egri chiziqlar.

5. [0,1] da $y(x)=\cos\frac{x}{n}$ va $y_1(x)=0$ egri chiziqlar.

Quyidagi egri chiziqlar orasidagi ρ_0 masofani ko‘rsatilgan oraliqlarda toping.

6. $y(x)=xe^{-x}$, $y_1(x)=0$, $x\in[0,2]$; 7. $y(x)=\sin 2x$; $y_1(x)=\sin x$, $x\in[0,\frac{\pi}{2}]$;

8. $y(x)=x$, $y_1(x)=\ln x$, $x\in[e^{-1},e]$. 9. $y(x)=\cos 2x$; $y_1(x)=\cos x$, $x\in[0,\frac{\pi}{2}]$;

10. $[e^{-1},e]$ oraliqda $y(x)=\ln x$, $y_1(x)=x$ funksiyalar orasidagi ρ_1 masofani toping.

11. $[0,\frac{\pi}{4}]$ oraliqda $y(x)=x$ va $y_1(x)=\sin x$ funksiyalar orasidagi ρ_2 masofani toping.

12. $[0,\frac{\pi}{3}]$ oraliqda $y(x)=x$ va $y_1(x)=-\cos x$ funksiyalar orasidagi ρ_2 masofani toping.

13. $[0,1]$ oraliqda $y(x)=e^x$ va $y_1(x)=x$ funksiyalar orasidagi ρ_{1001} masofani toping.

Quyidagi funksionallarni $y(x)=0$ to‘g‘ri chiziqning: a) kuchli atrofida; b) kuchsiz atrofida uzluksizlikka tekshiring.

14. $V\ [y]=\int_0^{\pi} y'^2\,dx.$ 15. $V\ [y]=\int_1^{2}|y'|\,dx.$

16. $V[y]=\int_0^{\pi}\sqrt{1+y'^2}\,dx.$ 17. $V[y]=\int_0^{\pi}(1+2y'^2)\,dx.$

18. Quyidagi $y_1(x)=x^2$ va $y_2(x)=x^3$ funksiyalar orasidagi masofani: $C[0,1]$ fazo normasida hisoblang.

19. Quyidagi $y_1(x)=x^2$ va $y_2(x)=x^3$ funksiyalar orasidagi masofani: $C^1[0,1]$ fazo normasida hisoblang.

20. $C^1[0,1]$ fazoda aniqlangan $J(y)=\int_0^1(y-y^1)dx$ funksionalning $y_0(x)=x^3$ funksiyada uzluksizligini ko’rsating.

21. $C^1[0,1]$ fazo normasida $J(y)=\int_0^1(y-y^1)dx$ funksionalning $y_0(x)=x^3$ funksiyada uzluksizligini ko’rsating.

<u>2 – masala.</u> Quyidagi funksionalning birinchi variatsiyasini toping.

1. $J(y)=\int_0^1(y^2-y'^2+5yx^2)dx$ 2. $J(y)=\int_{-1}^1(y^2-4yy'+y'^2e^{2x})dx$

3. $J(y)=\int_0^{\pi}(y'^2-4y^2+3y'Sinx)dx$ 4. $J(y)=\int_0^1(y^2+3y'^2+yx^3)dx$

5. $J(y)=\int_{-1}^{1}(3y^2+2yy'-y'^2e^x)dx$ 6. $J(y)=\int_{0}^{\pi}(2y'^2+y^2+y'Sin3x)dx$

7. $J(y)=\int_{0}^{1}(2y^2-4y'^2+yx^4)dx$ 8. $J(y)=\int_{-1}^{1}(y^2-5yy'-y'^2e^{3x})dx$

9. $J(y)=\int_{0}^{\pi}(y'^2+3y^2+y'Sin2x)dx$ 10. $J(y)=\int_{0}^{1}(y^2+7y'^2-2yx^2)dx$

11. $J(y)=\int_{-1}^{1}(y^2+2yy'+5y'^2e^x)dx$ 12. $J(y)=\int_{0}^{\pi}(2y'^2-6y^2-y'Sinx)dx$

13. $J(y)=\int_{0}^{1}(y^2+2y'^2+3yx^3)dx$ 14. $J(y)=\int_{-1}^{1}(5y^2-3yy'+y'^2e^{2x})dx$

15. $J(y)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(y'^2-4y^2+y'\cos 2x)dx$ 16. $J(y)=\int_{0}^{1}(3y^2-y'^2+5yx^4)dx$

17. $J(y)=\int_{-1}^{1}(y^2+4yy'+3y'^2e^{3x})dx$ 18. $J(y)=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(y'^2+2y^2-5y'\cos x)dx$

19. $J(y)=\int_{0}^{1}(y^2-2y'^2+6yx^2)dx$ 20. $J(y)=\int_{-1}^{1}(4y^2+2yy'+y'^2e^x)dx$

2-amaliy mashg‘ulot. Variatsion hisobning asosiy masalasi. Kuchsiz ekstremumning birinchi tartibli zaruriy sharti, Eyler tenglamasi. Eyler tenglamasining xususiy hollari.

1. Masalaning qo’yilishi.

$$V=\{y(x)\in C^1[x_0,x_1]:\ y(x_0)=y_0,\ y(x_1)=y_1\ (x,y(x),\ y'(x))\in Q, x\in[x_0,x_1]\} \quad (1)$$

To`plamida aniqlangan,

$$J[y]=\int_{x_0}^{x_1}F(x,y,y')dx \quad (2)$$

funksionalning ekstremumini topish masalasini qaraymiz. Bu masalaga *variatsion hisobning asosiy masalasi* deyiladi va u

$$J[y]=\int_{x_0}^{x_1}F(x,y,y')dx\to\min(\max),\ y(x_0)=y_0,\ y(x_1)=y_1,\ y(x)\in C^1[x_0,x_1] \quad (3)$$

1-t a ’ r i f. $y^0=y^0(x)$ – joyiz funksiya bo`lsin $(y^0\in V)$. Agar $y^0$ ning shunday $V_0(y^0,\varepsilon)$ nolinchi tartibli $\varepsilon$ – atrofi mavjud bo`lib, shu atrofga tegishli barcha $y=y(x)$ joyiz funksiyalar uchun,

$$J[y^0]\le J[y] \qquad (J[y^0]\ge J[y])$$

munosabat bajarilsa, $y^0(x)$ funksiya - (2) funksionalning *kuchli lokal minimum (maksimum) nuqtasi* deyiladi.

2-t a ’ r i f. Agar $y^0 = y^0(x)$ joyiz funksiyaning shunday $V_1(y^0, \varepsilon)$ birinchi tartibli ε - atrofi mavjud bo`lsaki, $J[y^0] \le J[y]$ $(J[y^0] \ge J[y])$ $\forall y \in V_1(y^0, \varepsilon) \cap V$ munosabat bajarilsa, $y^0(x)$ funksiya (2) funksionalning *kuchsiz lokal minimum (maksimum) nuqtasi* deyiladi.

3. Eyler tenglamasi.

$F(x, y, y')$ funksiyaning xususiy hosilalari uchun, quyidagi belgilashlardan foydalanamiz:

$$F_x = \frac{\partial F}{\partial x}, \quad F_y = \frac{\partial F}{\partial y}, \quad F_{y'} = \frac{\partial F}{\partial y'},$$

$$F_{xy'} = \frac{\partial^2 F}{\partial x \partial y'}, \quad F_{yy} = \frac{\partial^2 F}{\partial y^2}, \quad F_{yy'} = \frac{\partial^2 F}{\partial y \partial y'}, \quad F_{y'y'} = \frac{\partial^2 F}{\partial y'^2}$$

$$\delta J = \int_{x_0}^{x_1} \left[F_y\left(x, y^0(x), y^{0\prime}(x)\right) - \frac{d}{dx} F_{y'}\left(x, y^0(x), y^0(x)\right) \right] h(x) dx \qquad (13)$$

formulaga ega bo`lamiz.

$y^0 = y^0(x) -$ (3) masalada kuchsiz ekstremal bo`lsin. U vaqtda, ekstremumning zaruriy shartiga ko`ra, shu nuqtada hisoblangan birinchi variatsiya nolga teng, ya’ni (13) ga asosan,

$$\int_{x_0}^{x_1} \left[F_y\left(x, y^0(x), y^{0\prime}(x)\right) - \frac{d}{dx} F_{y'}\left(x, y^0(x), y^0(x)\right) \right] h(x) dx = 0 \qquad (14)$$

1-t e o r e m a. $F(x, y, y') \in C^{(2)}(Q)$ bo`lsin. Agar $y^0(x) \in C^{(2)}[x_0, x_1]$ (3) masalada kuchsiz ekstremal bo`lsa, u

$$F_{y'y'} y'' + F_{yy'} y' + F_{xy'} - F_y = 0 \qquad (15)$$

tenglamani qanoatlantiradi.

Hosil qilingan (15) differensial tenglamaga, *Eyler tenglamasi* deyiladi.

2-t e o r e m a. $F(x, y, y') \in C^{(1)}(Q)$ bo’lsin. Agar $y^0(x) \in C^{(1)}[x_0, x_1]$ joyiz funksiya (3) masalada kuchsiz ekstremal bo`lsa, u

$$F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0 \qquad (16)$$

tenglamani qanoatlantiradi.

3-t a ’ r if. Eyler tenglamasini qanoatlantiruvchi $y = y(x)$ joyiz funksiyalarga (2) *funksionalning joyiz stasionar funksiyalari* deyiladi.

M i s o l l a r. 1)
$$\left. \begin{array}{l} J[y] = \int_1^2 \left(y'^2 + 4xy\right) dx \to extr \\ y(1) = 0, \quad y(2) = -1 \end{array} \right\}$$

masalani qaraymiz. Uning uchun Eyler tenglamasini tuzamiz:

$$F = y'^2 + 4xy,\ \ F_y = 4x,\ \ F_y = 2y',\ \ \frac{d}{dx}F_{y'} = 2y''$$

$$F_y - \frac{d}{dx}F_{y'} = 4x - 2y'' = 0.$$

Demak, Eyler tenglamasi $y'' - 2x = 0$ tenglamadan iborat. Bu tenglamaning umumiy yechimi

$$y = y(x) = \frac{x^3}{3} + c_1 x + c_2$$

ko`rinishda bo`ladi. $y(1) = 0,\ y(2) = -1$ chegaraviy shartlarga ko`ra,

$$c_1 + c_2 = -\frac{1}{3},\ \ 2c_1 - c_2 = -\frac{11}{3}.$$

Bu yerdan, $c_1 = -\frac{10}{3}$, $c_2 = 3$. Shunday qilib, $y = \frac{x^3}{3} - \frac{10}{3}x + 3$ funksiya – yagona joyiz stasionar funksiyadir.

$$2)\quad \left. \begin{aligned} & J[y] = \int_1^{2\pi} \left(y'^2 - y^2\right)dx \to extr \\ & y(0) = 1,\quad y(2\pi) = 1 \end{aligned} \right\}$$

masalani qaraymiz. Unda $F = {}'^2 - y^2$, $F_y = 2y$, $F_{y'} = 2y''$. Demak, (16) tenglama $y'' + y = 0$ ko’rinishda bo`ladi. $y = c_1 \cos x + c_2 \sin x$ funksiya hosil qilingan Eyler tenglamasining umumiy yechimidir. $y(0) = y(2\pi) = 1$ chegaraviy shartlarga asosan, $y = \cos x + c \sin x$ funksiyaga ega bo`lamiz, bu yerda c – ixtiyoriy o`zgarmas. Demak, qaralayotgan masalada cheksiz ko`p joyiz stasionar funksiyalar mavjud.

$$3)\quad \left. \begin{aligned} & J[y] = \int_1^{3} (3x - y)y\,dx \to extr \\ & y(1) = 1,\quad y(3) = 4. \end{aligned} \right\}$$

Bu masalada $F = (3x - y)y,\ F_y = 3x - 2y,\ F_y = 0.$ Eyler tenglamasi, $F_y = 0$ tenglamadan iborat. Bu yerdan $y = \frac{3x}{2}$. Bu funksiya, $y(1) = 1,\ y(3) = 4$ chegaraviy shartlarni qanoatlantirmaydi. Qaralayotgan funksional joyiz stasionar funksiyalarga va, demak, ekstremumga ham ega emas.

Eyler tenglamasining xususiy hollari

Eyler tenglamasining umumiy ko’rinishi quydagicha hosil qilgan edik

$$F_y(x, y, y') - \frac{d}{dx}F_{y'}(x, y, y') = 0$$

Endi uning xususiy hollarini ko’rib chiqamiz

a) F funksiya faqat y' ga bog`liq, ya’ni $F = F(y')$ bo’lsin. Bu holda Eyler tenglamasi $\frac{d}{dx}F_{y'}(y') = 0$ yoki $F_{y''} = 0$ bo`ladi. Eyler tenglamasining yechimlari $y = C_1 x + C_2$ chiziqli funksiyalardan iboratdir.

b) $F = F(x, y')$, ya’ni F funksiya faqat x va y' larga bog`liq bo`lsin. Bu holda $F_y = 0$ bo`ladi va Eyler tenglamasi $\frac{d}{dx}F_{y'} = 0$ ko`rinishni oladi. Uni y' ga nisbatan yechish yoki biror parametr kiritish yo`li bilan integrallash mumkin.

c) F faqat y va y' ga bog’liq bo`lsin: $F = F(y, y')$. Bu holda $F \in C^{(2)}$ deb faraz qilib, (15) Eyler tenglamasini yozamiz: $F_{y'y'}y'' + F_{y'y'}y' - F_y = 0$ $(F_{xy'} = 0)$. Agar bu tenglamaning ikkala tomonini $y'$ ga ko`paytirsak, uni $\frac{d}{dx}(F - y'F_{y'}) = 0$ ko`rinishda yozish mumkin. Natijada Eyler tenglamasi

$$F - y'F_{y'} = C \tag{18}$$

ko`rinishdagi birinchi integralga ega bo`ladi.

5. Gilbert teoremasi.

3-t e o r e m a (Gilbert). $F(x, y, y') \in C^{(2)}(Q)$ bo`lsin. Agar (3) masalaning $y^0(x)$ kuchsiz lokal ekstremali uchun $F_{y'y'}(x, y^0(x), y^{0'}(x)) \neq 0,\ x \in [x_0, x_1]$ bo`lsa, $y^0(x) \in C^{(2)}[x_0, x_1]$ bo`ladi.

Mustaqil ishlash uchun misollar

1 – masala. Quyidagi variatsion hisob asosiy masalasida Eyler tenglamasini tuzing.

1. $J(y) = \int_0^2 (y'^2 - yy' + 9y^2)dx \to \min,\ y(0) = 0,\ y(2) = 1,\ y \in C^{(1)}[0;2]$

2. $J(y) = \int_{-1}^1 (y'^2 - 9y^2 + 2yx^2)dx \to \min;\ y(-1) = 0,\ y(1) = 1,\ y \in C^{(1)}[-1;1]$

3. $J(y) = \int_0^1 (y'^3 - 5ye^x)dx \to \min;\ y(0) = 0,\ y(1) = 1,\ y \in C^{(1)}[0;1]$

4. $J(y) = \int_0^2 (y'^2 + 3yy' + 4y^2)dx \to \min,\ y(0) = 0,\ y(2) = 1,\ y \in C^{(1)}[0;2]$

5. $J(y) = \int_{-1}^1 (y'^2 - y^2 + 6yx^3)dx \to \min;\ y(-1) = 0,\ y(1) = 1,\ y \in C^{(1)}[-1;1]$

6. $J(y) = \int_0^1 (y'^3 - ye^{2x})dx \to \min;\ y(0) = 0,\ y(1) = 1,\ y \in C^{(1)}[0;1]$

7. $J(y) = \int_0^2 (4y'^2 + 5yy' - y^2)dx \to \min,\ y(0) = 0,\ y(2) = 1,\ y \in C^{(1)}[0;2]$

8. $J(y) = \int_{-1}^1 (y'^2 - 4y^2 + 3yx^2)dx \to \min;\ y(-1) = 0,\ y(1) = 1,\ y \in C^{(1)}[-1;1]$

9. $J(y) = \int_0^1 (y'^3 + 6ye^x)dx \to \min;\ y(0) = 0,\ y(1) = 1,\ y \in C^{(1)}[0;1]$

10. $J(y) = \int_0^2 (y'^2 - 2yy' - 4y^2)dx \to \min,\ y(0) = 0,\ y(2) = 1,\ y \in C^{(1)}[0;2]$

11. $J(y)=\int\limits_{-1}^{1}\left(y'^2+2y^2+3yx^3\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}\left[-1;1\right]$

12. $J(y)=\int\limits_{0}^{1}\left(y'^3+3ye^{2x}\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}\left[0;1\right]$

13. $J(y)=\int\limits_{0}^{2}(y'^2-3yy'+8y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

14. $J(y)=\int\limits_{-1}^{1}\left(y'^2-6y^2+yx^2\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}\left[-1;1\right]$

15. $J(y)=\int\limits_{0}^{1}\left(y'^3-4ye^{x}+x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}\left[0;1\right]$

16. $J(y)=\int\limits_{0}^{2}(y'^2+4yy'-y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

17. $J(y)=\int\limits_{-1}^{1}\left(y'^2+y^2-4yx^3\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}\left[-1;1\right]$

18. $J(y)=\int\limits_{0}^{1}\left(y'^3+ye^{x}+2x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}\left[0;1\right]$

19. $J(y)=\int\limits_{0}^{2}(y'^2-6yy'+3xy^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

20. $J(y)=\int\limits_{-1}^{1}\left(y'^2-2xy^2+3y\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}\left[-1;1\right]$

21. $J(y)=\int\limits_{0}^{2}(y'^2-5yy'+4y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

22. $J(y)=\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(y'^2-y'\sin x\right)dx\to\min;\ y(\frac{\pi}{4})=1,\ y(\frac{\pi}{2})=0,\ y\in C^{(1)}\left[\frac{\pi}{4};\frac{\pi}{2}\right]$

23. $J(y)=\int\limits_{0}^{1}\left(y'^2+3x^2y\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}\left[0;1\right]$

24. $J(y)=\int\limits_{0}^{2}\left(y'^2-xyy'\right)dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}\left[0;2\right]$

25. $J(y)=\int\limits_{\frac{\pi}{2}}^{\pi}\left(y'^2-4y'\sin 2x\right)dx\to\min;\ y(\frac{\pi}{2})=1,\ y(\pi)=0,\ y\in C^{(1)}\left[\frac{\pi}{2};\pi\right]$

26. $J(y)=\int\limits_{0}^{1}\left(y'^2-2x^2y'\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}\left[0;1\right]$

27. $J(y)=\int\limits_{0}^{2}\left(y'^2-18xyy'\right)dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}\left[0;2\right]$

28. $J(y)=\int\limits_{0}^{1}\left(y'^2-3y'Cosx\right)dx\to\min;\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}\left[0;1\right]$

29. $J(y)=\int\limits_0^1\left(y'^2-7x^3y\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

30. $J(y)=\int\limits_0^{\sqrt{2}\pi}\left(y'^2+16xyy'\right)dx\to\min;\ y(0)=0,\ y(\sqrt{2}\pi)=0,\ y\in C^{(1)}\left[0;\sqrt{2}\pi\right]$

<u>2 – masala.</u> Quyidagi variatsion hisob asosiy masalasida joiz statsionar funksiyani toping.

1. $J(y)=\int\limits_0^2\left(y'^2-8xyy'\right)dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

2. $J(y)=\int\limits_0^{\pi}\left(y'^2+2y'Cosx\right)dx\to\min;\ y(0)=1,\ y(\pi)=0,\ y\in C^{(1)}[0;\pi]$

3. $J(y)=\int\limits_0^1\left(y'^2+12xy\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

4. $J(y)=\int\limits_0^{\frac{\pi}{\sqrt{2}}}\left(y'^2+4xyy'\right)dx\to\min;\ y(0)=0,\ y(\frac{\pi}{\sqrt{2}})=1,\ y\in C^{(1)}\left[0;\frac{\pi}{\sqrt{2}}\right]$

5. $J(y)=\int\limits_0^{\frac{\pi}{2}}\left(y'^2-y'Cos2x\right)dx\to\min;\ y(0)=0,\ y(\frac{\pi}{2})=0,\ y\in C^{(1)}\left[0;\frac{\pi}{2}\right]$

6. $J(y)=\int\limits_0^1\left(y'^2-6xy'\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

7. $J(y)=\int\limits_0^{\frac{\pi}{4}}\left(y'^2+2xyy'\right)dx\to\min;\ y(0)=0,\ y(\frac{\pi}{4})=1,\ y\in C^{(1)}\left[0;\frac{\pi}{4}\right]$

8. $J(y)=\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(y'^2-y'\sin x\right)dx\to\min;\ y(\frac{\pi}{4})=1,\ y(\frac{\pi}{2})=0,\ y\in C^{(1)}\left[\frac{\pi}{4};\frac{\pi}{2}\right]$

9. $J(y)=\int\limits_{-1}^1\left(y'^2-9y^2+2yx^2\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

10. $J(y)=\int\limits_0^1\left(y'^3-5ye^x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

11. $J(y)=\int\limits_0^2(y'^2+3yy'+4y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

12. $J(y)=\int\limits_{-1}^1\left(y'^2-y^2+6yx^3\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

13. $J(y)=\int\limits_0^1\left(y'^3-ye^{2x}\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

14. $J(y)=\int\limits_0^2(4y'^2+5yy'-y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

15. $J(y)=\int_{-1}^{1}\left(y'^2-4y^2+3yx^2\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

16. $J(y)=\int_{0}^{1}\left(y'^3+6ye^x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

17. $J(y)=\int_{0}^{1}\left(y'^2+3x^2y\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

18. $J(y)=\int_{0}^{2}\left(y'^2-xyy'\right)dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

19. $J(y)=\int_{\frac{\pi}{2}}^{\pi}\left(y'^2-4y'\sin 2x\right)dx\to\min;\ y(\frac{\pi}{2})=1,\ y(\pi)=0,\ y\in C^{(1)}\left[\frac{\pi}{2};\pi\right]$

20. $J(y)=\int_{0}^{1}\left(y'^2-2x^2y'\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

21. $J(y)=\int_{0}^{2}\left(y'^2-18xyy'\right)dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

22. $J(y)=\int_{0}^{1}\left(y'^2-3y'Cosx\right)dx\to\min;\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

23. $J(y)=\int_{0}^{1}\left(y'^2-7x^3y\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

24. $J(y)=\int_{0}^{\sqrt{2}\pi}\left(y'^2+16xyy'\right)dx\to\min;\ y(0)=0,\ y(\sqrt{2}\pi)=0,\ y\in C^{(1)}\left[0;\sqrt{2}\pi\right]$

25. $J(y)=\int_{-1}^{0}\left(y'^2-5y'Cos2x\right)dx\to\min;\ y(-1)=0,\ y(0)=1,\ y\in C^{(1)}[-1;0]$

26. $J(y)=\int_{0}^{1}\left(y'^2+2x^3y'\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

27. $J(y)=\int_{0}^{2}\left(y'^2+3x^2y'\right)dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

28. $J(y)=\int_{0}^{1}\left(y'^2+y'\sin x+yy'\right)dx\to\min;\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[1;3]$

29. $J(y)=\int_{0}^{\frac{\pi}{4}}\left(y'^2+8xyy'\right)dx\to\min;\ y(0)=0,\ y(\frac{\pi}{4})=1,\ y\in C^{(1)}\left[0;\frac{\pi}{4}\right]$

30. $J(y)=\int_{-1}^{1}\left(y'^2+y^2-4yx^3\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

Eyler tenglamalarni hususiy hollari bilan yeching.

1. $J(y)=\int_{0}^{2}(y'^2-yy'+9y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

2. $J(y)=\int_0^2 (y'^2+3yy'+4y^2)dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

3. $J(y)=\int_0^2 (4y'^2+5yy'-y^2)dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

4. $J(y)=\int_0^2 (y'^2-2yy'-4y^2)dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

5. $J(y)=\int_0^2 (y'^2-3yy'+8y^2)dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

6. $J(y)=\int_0^2 (y'^2+4yy'-y^2)dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

7. $J(y)=\int_0^2 (y'^2-5yy'+4y^2)dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

8. $J(y)=\int_0^1 y\sqrt{1+y'^2}\,dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

9. $J(y)=\int_0^1 \left(y'^2-7y\right)dx \to \min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

10. $J(y)=\int_{-1}^1 \left(y'^2+y^2-4yx^3\right)dx \to \min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

11.
$J(y)=\int_{-1}^1 \left(y'^2+3x^2\right)dx \to \min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

12. $J(y)=\int_0^1 \left(y'^3+6e^x\right)dx \to \min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

13. $J(y)=\int_0^1 \left(y'^2+3x^2\right)dx \to \min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

14. $J(y)=\int_0^2 \left(y'^2-xy'\right)dx \to \min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

15. $J(y)=\int_{\frac{\pi}{2}}^{\pi} \left(y'^2-4y'\sin 2x\right)dx \to \min;\ y(\frac{\pi}{2})=1,\ y(\pi)=0,\ y\in C^{(1)}\left[\frac{\pi}{2};\pi\right]$

16. $J(y)=\int_0^1 \left(y'^2-2x^2y'\right)dx \to \min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

17. $J(y)=\int_0^2 \left(y'^2-18xy'\right)dx \to \min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

18. $J(y)=\int_0^1 \left(y'^2-3y'Cosx\right)dx \to \min;\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

19. $J(y)=\int_0^1 \left(y'^2-7x^3\right)dx \to \min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

20. $J(y)=\int\limits_0^{\sqrt{2}\pi}\left(y'^2+16xy'\right)dx\to\min;\ y(0)=0,\ y(\sqrt{2}\pi)=0,\ y\in C^{(1)}\left[0;\sqrt{2}\pi\right]$

21. $J(y)=\int\limits_{-1}^{0}\left(y'^2-5y'Cos2x\right)dx\to\min;\ y(-1)=0,\ y(0)=1,\ y\in C^{(1)}[-1;0]$

22. $J(y)=\int\limits_0^2\left(y'^2-8y'\right)dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

23. $J(y)=\int\limits_0^{\pi}\left(y'^2+2y'\right)dx\to\min;\ y(0)=1,\ y(\pi)=0,\ y\in C^{(1)}[0;\pi]$

24. $J(y)=\int\limits_0^1\left(y'^2+y\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

25. $J(y)=\int\limits_0^{\frac{\pi}{\sqrt{2}}}\left(y'^2+4y'\right)dx\to\min;\ y(0)=0,\ y(\frac{\pi}{\sqrt{2}})=1,\ y\in C^{(1)}\left[0;\frac{\pi}{\sqrt{2}}\right]$

26. $J(y)=\int\limits_0^{\frac{\pi}{2}}\left(y'^2-y'\right)dx\to\min;\ y(0)=0,\ y(\frac{\pi}{2})=0,\ y\in C^{(1)}\left[0;\frac{\pi}{2}\right]$

27. $J(y)=\int\limits_0^1\left(y'^2-6y'\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

28. $J(y)=\int\limits_0^{\frac{\pi}{4}}\left(y'^2+2y'\right)dx\to\min;\ y(0)=0,\ y(\frac{\pi}{4})=1,\ y\in C^{(1)}\left[0;\frac{\pi}{4}\right]$

29. $J(y)=\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(y'^2-y'\right)dx\to\min;\ y(\frac{\pi}{4})=1,\ y(\frac{\pi}{2})=0,\ y\in C^{(1)}\left[\frac{\pi}{4};\frac{\pi}{2}\right]$

30. $J(y)=\int\limits_{-1}^{1}\left(y'^2-x^2\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

3-amaliy mashg‘ulot. Variatsion hisobning asosiy masalasida ikkinchi tartibli zaruriy shartlar va yetarli shartlar. Ikkinchi variatsiyani tekshirish. Lejandr-Klebsh, Yakobi shartlari. Ekstremumning yetarli sharti**.**

1. Masalaning qo’yilishi.

$$V=\left\{y(x)\in C^1[x_0,x_1]:\ y(x_0)=y_0,\ y(x_1)=y_1\ (x,y(x),\ y'(x))\in Q,x\in[x_0,x_1]\right\}\quad(1)$$

To`plamida aniqlangan,

$$J[y]=\int\limits_{x_0}^{x_1}F(x,y,y')dx\qquad(2)$$

funksionalning ekstremumini topish masalasini qaraymiz. Bu masalaga *variatsion hisobning asosiy masalasi* deyiladi va u

$$J[y]=\int_{x_0}^{x_1} F(x,y,y')dx \to \min(\max),\ y(x_0)=y_0,\ y(x_1)=y_1,\ y(x)\in C^1[x_0,x_1] \qquad (3)$$

2. Lejandr sharti.

$$J[y]=\int_{x_0}^{x_1} F(x,y,y')dx \qquad (1)$$

funksionalning

$$V=\{y=y(x)\in C^{(1)}[x_0,x_1]: y(x_0)=y_0, y(x_1)=y_1\} \qquad (2)$$

to'plamdagi ekstremumini topish masalasi, ya'ni variatsion hisob asosiy masalasi, berilgan bo'lsin.

$y^0=y^0(x)$ joyiz funksiya bo'lsin $(y^0\in V)$. Shu nuqtada (1) funksionalning ikkinchi variatsiyasini hisoblaymiz. Ta'rifga ko'ra, bu variatsiya,

$$\delta^2 J[y^2,h]=\frac{d^2 J[y^0+\alpha h]}{d\alpha^2}\ /_{\alpha=0}$$

formula bo'yicha hisoblanadi, bu yerda

$$h=h(x)\in C^{(1)}[x_0,x_1], h(x_0)=h(x_1)=0\ .$$

Agar $F(x,y,y')\in C^{(2)}(Q)$ deb faraz qilsak, $\varphi(\alpha)=J[y^0+\alpha h]$ funksiya α=0 nuqta atrofida uzluksiz ikkinchi tartibli hosilaga ega. Demak,

$$\delta^2 J[y^0,h]=\frac{d^2}{d\alpha^2}\int_{x_0}^{x_1} F(x,y^0(x)+\alpha h(x),y^{0'}(x)+\alpha h'(x))dx\,|_{\alpha=0}=$$

$$=\int_{x_0}^{x_1}\frac{\partial^2}{\partial\alpha^2}F(x,y^0(x)+\alpha h(x),y^{0'}(x)+\alpha h'(x))dx\,|_{\alpha=0}=$$

$=$

$$=\int_{x_0}^{x_1}[F_{yy}(x,y^0(x),y^{0'}(x))h^2(x)+2F_{yy'}(x,y^0(x),y^{0'}(x))h(x)h'(x)+ \qquad (3)$$

$$+F_{y'y'}(x,y^0(x),y^{0'}(x))h'^2(x)]dx,\quad h=h(x)\in C^{(1)}[x_0,x_1],\qquad h(x_0)=h(x_1)=0$$

1-teorema(Lejandr). $F(x,y,y')\in C^{(2)}(Q)$ bo'lsin. Agar $y^0(x)\in C^{(1)}[x_0,x_1]-(1)$ funksionalning (2) to'plamdagi kuchsiz minimali (maksimali) bo'lsa,

$$F_{y'y'}(x,y^0(x),y^{0'}(x))\ge 0\quad (\le 0),\quad \forall x\in[x_0,x_1] \qquad (4)$$

tengsizlik bajariladi. (4) munosabatga Lejandr sharti deyiladi

3.Yakobi sharti.

$F(x,y,y')\in C^{(3)}(Q), y^0(x)\in C^{(2)}[x_0x_1]$ - joyiz stasionar funksiya $F_{y'y'}(x,y^0(x),y^{0'}(x))\ne 0\ \forall x\in[x_0x_1]$ bo'lsin. U vaqtda, (8) masala uchun tuzilgan,

$$\omega_h(x,h,h')-\frac{d}{dx}\omega_{h'}(x,h,h')=0$$

Eyler tenglamasiga, variason hisob asosiy masalasi uchun *Yakobi tenglamasi* deyiladi. $\omega(x,h,h')$ funksiyaning ko'rinishini hisobga olib, Yakobi tenglamasini

$$A(x)h''+B(x)h'+C(x)h=0 \qquad (9)$$

ikkinchi tartibli bir jinsli chiziqli differensial tenglama ko'rinishida yozish mumkin, bu yerda

$$A(x)=F_{y'y'}(x,y^0(x),y^{0'}(x)),\quad B(x)=\frac{d}{dx}F_{y'y'}(x,y^0(x),y^{0'}(x)),$$

$$C(x)=\frac{d}{dx}F_{yy'}(x,y^0(x),y^{0'}(x))-F_{yy}(x,y^0(x),y^{0'}(x))$$

(9) tenglama $h(x_0)=0,\ h'(x_0)=1$ chegaraviy shartlarni qanoatlantiruvchi (aynan noldan farqli) yagona yechimga ega. Shu yechimning x_0 dan farqli nollariga, *x_0 nuqtaga qo'shma nuqta* deyiladi.

T a' r i f. Agar (9) Yakobi tenglamasi $h(x_0)=0,\ h(x^*)=0 \quad x^*\neq x_0$ shartlarni qanoatlantiruvchi trivial (aynan nol) bo'lmagan $h(x)$, $x\in[x_0,x_1]$ yechimga ega bo'lsa, x^* nuqtaga $y^0(x)$ joyiz chiziq bo'ylab x_0 nuqtaga qo'shma nuqta deyiladi.

2-t e o r e m a (Yakobi). Faraz qilaylik:

a) $F(x,y,y')\in C^{(3)}(Q)$, б) $y^0(x)\in C^{(2)}[x_0x_1]$ - kuchsiz minimal (maksimal)

$F_{y'y'}(x,y^0(x),y^{0'}(x))>0 \quad (<0) \quad \forall x\in[x_0x_1]$ bo'lsin. U holda, $y^0(x)$ funksiya Yakobi shartini qanoatlantiradi: (x_0,x_1) intervalda $y^0(x)$ chiziq bo'ylab x_0 nuqtaga qo'shma bo'lgan nuqta mavjud emas.

3.Kuchsiz ekstremumning yetarli shartlari.

3-t e o r e m a.

a) $F(x,y,y')\in C^{(3)}(Q), y^0(x)\in C^{(2)}[x_0x_1]$ -joyiz stasionar funksiya bo'lsin;

b) kuchaytirilgan Lejandr sharti bajarilsin:

$F_{y'y'}(x,y^0(x),y^{0'}(x))>0 \quad (<0) \quad \forall x\in[x_0x_1]$;

c) kuchaytirilgan Yakobi sharti o'rinli bo'lsin: $y^0(x)$ joyiz chiziq bo'ylab $(x_0,x_1]$ da x_0 nuqtaga qo'shma bo'lgan x_1 nuqta mavjud emas.

U holda, $y^0(x)$ - variatsion hisobning asosiy masalasida kuchsiz lokal minimal (maksimal) bo'ladi.

Eyler tenglamasi,

$$(A(x)-q)h''+B(x)h'+C(x)h=0 \qquad (24)$$

ko'rinishda bo'ladi. $A(x)=F_{y'y'}(x,y^0(x),y^{0'}(x)>0,\ x\in[x_0,x_1]$.bo'lgani uchun, shunday $q>0$ topiladiki, $\cdot A(x)q>0,\ x\in[x_0,x_1]$ bajariladi. Farazga ko'ra, (9) Yakobi tenglamasining ,

$$h(x_0)=0,\ h'(x_0)=1 \qquad (25)$$

boshlang'ich shartlardagi yechimi (x_0,x_1) intervalda nolga aylanmaydi.

4. Kuchli ekstremumning zaruriy va yetarli shartlari.

$Q=S\times R^2, S\subset R^2$ berilgan ochiq to'plam, $F(x,y,y')\in C'(Q)$ bo'lsin.

Quyidagi

$$E(x,y,y',u,)=F(x,y,u)-F(x,y,y')-(u-y')F_{y'}(x,y,y'),(x,y,y',u)\in Q\times R^1 \qquad (31)$$

funksiyani qaraymiz. $E(x,y,y',u)$ funksiyaga Veyershtrass funksiyasi deyiladi.

4-teorema. Agar $y^0(x)=D^1[x_0,x_1]$- (1) funksionalning
$\widetilde{V}=\{y(x)\in D'[x_0,x_1]: y(x_0)=y_0, y(x_1)=y_1\}$
to'plamdagi kuchli lokal minimum (maksimum) nuqtasi bo'lsa, $y^0(x)$ mavjud bo'lgan barcha $x\in[x_0,x_1]$ nuqtalarda

$$E(x,y^0(x),y^0{}'(x),u)\geq 0 \quad (\leq 0) \quad \forall u\in R^1 \qquad (32)$$

Veyershtrass sharti bajariladi. $y^0(x)$ ning ξ burchak nuqtalarida esa, (32) shart,

$$E(\xi,y^0(\xi),y^0(\xi\pm 0),u)\geq 0(\leq 0)\forall u\in R \qquad (33)$$

ko'rinishda bo'ladi.

5-t e o r e m a**.** Quyidagi shartlar bajarilsin:

1). $F(x,y,y')\in C^{(4)}(Q), Q=S\times R^1,$

2). $F_{y'y'}(x,y,y')\geq 0 \quad (\leq 0), \ \forall(x,y,y')\in Q;$

3). $y^0(x)\in C^{(3)}[x_0,x_1]-$ (1) funksionalning joyiz stasionar funksiyasi;

4). $y^0(x)$ funksiya uchun kuchaytirilgan Lejandr va Yakobi shartlari o'rinli.

U holda, $y^0(x)$- (1) funksionalning (2) to'plamdagi kuchli lokal minimum (maksimum) nuqtasi bo'ladi.

Veyershtrass-Erdman shartlarini qisqacha

$$F_y\Big|_{x=\xi-0}=F_{y'}\Big|_{x=\xi+0},(F-y'F_{y'})\Big|_{x=\xi-0}=(F-y'F_{y'})\Big|_{x=\xi+0},$$

ko'rinishda yozish mumkin.

5. Kvadratik funksional bo'lgan hol

, (1) funksional quyidagi

$$J[y]=\int_{x_0}^{x_1}\left[p(x)y^2+2q(x)yy'+r(x)y'^2\right]dx \qquad (37)$$

ko'rinishdagi kvadratik funksionaldan iborat bo'lsin.

6- t e o r e m a.

$$p(x)\in C[x_0,x_1]q(x)\in C^1[x_0,x_1], r(x)\in C^1[x_0,x_1], r(x)>0(<0)\,\forall x\in[x_0,x_1]$$

bo'lsin. Agar Yakobi sharti bajarilmasa, u holda,

$$\inf_{y\in V} J[y]=-\infty(\sup_{u\in V} J[y]=+\infty),$$

ya'ni (37) funksionalning (2) to'plamda global ekstremumi mavjud emas.Agar kuchaytirilgan Yakobi sharti bajarilsa, (37) funksionalning yagona joyiz stasionar funksiyasi mavjud va bu stasionar funksiya funksionalning global minimum (maksimum) nuqtasi bo'ladi.

Misollar.

1-misol

$$\int_0^b (y'^2-y^2+2y\sin x)dx\to \min,\ y(0)=0,\ y(b)=1.$$

$$F(x,y,y')=y'^2-y^2+2\sin x,\ F_y=-2y+2\sin x,\ F_{y'}=2y'$$

Eyler tenglamasini yozamiz:

$$y''+y-\sin x=0$$

Bu tenglamaning umumiy yechimi $y=-\frac{1}{2}x\cos x+c_1\cos x+c_2\sin x$ ko'rinishda bo'ladi.

y(0)=0, y(b)=1

shartlardan foydalanib c_1 va c_2 o'zgarmaslarni topamiz:

$$c_1 = 0, \quad c_2 = \frac{1 + \frac{b}{2}\cos b}{\sin b}, b \neq \pi k, k = 1, 2, ...$$

Demak, $b \neq \pi k$, $k=1,2,..$ bo'lganda ,

$$y^0(x) = -\frac{1}{2}x\cos x - \frac{2 + b\cos b}{2\sin b}\sin x, x \in [0, b]$$

Agar $b=\pi k$, $k=1,2,..$ bo'lsa, masalada stasionar funksiyalar yo'q. Demak, bu holda ekstremum mavjud emas.

$F_{y'y'} = 2 > 0$,demak, kuchaytirilgan Lejandr sharti bajariladi.

Yakobi tenglamasini tuzamiz: $A(x)= F_{y'y'} = 2 > 0$,

$$B(x)\frac{d}{dx}F_{y'y'} = 0, C(x)\frac{d}{dx}F_{y'y'} - F_{yy} = 2.$$

$h''+h=0$ - Yakobi tenglamasi, $h(x) = \gamma_1 \cos x + \gamma_2 \sin x -$ uning umumiy yechimi. Demak, $x^*=\pi$ nuqta $x_0=0$ nuqtaga qo'shma nuqtadir. $(0,\pi)$ oraliqda $x_0=0$ ga qo'shma nuqta yo'q. Shunday qilib, $b\leq\pi$ bo'lganda Yakobi sharti, $b<\pi$ bo'lganda esa kuchaytirilgan Yakobi sharti bajariladi. $b>\pi$ bo'lganda esa Yakobi sharti bajarilmaydi. 5-teoremaga ko'ra, $0<b<\pi$ bo'lganda formula bilan aniqlanuvchi $y^0(x)$ funksiya masalada kuchli minimal bo'ladi. $F = F(y') = y'^3 - 3y'^2$ bo'lgani uchun Eyler tenglamasi $(y'-1)y'' = 0$ ko'rinishda bo'ladi. Bu yerdan $y=x+c$ va $y=c_1x+c_2$ ko'rinishdagi yechimlarga ega bo'lamiz. $y=x+c$ funksiya chegaraviy shartlarni qanoatlantirmaydi. $y=c_1x+c_2$ funksiya uchun $y(0)=y(2)=0$ chegaraviy shartlardan $c_1=c_2=0$ bo'lishi kelib chiqadi. Demak, $y^0(x)=0, x\in[0.2]$ qaralayotgan masalada yagona silliq joyiz stasionar fnksiyadir.

$F_{y'y'}(y^0(x)) = 6(y^{0\prime}(x) - 1) = -6,$ ya'ni kuchaytirilgan Lejandr sharti bajariladi. Endi Yakobi tenglamasini tuzamiz:

$$A(x) = F_{y'y'}|_{y=y^0} = -6, \; B(x) = \frac{d}{dx}F_{y'y'}|_{y=y^0} = 0, \; C(x) = \frac{d}{dx}F_{y'y'} - F_{yy} = 0.$$

$h'' = 0$ - Yakobi tenglamasi, $h(x) = \gamma_1 x + \gamma_2 -$ uning umumiy yechimidir. $h(0) = 0, h(x^*) = 0, x^* > 0$ shartlardan $h(x)\equiv 0$ bo'lishi kelib chiqadi. Demak, $[0,2]$ da $x_0=0$ nuqtaga qo'shma nuqta mavjud emas, ya'ni kuchaytirilgan Yakobi sharti bajariladi. Demak, 3-teoremaga ko'ra , $y^0(x) \equiv 0$ kuchsiz lokal maksimal bo'ladi.

Ammo $y^0(x) \equiv 0$ uchun Veyershtrass sharti bajarilmaydi, ya'ni

$$E(x, y^0, y^{0\prime}, u) = F(x, y^0, u) - F(x, y^0, y^{0\prime}) - (u - y^0)F_{y'}(x, y^0, y^{0\prime}) = u^3 - 3u^2$$

funksiya ishorasini o'zgartiradi. 4-teoremaga asosan, $y^0(x)\equiv 0$- kuchli lokal maksimal bo'la olmaydi.

2-misol $J(y) = \int_{-1}^{1}(y'^3 + 5xyy' - x^2 y)dx$ funksionalning ikkinchi variatsiyasini hisoblang.

Yechilishi: Funksionalning ikkinchi variatsiyasi

$$\delta^2 J(y,h) = \int_a^b [F_{yy}h^2(x) + 2F_{yy'}h(x)h'(x) + F_{y'y'}h'^2(x)]dx$$

formula yordamida topiladi. Ikkinchi variatsiyani topish uchun integral ostidagi funksiя $F = y'^3 + 5xyy' - x^2 y$ dan y bo'yicha ikki marta xususiy hosila olamiz va bu hosilalar $F_y = 5xy' - x^2$, $F_{yy} = 0$ larga teng bo'ladi. Bundan keyin y bo'yicha olingan xususiy hosiladan y' bo'yicha xususiy hosila olib, $F_{yy'} = 5x$ ga ega bo'lamiz. Endi integral ostidagi funksiyadan y' bo'yicha ikki marta xususiy hosila olib, $F_{y'} = 3y'^2 + 5xy$, $F_{y'y'} = 6y'$ ega bo'lamiz. Natijada berilgan funksionalning ikkinchi variatsiyasi quyidagi ko'rinishda bo'ladi:

$$\delta^2 J(y,h) = \int_{-1}^{1} [10xh(x)h'(x) + 6y'h'^2(x)]dx.$$

3-misol $J(y) = \int_0^3 \left(y'^2 - 2y'x\right)dx \to \min;\ y(0) = 0,\ y(3) = 1,\ y \in C^{(1)}[0;3]$ masalaning joiz statsionar funksiyasida Lejandr shartining bajarilishini ko'rsating.

Yechilishi: Eyler tenglamasini tuzamiz: $F_y - \frac{d}{dx}F_{y'} = 0$. Integral ostidagi funksiya $F = y'^2 - 2y'x$ dan y va y' bo'yicha xususiy hosilalar olib, $F_y = 0$ va $F_{y'} = 2y' - 2x$ larga ega bo'lamiz. $F_{y'} = 2y' - 2x$ funksiyadan x bo'yicha to'la hosila olib, $\frac{d}{dx}F_{y'} = 2y'' - 2$ ega bo'lamiz. Bu funksiyalarni yuqoridagi tenglamaga keltirib qo'yib,

$$2y'' - 2 = 0 \text{ yoki } y'' = 1$$

Eyler tenglamasiga kelamiz. Bu tenglamaning ikkala tomonini x bo'yicha ikki marta integrallab, Eyler tenglamasining quyidagi:

$$y' = x + c_1,$$

$$y = \frac{x^2}{2} + c_1 x + c_2$$

umumiy yechimiga ega bo'lamiz. Endi yuqoridagi chegaraviy shartlardan foydalanib, c_1 va c_2 larni topamiz:

$$\begin{cases} y(0) = \frac{0^2}{2} + c_1 0 + c_2 = 0 \\ y(3) = \frac{3^2}{2} + 3c_1 + c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_2 = 0 \\ 3c_1 = 1 - \frac{9}{2} \end{cases} \Rightarrow \begin{cases} c_2 = 0 \\ 3c_1 = -\frac{7}{2} \end{cases} \Rightarrow \begin{cases} c_2 = 0 \\ c_1 = -\frac{7}{6} \end{cases}$$

Demak, joiz statsionar funksiya $y = \frac{x^2}{2} - \frac{7}{6}x$ ko'rinishda bo'ladi.

Endi $F_{y'} = 2y' - 2x$ funksiyadan y' bo'yicha xususiy hosila olib, $F_{y'y'} = 2$ ifodaga ega bo'lamiz. $F_{y'y'} = 2 > 0$ bo'lgani uchun kuchaytirilgan Lejandr sharti bajariladi (kuchsiz lokal minimum uchun).

4-misol. $J(y) = \int_0^1 \left(y'^2 + 9y^2 + y'e^{4x}\right)dx \to \min;\ y(0) = 0,\ y(1) = 1,\ y \in C^{(1)}[0;1]$ masalada $y^0(x) = x$ joiz funksiyalar bo'ylab Yakobi tenglamasini tuzing.

Yechilishi: Yakobi tenglamasi $A(x)h''+B(x)h'+C(x)h=0$ ko'rinishda bo'lib, bu yerda $A(x)=F_{y'y'}\big|_{y=y_0}$, $B(x)=\frac{d}{dx}F_{y'y'}\big|_{y=y_0}$, $C(x)=\frac{d}{dx}F_{yy'}-F_{yy}\big|_{y=y_0}$. Bu tenglamani tuzish uchun integral ostidagi funksiya $F=y'^2+9y^2+y'e^{4x}$ dan y va y' bo'yicha xususiy hosilalar olamiz va bu hosilalar $F_y=18y,\quad F_{y'}=2y'+e^{4x}$ lardan iborat bo'ladi. Bundan keyin $F_y=18y$ dan y va y' bo'yicha xususiy hosilalar olamiz: $F_{yy}\big|_{y=y_0}=18$, $F_{yy'}=0$. Bundan $\frac{d}{dx}F_{yy'}\big|_{y=y_0}=0$. Endi $F_{y'}$ dan y' bo'yicha xususiy hosila olib, $F_{y'y'}=2$ ga ega bo'lamiz. Bu funksiyadan x bo'yicha to'la hosila olsak, $\frac{d}{dx}F_{y'y'}\big|_{y=y_0}=0$ bo'ladi. Natijada, $A(x)=2,\quad B(x)=0,\quad C(x)=-18$ va berilgan masala uchun y_0 bo'ylab Yakobi tenglamasi,

$$2h''-18h=0$$

$$h''-9h=0$$

ko'rinishda bo'ladi.

<u>**5-misol.**</u> $J(y)=\int_0^1 (y'^2+4y^2-3y'x^2)dx\to\min(\max),\quad y(0)=0,\ y(1)=1$ masalada kuchsiz lokal ekstremalni toping.

Yechilishi: 1) Avvalo joiz statsionar funksiyani, ya'ni

$$F_y-\frac{d}{dx}F_{y'}=0$$

Eyler tenglamasining masaladagi chegaraviy shartlarni qanoatlantiruvchi xususiy yechimini topamiz.

$$F=y'^2+4y^2-3y'x^2$$

bo'lgani uchun, $F_y=8y,\quad F_{y'}=2y'-3x^2,\quad \frac{d}{dx}F_{y'}=2y''-6x$ va Eyler tenglamasi

$$8y-2y''+6x=0$$

yoki

$$y''-4y=3x$$

ko'rinishga ega bo'ladi. Uning umumiy yechimini $y=y_1+\tilde{y}$ ko'rinishda izlaymiz, bunda y_1 - mos bir jinsli tenglamaning umumiy yechimi, $\tilde{y}$ esa qaralayotgan tenglamaning birorta xususiy yechimi. Endi

$$y_1''-4y_1=0.$$

bir jinsli tenglamani qaraymiz.Bu tenglamaning yechimini $y_1=e^{\lambda x}$ ko'rinishda izlaymiz:

$$\lambda^2e^{\lambda x}-4e^{\lambda x}=0,$$

$$\lambda^2-4=0$$

$$\lambda_{1,2}=\pm 2.$$

Demak, $y_1(x)=c_1e^{2x}+c_2e^{-2x}$ bo'ladi. $\tilde{y}$ ni $\tilde{y}=ax+b$ ko'rinishda izlaymiz. $\tilde{y}''=0$

$$0-4(ax+b)=3x.$$

Bundan $a=-\frac{3}{4}, \quad b=0.$ Demak, $\tilde{y}=-\frac{3}{4}x$ va Eyler tenglamasining umumiy yechimi

$$y(x)=c_1e^{2x}+c_2e^{-2x}-\frac{3}{4}x$$

bo'ladi. $y(0)=0, \quad y(1)=1$ shartlarga ko'ra,

$$\begin{cases} c_1+c_2=0 \\ c_1e^2+c_2e^{-2}-\frac{3}{4}=1 \end{cases} \Rightarrow \begin{cases} c_1=-c_2 \\ c_1e^2-c_1e^{-2}-\frac{3}{4}=1 \end{cases} \Rightarrow \begin{cases} c_1=-c_2 \\ c_1(e^2-e^{-2})=\frac{7}{4} \end{cases}$$

$$\Rightarrow \begin{cases} c_1=-c_2 \\ c_1=\frac{7}{4(e^2-e^{-2})} \end{cases} \Rightarrow \begin{cases} c_2=-\frac{7}{4(e^2-e^{-2})} \\ c_1=\frac{7}{4(e^2-e^{-2})} \end{cases}$$

Demak, $y^0(x)=\frac{7\left(e^{2x}-e^{-2x}\right)}{4\left(e^2-e^{-2}\right)}-\frac{3}{4}x$ joiz statsionar funksiya bo'ladi.

2) $y^0(x)$ bo'ylab Lejandr shartini tekshiramiz: $F_{y'y'}\Big|_{y=y^0}=2>0, \quad x\in[0;1].$

Demak, $y^0(x)$ bo'ylab kuchsiz minimum uchun kuchaytirilgan Lejandr sharti bajariladi.

3) Endi $y^0(x)$ bo'ylab Yakobi shartining bajarilishini tekshiramiz. Buning uchun, oldin $A(x)h''+B(x)h'+C(x)h=0$ Yakobi tenglamasini tuzamiz:

$$A(x)=F_{y'y'}\Big|_{y=y^0}=2, \quad B(x)=\frac{d}{dx}\left(F_{y'y'}\right)\Big|_{y=y^0}=0.$$

$$C(x)=\left(\frac{d}{dx}F_{yy'}-F_{yy}\right)\Big|_{y=y^0}=-8.$$

Natijada Yakobi tenglamasi

$$2h''-8h=0, \quad h''-4h=0.$$

ko'rinishda bo'ladi. Uning yechimi $h=e^{\mu x}$ ko'rinishda izlanadi va $h''=\mu^2e^{\mu x}.$ U holda

$$\mu^2e^{\mu x}-4e^{\mu x}=0$$

$$\mu^2-4=0$$

$$\mu=\pm 2.$$

Shunday qilib, Yakobi tenglamasining umumiy yechimi

$$h(x)=d_1e^{2x}+d_2e^{-2x}$$

ko'rinishga ega.Boshlang'ich shartdan, $d_1\neq 0$ bo'lganda

$$h(0)=0\Rightarrow \ d_1+d_2=0\Rightarrow d_2=-d_1\Rightarrow h(x)=d_1\left(e^{2x}-e^{-2x}\right)\neq 0,$$

bo'lishi kelib chiqadi.

Agar $h\left(x^*\right)=0$ desak, u holda

$$e^{2x_*}-e^{-2x_*}=0, \ e^{2x_*}=e^{-2x_*}, \ x_*=0.$$

Demak, $y^0(x)$ bo'ylab $(0,1]$ kesmada nol nuqtaga qo'shma nuqta yo'q, ya'ni kuchaytirilgan Yakobi sharti bajariladi.

Shunday qilib, $y^0(x)$ - kuchsiz lokal minimal bo'ladi.

Mustaqil ishlash uchun misollar

1 – masala. Quyidagi funksionalning ikkinchi variatsiyasini hisoblang.

1. $J(y)=\int_0^1 (y'^2+3y^2-y'(y+x^2))dx$ 2. $J(y)=\int_{-1}^1 (y'^3-6xyy')dx$

3. $J(y)=\int_\pi^{2\pi} (y'^2+3y(y'+2yCosx))dx$ 4. $J(y)=\int_0^1 (y'^2+y^2-2y'(y+x^3))dx$

5. $J(y)=\int_{-1}^1 (y'^3+3x^2yy')dx$ 6. $J(y)=\int_\pi^{2\pi} (y'^2-y(y'+5yCos2x))dx$

7. $J(y)=\int_0^1 (y'^2-4y^2-y(y'+2x^2))dx$ 8. $J(y)=\int_{-1}^1 (y'^3-2xyy'+x^3)dx$

9. $J(y)=\int_\pi^{2\pi} (y'^2+2y(y'-y\sin x))dx$ 10. $J(y)=\int_0^1 (y'^2-3y^2-y(2y'+x^3))dx$

11. $J(y)=\int_{-1}^1 (y'^3+3xyy'-ye^x)dx$ 12. $J(y)=\int_\pi^{2\pi} (y'^2-5y(y'-2y\sin 2x))dx$

13. $J(y)=\int_0^1 (y'^2-5y^2+2y'(y+x^4))dx$ 14. $J(y)=\int_{-1}^1 (y'^3-4xyy'+y'x^2)dx$

15. $J(y)=\int_\pi^{2\pi} (y'^2+3y(y'-4yCos2x))dx$ 16. $J(y)=\int_0^1 (y'^2+3y^2+2y(y'-x^4))dx$

17. $J(y)=\int_{-1}^1 (y'^3-3xyy'+4x^3)dx$ 18. $J(y)=\int_\pi^{2\pi} (y'^2+4y(y'+3y\sin 2x))dx$

19. $J(y)=\int_0^1 (y'^2-y^2-yy'(1+x^2))dx$ 20. $J(y)=\int_{-1}^1 (y'^3+2xyy'-y^2)dx$

2 – masala. Quyidagi variatsion masalaning kuchsiz lokal ekstremalida Lejandr shartining bajarilishini ko'rsating.

1. $J(y)=\int_1^3 (y'^2-3y'Cosx)dx\to\min;\ y(1)=0,\ y(3)=1,\ y\in C^{(1)}[1;3]$

2. $J(y)=\int_0^1 (y'^2-7xy)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

3. $J(y)=\int_0^2 (y'^2+5yy')dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

4. $J(y)=\int_1^3 (y'^2-5y'Cosx)dx\to\min;\ y(1)=0,\ y(3)=1,\ y\in C^{(1)}[1;3]$

5. $J(y)=\int_0^1 (y'^2+2x^2y)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

6. $J(y)=\int_0^2 (y'^2+3xy')dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

7. $J(y)=\int_1^3 (y'^2+y'Cosx+y)dx\to\min;\ y(1)=0,\ y(3)=1,\ y\in C^{(1)}[1;3]$

8. $J(y)=\int_{-1}^{1}\left(y'^2-6y^2+y'x^2\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

9. $J(y)=\int_{0}^{1}\left(y'^2-4ye^x+x^3\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

10. $J(y)=\int_{0}^{2}(y'^2+3yy'-y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}(0;2)$

11. $J(y)=\int_{-1}^{1}\left(y'^2+y^2-4yx^2\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

12. $J(y)=\int_{0}^{1}\left(y'^2-ye^x+2x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

13. $J(y)=\int_{0}^{2}(y'^2-6yy'+3xy)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}(0;2)$

14. $J(y)=\int_{-1}^{1}\left(y'^2-2xy+3y\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

15. $J(y)=\int_{0}^{1}\left(y'^2+3y^2+y'e^{4x}\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

16. $J(y)=\int_{-1}^{1}\left(y'^2-4yy'+9y^2-yx^3\right)dx\to\min(\max),\ y(-1)=0,\ y(1)=-1,\ y\in C^{(1)}[-1;1]$

17. $J(y)=\int_{\pi}^{2\pi}\left(y'^2+2y'(y-\cos x)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=1,\ y\in C^{(1)}[\pi;2\pi]$

18. $J(y)=\int_{0}^{1}\left(y'^2-4y^2-y'e^{2x}\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

19. $J(y)=\int_{-1}^{1}\left(y'^2+3yy'+16y^2-2yx^3\right)dx\to\min(\max),\ y(-1)=0,\ y(1)=-1,\ y\in C^{(1)}[-1;1]$

20. $J(y)=\int_{\pi}^{2\pi}\left(y'^2+3y'(y+Sin2x)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=1,\ y\in C^{(1)}[\pi;2\pi]$

21. $J(y)=\int_{0}^{1}\left(y'^2-6y^2+y'e^x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

22. $J(y)=\int_{-1}^{1}\left(y'^2-5yy'-4y^2-yx^3\right)dx\to\min(\max),\ y(-1)=0,\ y(1)=-1,\ y\in C^{(1)}[-1;1]$

23. $J(y)=\int_{\pi}^{2\pi}\left(y'^2-4y(y'-\cos 2x)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=1,\ y\in C^{(1)}[\pi;2\pi]$

24. $J(y)=\int_{0}^{1}\left(y'^2-y^2+2y'e^{3x}\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

25. $J(y)=\int_{-1}^{1}\left(y'^2-6yy'+4y^2+3yx^3\right)dx\to\min(\max),\ y(-1)=0,\ y(1)=-1,\ y\in C^{(1)}[-1;1]$

26. $J(y)=\int_0^2\left(y'^2+8xyy'\right)dx\to\min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

$J(y)=\int_1^3\left(y'^2-2y'Cosx\right)dx\to\min;\ y(1)=0,\ y(3)=1,\ y\in C^{(1)}[1;3]$

27. $J(y)=\int_0^1\left(y'^2+12xy\right)dx\to\min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

28. $J(y)=\int_0^2(y'^2-5yy'+4y^2)dx\to\min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}(0;2)$

29. $J(y)=\int_{-1}^1\left(y'^2-9y^2+3yx^2\right)dx\to\min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

30. $J(y)=\int_0^1\left(y'^2-5ye^x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

<u>3 – masala.</u> Quyidagi variatsion masalaning $y^0(x)=x$ joiz funksiyalar bo'ylab Yakobi tenglamasini tuzing.

1. $J(y)=\int_\pi^{2\pi}\left(y'^2+9y^2+3y'(y+Sinx)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

2. $J(y)=\int_0^1\left(y'^2+2y'x^2\right)dx\to\min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

3. $J(y)=\int_0^1\left(y'^2+y^2+4y(y'-x^3)\right)dx\to\min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

4. $J(y)=\int_\pi^{2\pi}\left(y'^2+y^2-y'(y+\cos x)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

5. $J(y)=\int_0^1\left(y'^2+4y'x^2\right)dx\to\min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

6. $J(y)=\int_0^1\left(y'^2+9y^2-3y(y'-x^3)\right)dx\to\min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

7. $J(y)=\int_\pi^{2\pi}\left(y'^2+4y^2-5y'(y-\cos x\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

8. $J(y)=\int_0^1\left(y'^2-5y'x^2\right)dx\to\min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

9. $J(y)=\int_0^1\left(y'^2-16y^2+2y(y'+x^3)\right)dx\to\min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

10. $J(y)=\int_\pi^{2\pi}\left(y'^2+9y^2-4y'(y-\cos x)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

11. $J(y)=\int_0^1\left(y'^2-9y^2+2y'e^x\right)dx\to\min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

12. $J(y)=\int_{-1}^{1}\left(y'^2-3yy'+4y^2-yx^3\right)dx\to \min(\max),\ y(-1)=0, y(1)=-1, y\in C^{(1)}[-1;1]$

13. $J(y)=\int_{\pi}^{2\pi}\left(y'^2-2y'(y+Sinx)\right)dx\to \min,\ y(\pi)=-1, y(2\pi)=1, y\in C^{(1)}[\pi;2\pi]$

14. $J(y)=\int_{0}^{1}\left(y'^2+y^2+y'e^{2x}\right)dx\to \min;\ y(0)=0, y(1)=1, y\in C^{(1)}[0;1]$

15. $J(y)=\int_{-1}^{1}\left(y'^2+yy'+4y^2-2yx^3\right)dx\to \min(\max),\ y(-1)=0, y(1)=-1, y\in C^{(1)}[-1;1]$

16. $J(y)=\int_{\pi}^{2\pi}\left(y'^2+4y'(y-Sinx)\right)dx\to \min,\ y(\pi)=-1, y(2\pi)=1, y\in C^{(1)}[\pi;2\pi]$

17. $J(y)=\int_{0}^{1}\left(y'^2+9y^2+y'e^{3x}\right)dx\to \min;\ y(0)=0, y(1)=1, y\in C^{(1)}[0;1]$

18. $J(y)=\int_{-1}^{1}\left(y'^2-2yy'+y^2-3yx^3\right)dx\to \min(\max),\ y(-1)=0, y(1)=-1, y\in C^{(1)}[-1;1]$

19. $J(y)=\int_{\pi}^{2\pi}\left(y'^2-3y'(y+\cos x)\right)dx\to \min,\ y(\pi)=-1, y(2\pi)=1, y\in C^{(1)}[\pi;2\pi]$

20. $J(y)=\int_{0}^{1}\left(y'^2+3y^2+y'e^{4x}\right)dx\to \min;\ y(0)=0, y(1)=1, y\in C^{(1)}[0;1]$

21. $J(y)=\int_{-1}^{1}\left(y'^2-4yy'+9y^2-yx^3\right)dx\to \min(\max),\ y(-1)=0, y(1)=-1, y\in C^{(1)}[-1;1]$

22. $J(y)=\int_{\pi}^{2\pi}\left(y'^2+2y'(y-\cos x)\right)dx\to \min,\ y(\pi)=-1, y(2\pi)=1, y\in C^{(1)}[\pi;2\pi]$

23. $J(y)=\int_{0}^{1}\left(y'^2-4y^2-y'e^{2x}\right)dx\to \min;\ y(0)=0, y(1)=1, y\in C^{(1)}[0;1]$

24. $J(y)=\int_{-1}^{1}\left(y'^2+3yy'+16y^2-2yx^3\right)dx\to \min(\max),\ y(-1)=0, y(1)=-1, y\in C^{(1)}[-1;1]$

25. $J(y)=\int_{\pi}^{2\pi}\left(y'^2+3y'(y+Sin2x)\right)dx\to \min,\ y(\pi)=-1, y(2\pi)=1, y\in C^{(1)}[\pi;2\pi]$

26. $J(y)=\int_{0}^{1}\left(y'^2-6y^2+y'e^{x}\right)dx\to \min;\ y(0)=0, y(1)=1, y\in C^{(1)}[0;1]$

27. $J(y)=\int_{-1}^{1}\left(y'^2-5yy'-4y^2-yx^3\right)dx\to \min(\max),\ y(-1)=0, y(1)=-1, y\in C^{(1)}[-1;1]$

28. $J(y)=\int_{\pi}^{2\pi}\left(y'^2-4y(y'-\cos 2x)\right)dx\to \min,\ y(\pi)=-1, y(2\pi)=1, y\in C^{(1)}[\pi;2\pi]$

29. $J(y)=\int_{0}^{1}\left(y'^2-y^2+2y'e^{3x}\right)dx\to \min;\ y(0)=0, y(1)=1, y\in C^{(1)}[0;1]$

30. $J(y)=\int_{-1}^{1}\left(y'^2-6yy'+4y^2+3yx^3\right)dx\to \min(\max),\ y(-1)=0, y(1)=-1, y\in C^{(1)}[-1;1]$

4 – masala. Quyidagi variatsion masalada kuchsiz lokal ekstremalni toping.

1. $J(y)=\int_0^1\left(y'^2-4y^2+2y(y'-x^3)\right)dx\to\min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

2. $J(y)=\int_\pi^{2\pi}\left(y'^2+y^2-3y'(y+Sinx)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

3. $J(y)=\int_0^1\left(y'^2-6y'x^2\right)dx\to\min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

4. $J(y)=\int_0^1\left(y'^2-y^2-3y(y'-x^3)\right)dx\to\min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

5. $J(y)=\int_\pi^{2\pi}\left(y'^2+4y^2-y'(y+Sinx)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

6. $J(y)=\int_0^1\left(y'^2+y'x^2\right)dx\to\min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

7. $J(y)=\int_0^1\left(y'^2-9y^2+y(y'-x^3)\right)dx\to\min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

8. $J(y)=\int_\pi^{2\pi}\left(y'^2+16y^2-5y'(y+Sinx)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

9. $J(y)=\int_0^1\left(y'^2-3y'x^2\right)dx\to\min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

10. $J(y)=\int_0^1\left(y'^2+4y^2-2y(y'+x^3)\right)dx\to\min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

11. $J(y)=\int_\pi^{2\pi}\left(y'^2+9y^2+3y'(y+Sinx)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

12. $J(y)=\int_0^1\left(y'^2+2y'x^2\right)dx\to\min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

13. $J(y)=\int_0^1\left(y'^2+y^2+4y(y'-x^3)\right)dx\to\min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

14. $J(y)=\int_\pi^{2\pi}\left(y'^2+y^2-y'(y+\cos x)\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

15. $J(y)=\int_0^1\left(y'^2+4y'x^2\right)dx\to\min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

16. $J(y)=\int_0^1\left(y'^2+9y^2-3y(y'-x^3)\right)dx\to\min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

17. $J(y)=\int_\pi^{2\pi}\left(y'^2+4y^2-5y'(y-\cos x\right)dx\to\min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

18. $J(y)=\int_0^1\left(y'^2-5y'x^2\right)dx\to\min,\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

19. $J(y)=\int_0^1 \left(y'^2-16y^2+2y(y'+x^3)\right)dx \to \min,\ y(0)=1,\ y(1)=0,\ y\in C^{(1)}[0;1]$

20. $J(y)=\int_\pi^{2\pi} \left(y'^2+9y^2-4y'(y-\cos x)\right)dx \to \min,\ y(\pi)=-1,\ y(2\pi)=0,\ y\in C^{(1)}[\pi;2\pi]$

21. $J(y)=\int_0^1 \left(y'^2+2x^2y\right)dx \to \min;\ y(0)=0,\ y(1)=2,\ y\in C^{(1)}[0;1]$

22. $J(y)=\int_0^2 \left(y'^2+3xy'\right)dx \to \min;\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}[0;2]$

23. $J(y)=\int_1^3 \left(y'^2+y'Cosx+y\right)dx \to \min;\ y(1)=0,\ y(3)=1,\ y\in C^{(1)}[1;3]$

24. $J(y)=\int_{-1}^1 \left(y'^2-6y^2+y'x^2\right)dx \to \min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

25. $J(y)=\int_0^1 \left(y'^2-4ye^x+x^3\right)dx \to \min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

26. $J(y)=\int_0^2 (y'^2+3yy'-y^2)dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}(0;2)$

27. $J(y)=\int_{-1}^1 \left(y'^2+y^2-4yx^2\right)dx \to \min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

28. $J(y)=\int_0^1 \left(y'^2-ye^x+2x\right)dx \to \min;\ y(0)=0,\ y(1)=1,\ y\in C^{(1)}[0;1]$

29. $J(y)=\int_0^2 (y'^2-6yy'+3xy)dx \to \min,\ y(0)=0,\ y(2)=1,\ y\in C^{(1)}(0;2)$

30. $J(y)=\int_{-1}^1 \left(y'^2-2xy+3y\right)dx \to \min;\ y(-1)=0,\ y(1)=1,\ y\in C^{(1)}[-1;1]$

4-amaliy mashg‘ulot. Variatsion hisobning asosiy masalasining umumlashmalari. Eyler tenglamalari sistemasi, Eyler-Puasson, Eyler-Ostrogradskiy tenglamalari

2. Bir necha funksiyalarga bog’liq bo’lgan funksionalning ekstremumi.

Variatsion hisob asosiy masalasining umumlashmasi sifatida, dastlab, bir necha, $y_1=y_1(x),...,y_n=y_n(x)$ funksiyalarga bog’liq funksionalning ekstremumi haqidagi masalani qaraymiz.

$$J[y_1,...,y_n]=\int_{x_0}^{x_1} F(x,y_1...,y_n,y_1'...,y_n')dx \to \min(\max) \qquad (1)$$

$$\left.\begin{array}{c} y_1(x_0)=y_{01}, y_2(x_0)=y_{02},...,y_n(x_0)=y_{0n}, \\ y_1(x_1)=y_{11}, y_2(x_1)=y_{12},...,y_n(x_1)=y_{2n}, \\ (x, y_1(x),...,y_n(x), y_1'(x),...,y_n'(x))\in Q, x\in[x_0,x_1] \\ y_i(x)\in C^1[x_0,x_1], i=1,2,...,n. \end{array}\right\} \qquad (2)$$

ekstremal masalani qaraymiz.

Quyidagi belgilashlarni kiritamiz: $y=(y_1,...,y_n)$, $y'=(y_1',...,y_n')$, $y_0=(y_{01},...,y_{0n})$, $C_n^{(1)}[x_0,x_1]-$ $[x_0,x_1]$ kesmada uzluksiz differensiallanuvchi $y(x)=(y_1(x),...,y_n(x))-n-$ vektor funksiyalar fazosi. U holda (1), (2) masalani

$$J[y]=\int_{x_0}^{x_1}F(x,y,y')dx\to\min(\max), \qquad (1^/)$$

$$y(x_0)=x_0, y(x_1)=x_1,\ y(x)\in C_n^{(1)}[x_0,x_1] \qquad (2^/)$$

<u>1-t a’ r i f.</u> Agar $y^0(x)$ joiz funksiyaning shunday $V_0(y^0,\varepsilon)$ nolinchi tartibli $\varepsilon-$ atrofiga tegishli barcha $y=y(x)$ joiz funksiyalar uchun

$$J[y^0]\le J[y] \quad (J[y^0]\ge J[y]) \qquad (3)$$

munosabat bajarilsa, $y^0(x)$ funksiya (1) funksionalning kuchli lokal minimum (maksimum) nuqtasi deyiladi.

2-t a’ r i f. Agar (3) munosabat $y^0(x)$ joiz funksiyaning biror $V_1(y^0,\varepsilon)$ birinchi tartibli ε- atrofiga tegishli $y=y(x)$ joiz funksiyalar uchun bajarilsa, $y^0(x)-$ (1) funksionalning *kuchsiz lokal minimum (maksimum)* nuqtasi deyiladi.

<u>1-teorema.</u> $F(x,y,y')\in C^1(Q)$ bo’lsin. Agar (1) funksional $y^0(x)=(y_1^0(x),...,y_n^0(x))\in C_n^{(1)}[x_0,x_1]$ joyiz funksiyada kuchsiz lokal ekstremumga erishsa, $[x_0,x_1]$ kesmada

$$F_{y_i}\left(x,y^0(x),y^{0/}(x)\right)-\frac{d}{dx}F_{y_i'}\left(x,y^0(x),y^{0/}(x)\right)=,\ i=\overline{1,n} \qquad (4)$$

tengliklar bajariladi

<u>3-t a’ r i f.</u> Eyler tenglamalar sistemasini qanoatlantiruvchi $y(x)=(y_1(x),...,y_n(x))$ joyiz funksiyalarga (1) funksionalning *stasionar funksiyalari* deyiladi.

<u>2-t e o r e m a.</u> $F(x,y,y')\in C^{(2)}(Q)$ bo’lsin. Agar (1) funksional $y^0(x)\in C_n^1[x_0,x_1]$ joyiz funksiyada kuchsiz lokal minimum (maksimum) ga erishsa, quyidagi:

$$F^0_{y'y'}(x)\ge 0\ \ (\le 0),\ \forall x\in[x_0,x_1] \qquad (10)$$

Lejandr sharti bajariladi.

$F(x,y,y')\in C^{(3)}(Q),\ \ y^0(x)\in C_n^2[x_0,x_1]$ deb hisoblab, shu kvadratik funksional uchun Eyler tenglamalari sistemasini yozamiz:

$$\sum_{i,j=1}^{n}F^0_{y_iy_j}(x)h_j+\sum_{j=1}^{n}F^0_{y_iy_j'}(x)h_j'-\frac{d}{dx}\left[\sum_{j=1}^{n}F^0_{y_i'y_j'}h_j'+\sum_{j=1}^{n}F^0_{y_i'y_j}(x)h_j\right]=0,\ \ i=\overline{1,n}. \ (11)$$

(11)–(1), (2) masala uchun *Yakobi tenglamalari sistemasi* deyiladi.

4-t a' r i f. Agar (11) sistema $h_i(x_0)=h_i(x_*)=0,\ i=\overline{1,n},$ shartlarni qanoatlantiruvchi trivial (aynan nol) bo'lmagan yechimga ega bo'lsa, x_* nuqtaga $y^0(x)$ joyiz chiziq bo'ylab x_0 nuqtaga qo'shma nuqta deyiladi.

4-t e o r e m a. Quyidagi shartlar bajarilsin:

1) $F(x,y,y')\in C^{(3)}(Q)$; 2) $y^0(x)\in C_n^2[x_0,x_1]$ – joyiz stasionar funksiya; 3) Kuchaytirilgan Lejandr sharti: $F^0_{y'y'}(x)>0\ \ (<0)\ \forall x\in[x_0,x_1]$; 4) Kuchaytirilgan Yakobi sharti: $(x_0,x_1]$ intervalda x_0 nuqta bilan qo'shma bo'lgan x_* nuqta mavjud emas. U holda (1) funksional $y^0(x)$da kuchsiz lokal minimum (maksimum)ga erishadi.

5-t e o r e m a. $p_{ij}(x)\in C[x_0,x_1],\ q_{ij}(x)\in C^{(1)}[x_0,x_1],\ r_{ij}(x)\in C^{(1)}[x_0,x_1]\ ,\ \ j,i=\overline{1,n}$ $r(x)=\left(r_{ij}(x),\ i,j=\overline{1,n}\right)>0\ \ (<0)\ \ \forall x\in[x_0,x_1]$ bo'lssin. Agar Yakobi sharti bajarilmasa, $\inf J(x)=-\infty\left(\sup J(x)=+\infty\right)$, ya'ni masala yechimga ega emas. Agar kuchaytirilgan Yakobi sharti bajarilsa, yagona stasionar funksiya mavjud va bu funksiya (12) funksional uchun global minimal (maksimal) bo'ladi.

M i s o l. $J[y_1,y_2]=\int\limits_0^{\frac{\pi}{2}}\left[y'^2_1+y'^2_2+2y_1y_2\right]dx\to\min(\max),$

$$y_1(0)=0,\ \ y_2(0)=0,$$

$$y_1\left(\frac{\pi}{2}\right)=1,\ \ y_2\left(\frac{\pi}{2}\right)=-1.$$

Bu masalada $F=y'^2_1+y'^2_2+2y_1y_2,\ F_{y_1}=2y_2,\ F_{y_2}=2y_1,\ F_{y_1'}=2y_1',\ F_{y_2'}=2y_2'$ va Eyler tenglamalar sistemasi ikkita tenglamadan iborat bo'ladi:

$$\left.\begin{array}{l} F_{y_1}-\dfrac{d}{dx}F_{y_1'}=0\Leftrightarrow y_1''-y_2=0,\\ F_{y_2}-\dfrac{d}{dx}F_{y_2'}=0\Leftrightarrow y_2''-y_1=0 \end{array}\right\}$$

bu yerdan

$$y_2=y_1'',\ \ y_1^{IV}-y_1=0$$

sistemaga kelamiz. Hosil qilingan sistemaning umumiy yechimi

$$y_1=c_1e^x+c_2e^{-x}+c_3\cos x+c_4\sin x,$$

$$y_2=c_1e^x+c_2e^{-x}-c_3\cos x-c_4\sin x$$

ko'rinishda bo'ladi. Chegaraviy shartlardan foydalanib, $C_1=0, C_2=0,\ C_3=0,\ C_4=1$ ekanligini topamiz. Demak, qaralayotgan masalada joyiz stasionar funksiya

$$y_1^0(x)=\sin x,\ \ y_2^0(x)=-\sin x$$

bo'ladi. Lejandr sharti kuchaytirilgan shaklda bajariladi:

$$F_{y'y'}=\begin{pmatrix} F_{y_1'y_1'} & F_{y_1'y_2'}\\ F_{y_2'y_1'} & F_{y_2'y_2'} \end{pmatrix}>0 \qquad (13)$$

Yakobe tenglamalar sistemasi esa Eyler tenglamalar sistemasi kabi bo'ladi:

$$h_1''-h_2=0,\ \ h_2''-h_1=0.$$

Uning umumiy yechimini yozamiz:

$$h_1=\gamma_1e^x+\gamma_2e^{-x}+\gamma_3\cos x+\gamma_4\sin x,$$

$$h_2=\gamma_1e^x+\gamma_2e^{-x}-\gamma_3\cos x-\gamma_4\sin x$$

$h_1(0)=0,\ h_2(x_*)=0,\ 0<x_*\le\frac{\pi}{2}$ chegaraviy shartlardan $\gamma_1=\gamma_2=\gamma_3=\gamma_4=0$, ya'ni $h_1(x)=0,\ h_2(0)=0$ bo'lishi kelib chiqadi. Demak, Yakobi sharti kuchaytirilgan shaklda bajariladi. Shunday qilib, kuchsiz lokal minimumning yetarli shartlari bajariladi. Bundan tashqari, (13) shart $F^0_{y'y'}(x,y,y')>0\quad\forall(x,y,y')\in R^5$ ekanligini bildirganligidan, 4-teoremaga ko'ra $y^0(x)=(\sin x,-\sin x)$ joyiz funksiya kuchli minimal ham bo'ladi.

2. Yuqori tartibli hosilalarga bog'liq bo'lgan funksionalning ekstremumi.

Quyidagi

$$J[y]=\int_{x_0}^{x_1}F\left(x,y,y',...,y^{(n)}\right)dx\to\min(\max),\tag{14}$$

$$\left.\begin{aligned}y(x_0)=y_{00},\ y'(x_0)=y_{01},...,\ y^{(n-1)}(x_0)=y_{0.n-1},\\ y(x_0)=y_{10},\ y'(x_1)=y_{11},...,\ y^{(n-1)}(x_1)=y_{1.n-1},\\ y(x)=\in C^{(n)}[x_0,x_1],\ \left(x,y(x),y^1(x),...,y^n(x)\right)\in Q,\ x\in[x_0,x_1]\end{aligned}\right\}\tag{15}$$

ekstremal masalani qaraymiz.

5-t a' r i f. Agar biror *ε>0* son topilib, $\left\|y-y^0\right\|_{C^{(n)}[x_0,x_1]}<\varepsilon$ shartni qanoatlantiruvchi barcha *y=y(x)* joyiz chiziqlar uchun $J\left[y^0\right]\le J[y]\ \left(J\left[y^0\right]\ge J[y]\right)$ tengsizlik bajarilsa, (14) funksional $y^0=y^0(x)$ joyiz chiziqda kuchsiz lokal minimumga (maksimumga) erishadi, deyiladi. Bunda $y^0(x)$ - (15) masalaning kuchsiz minimali (maksimali) deyiladi.

<u>6-t e o r e m a.</u> $F(x,y,z_1,...,z_n)\in C^{(n+1)}(Q)$ bo'lsin. Agar $y^0(x)$ – (14), (15) masalada kuchsiz ekstremal bo'lsa, barcha $x\in[x_0,x_1]$ uchun

$$F^0_y(x)-\frac{d}{dx}F^0_{y'}(x)+\frac{d^2}{dx^2}F^0_{y''}(x)-...+(-1)\frac{d^n}{dx^n}F^0_{y^{(n)}}(x)=0\tag{16}$$

tenglik bajariladi, bu yerda

$$F_{y^{(i)}}(x)=F_{y^{(i)}}\left(x,y^0(x),y^{0'}(x),...,y^{0^{(n)}}(x)\right),\quad i=\overline{0,n}.\tag{17}$$

Noma'lum $y=y(x)\in C^{(n)}[x_0,x_1]$ funksiyaga nisbatan

$$F_y-\frac{d}{dx}F_{y'}+\frac{d^2}{dx^2}F_{y''}-...+\frac{d^n}{dx^n}F_{y^{(n)}}=0\tag{20}$$

tenglamaga *Eyler–Puasson tenglamasi* deyiladi. $F(x,y,z_1,...,z_n)\in C^{(n+1)}(Q)$, bo'lganda Eyler-Puasson tenglamasi *2n*– tartibli oddiy differensial tenglamadan iborat.

<u>6-t a ' r i f.</u> Eyler-Puasson tenglamasini qanoatlantiruvchi $y^0(x)$ joyiz funksiyaga (14), (15) masalaning *stasionar funksiyasi* deyiladi.

<u>-t a' r i f.</u> Agar Yakobi tenglamasi $h^{(i)}(x_0)=h^{(i)}(x_*)=0,\ i=\overline{0,n-1}$, shartlarni qanoatlantiruvchi trivial (aynan nol) bo'lmagan yechimga ega bo'lsa, x_* nuqta- $y^0(x)$ joyiz chiziq bo'ylab x_0 nuqtaga *qo'shma nuqta* deyiladi.

<u>7-t e o r e m a.</u> $F(x,y,z_1,...,z_n)\in C^{(n+2)}(Q)$ bo'lsin. Agar $y^0(x)\in C^{(2n)}[x_0,x_1]$ (14), (15) masalada kuchsiz minimal (maksimal) bo'lsa, quyidagi shartlar bajariladi:

a) Lejandr sharti: $F^0_{y^{(n)}y^{(n)}}(x)\ge 0\ \ (\le 0),\ \forall x\in[x_0,x_1]$

b) Yakobi sharti: (x_0, x_1) intervalda *y⁰(x)* chiziq bo'ylab x_0 nuqtaga qo'shma nuqta mavjud emas.

<u>8-t e o r e m a.</u> Agar:

a) $F(x, y, z_1, ..., z_n) \in C^{(n+2)}(Q)$, $Q = S \times R$, $S \subset R^{n+1}$–ochiq to'plam;

b) $F^0_{y^{(n)}y^{(n)}}(x, y, z_1, ..., z_n) \ge 0 \ \ (\le 0),\ \forall (x, y, z_1, ..., z_n) \in Q$;

c) $y^0(x) \in C^{(2n)}[x_0, x_1]$ joyiz stasionar funksiya;

d) kuchaytirilgan Lejandr sharti bajarilsa: $F^0_{y^{(n)}y^{(n)}}(x) > 0 \ \ (< 0),\ \forall x \in [x_0, x_1]$;

e) kuchaytirilgan Yakobi sharti bajarilsa: $[x_0, x_1]$ oraliqda $y^0(x)$ chiziq bo'ylab x_0 nuqtaga qo'shma nuqta mavjud emas. U holda, $y^0(x)$ – (14), (15) masalada kuchli lokal minimal (maksimal) bo'ladi.

Endi (14) funksional

$$J[y] = \int_{x_0}^{x_1} \sum_{i=0}^{n} P_i(x) \left[y^{(i)}(x) \right]^2 dx \tag{22}$$

ko'rinishdagi kvadratik funksionaldan iborat bo'lsin. U holda, Yakobi tenglamasi,

$$\sum_{i=0}^{n} (-1)^{(i)} \frac{d^i}{dx^i} (P_i(x) h^{(i)}) = 0$$

ko'rinishda bo'ladi.

<u>9-t e o r e m a.</u> Faraz qilaylik, $P_i(x) \in C^{(i)}[x_0, x_1]$, $P_n(x) > 0 \ \ (< 0),\ \forall x \in [x_0, x_1]$ bo'lsin. Agar Yakobi sharti bajarilmasa, (22) funksional uchun $\inf J[y] = -\infty$ $(\sup J[y] = +\infty)$ bo'ladi. Agar kuchaytirilgan Yakobi sharti bajarilsa, (22) funksional uchun yagona joyiz stasionar funksiya mavjud, bu funksiya funksionalga global minimum (maksimum) beradi.

<u>M i s o l.</u>

$$\left. \begin{aligned} J[y] = \int_0^{\pi} (y''^2 - 16 y^2) dx \to \min, \\ y(0) = 0,\ y'(0) = 0,\ y(\pi) = 0,\ y'(\pi) = 1. \end{aligned} \right\}$$

Bu masalada qatnashayotgan hosilalarning yuqori tartibi *n=2* bo'lgani uchun, (20) tenglama,

$$F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''} = 0$$

ko'rinishda yoziladi. $F = y''^2 - 16y^2$, $F = -32y$, $F_{y'} = 0$, $F_{y''} = 2y''$ bo'lgani uchun, Eyler-Puasson tenglamasi,

$$-32y + \frac{d^2}{dx^2}(2y'') = 0 \Leftrightarrow y^{IV} - 16 = 0$$

ko'rinishida bo'ladi. Uning umumiy yechimi,

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x.$$

Chegaraviy shartlardan

$$c_1 = \frac{1}{2(e^{2n} - 1)},\quad c_2 = \frac{1}{2(1 - e^{-2n})},\quad c_3 = \frac{1 + e^{2n}}{2(e^{2n} - 1)},\quad c_4 = \frac{1}{2}$$

bo'lishi kelib chiqadi. Qaralayotgan masaladagi funksional (22) ko'rinishdagi kvadratik funksionaldir: $n=2,\ P_0(x)=-16,\ P_1(x)=0,\ P_2(x)=1$. Uning uchun Yakobi tenglamasi $h^{IV}-16h=0$ bo'ladi. Bu tenglamaning umumiy yechimi

$$h=\gamma_1 e^{2x}+\gamma_2 e^{-2x}+\gamma_3\cos 2x+\gamma_4\sin 2x.$$

uchun $h(0)=0,\ h(x_*)=0,\ h'(x)=0,\ h'(x_*)=0,\ x_*>0$ shartlardan $h(x)=0$ bo'lishi kelib chiqadi, ya'ni kuchaytirilgan Yakobi sharti bajariladi. $F_{y''y'}=2>0-$ kuchaytirilgan Lejandr sharti ham bajariladi. 9-teoremaga asosan,

$$y^0(x)=\frac{e^{2x}}{2(e^{2n}-1)}+\frac{e^{-2x}}{2(1-e^{-2n})}-\frac{1+e^{2n}}{2(e^{2n}-1)}\cos 2x+\frac{1}{2}\sin 2x$$

funksiya masalaning global yechimidir.

3.Ekstremumining zaruriy sharti. Eyler-Ostrogradskiy teglamasi.

, agar $z(x,y)$ funksiya ekstremal bo'lsa, ixiyoriy $h=h(x,y)$ variatsiya uchun,

$$\delta J[z(x,y)]\equiv 0 \Leftrightarrow \iint\left[F_z\cdot h+F_p\cdot h_x+F_q\cdot h_y\right]dxdy\equiv 0 \qquad (30)$$

munosabat o'rinli bo'ladi. Oxirga munosabatning chap tomonidagi ikkinchi va uchunchi qo'shiluvchilarning shaklini o'zgartiramiz. Buning uchun,

$$\frac{\partial}{\partial ч}\left[F_p\cdot h\right]=\frac{\partial}{\partial x}\{F_p\}\cdot h+F_p\cdot h_x, \qquad F_p\cdot h_x=\frac{\partial}{\partial x}\left[F_p\cdot h\right]-\frac{\partial}{\partial x}\{F_p\}\cdot h$$

ifodalardan

$$\frac{\partial}{\partial y}\left[F_q\cdot h\right]=\frac{\partial}{\partial y}\{F_q\}\cdot h+F_q\cdot h_y \qquad F_q\cdot h_y=\frac{\partial}{\partial y}\left[F_q\cdot h\right]-\frac{\partial}{\partial y}\{F_q\}\cdot h$$

ifodalarni olamiz va (30) munosabatga keltirib qo'yamiz:

$$\iint\left[F_z-\frac{\partial}{\partial x}\{F_p\}-\frac{\partial}{\partial y}\{F_q\}\right]\cdot hdxdy+\iint_D\left\{\frac{\partial}{\partial x}\left[F_p\cdot h\right]+\frac{\partial}{\partial y}\left[F_q\cdot h\right]\right\}dxdy=0.$$

Oxirgi munosabatdagi ikkinchi integralda ikki karrali integralni egri chiziqli integral orqali ifodalash formulasi- Grin formulasi

$$\iint_D\left(\frac{\partial q}{\partial x}-\frac{\partial y}{\partial y}\right)dxdy=\oint_{\partial D}Px+Qdy \qquad (31)$$

dan foydalanib va (24) chegaraviy shartlardan, $h[(x,y)]_{\partial D}=0$ ekanligini hisobga olib, (30) shartni,

$$\iint_D\left[F_z-\frac{\partial}{\partial x}\{F_P\}-\frac{\partial}{\partial y}\{F_q\}\right]h(x,y)dxdy\equiv 0 \qquad (32)$$

funksional ekstremumga erishadigan $z=z(x,y)$ funksiya quyidagi,

$$F_z-\frac{\partial}{\partial x}\{F_p\}-\frac{\partial}{\partial y}\{F_q\}=0 \qquad (33)$$

tenglamani qanoatlantirishi zarur. (33) tenglama, *Eyler – Ostrogradskiy tenglamasi* deyiladi

bunda

$$\frac{\partial}{\partial x}\{F_p\}=\frac{\partial}{\partial x}\left\{F_p(x,y,z,p,q)\right\}=F_{xp}+F_{zp}\cdot p+F_{pp}\cdot\frac{\partial p}{\partial x}+F_{qp}\frac{\partial q}{\partial x},$$

$$\frac{\partial}{\partial y}\{F_q\}=\frac{\partial}{\partial y}\left\{F_q(x,y,z,p,q,)\right\}=F_{yq}+F_{zq}\cdot q+F_{pq}\frac{\partial p}{\partial y}+F_{qq}\frac{\partial q}{\partial y},$$

$$\frac{\partial p}{\partial y}=\frac{\partial^2 z}{\partial y dx},\frac{\partial q}{\partial x}=\frac{\partial^2 z}{\partial x\partial y}.$$

3.4. misol. Ushbu

$$J[z]=\iint\limits_D\left(z^2{}_x+z_y{}^2\right)dxdy$$

funksional uchun Eyler-Ostrogradskiy tenglamasini yozing.

Yech i l i sh i. Berilgan funksional ikki o'zgaruvchili funksiyaga bog`liq bo'lib, unda $F=F(p,q)=p^2+p^2$ bo'lganligidan, $F_z=0,F_p=2p,F_q=2q$ va Eyler-Ostrogradskiy tenglamasi

$$\frac{\partial}{\partial ч}\{2p\}+\frac{\partial}{\partial D}\{2q\}=0,$$

yoki, $z^2{}_{xx}+z^2{}_{yy}=o\Leftrightarrow\Delta z=o$ Laplas teglamasidan iborat. Ma'lumki, uning echimlari *garmonik funksiyalar* deb ataladi.

Misollar

1-misol. $J(y_1,y_2)=\int\limits_0^1\left(y_1'^2+y_2'^2-3y_1'y_2'+4y_1{}^2\right)dx\to\min,$

$y_1(0)=1,\ \ y_2(0)=1,\ \ y_1(1)=0,\ \ y_2(1)=-1,\ \ y_1,y_2\in C^{(1)}[0:1].$

masala uchun Eyler tenglamalari sistemasini tuzing.

Yechilishi: Bu masala funktsionali uchun Eyler tenglamalari sistemasi

$$\begin{cases}F_{y_1}-\dfrac{d}{dx}F_{y'_1}=0\\ F_{y_2}-\dfrac{d}{dx}F_{y_2'}=0\end{cases}$$

ko'rinishda bo'lishi kerak. Integral ostidagi funksiya $F=y_1'^2+y_2'^2-3y_1'y_2'+4y_1^2$. Bu funksiyadan y_1, y_1' va x lar bo'yicha hosilalar olib, quyidagilarga ega bo'lamiz:

$$F_{y_1}=8y_{1,}\quad F_{y_1'}=2y_1'-3y_2',\ \frac{d}{dx}F_{y_1'}=2y_1''-3y_2''.$$

Bundan y_1 bo'yicha Eyler tenglamasi $8y_1-2y_1''-3y_2''=0$ ko'rinishga keladi. Endi integral ostidagi funksiyadan y_2, y'_2 va x lar bo'yicha hosilalar olib,

$$F_{y_2}=0,\ F_{y_2'}=2y_2'-3y_1',\ \frac{d}{dx}F_{y_2'}=2y_2''-3y_1''$$

larga ega bo'lamiz. Bundan y_2 bo'yicha Eyler tenglamasi $-2y_2''+3y_1''=0$ yoki $2y_2''-3y_1''=0$ ko'rinishda bo'ladi. Natijada, Eyler tenglamalari sistemasi,

$$\begin{cases} 2y_1'' - 8y_1 - 3y_2'' = 0 \\ 2y_2'' - 3y_1'' = 0 \end{cases}$$

ko’rinishda bo’ladi.

2-misol.Ushbu

$J(y) = \int_0^1 \left(y''^2 + y'^2 - 2y'x^3\right)dx \to \min$, $y_0(0) = 0$, $y_0'(0) = 0$, $y_0(1) = 1$, $y_0'(1) = -1$,

$y \in C^{(2)}[0:1]$,

variatsion masala uchun Eyler – Puasson tenglamasini tuzing.

Yechilishi: Eyler – Puasson tenglamasi

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F^0_{y''} + ... + (-1)\frac{d^n}{dx^n}F^0_{y^{(n)}} = 0$$

tenglamadan iborat. Integral ostidagi funksiya $F = y''^2 + y'^2 - 2y'x^3$. Bu funksiyadan y bo’yicha xususiy hosila olamiz: $F_y = 0$. Endi integral ostidagi funksiyadan y' bo’yicha xususiy hosila olib, $F_{y'} = 2y' - 2x^3$ ifodaga ega bo’lamiz. $F_{y'} = 2y' - 2x^3$ funksiyadan x bo’yicha to’la hosila olib, $\frac{d}{dx}F_{y'} = 2y'' - 6x^2$ ifodaga ega bo’lamiz. Endi integral ostidagi funksiyadan y'' bo’yicha xususiy hosila olib, $F_{y''} = 2y''$ ifodaga ega bo’lamiz. $F_{y''} = 2y''$ funksiyadan x bo’yicha hosilalar olib, $\frac{d}{dx}F_{y''} = 2y'''$ va $\frac{d^2}{dx^2}F_{y''} = 2y^{IV}$ ifodalarga ega bo’lamiz. Bu olingan hosilalarni yuqoridagi tenglamaga keltirib qo’yib,

$$-2y'' + 6x^2 + 2y^{IV} = 0$$

yoki

$$y^{IV} - y'' + 3x^2 = 0$$

Eyler – Puasson tenglamasiga ega bo’lamiz.

3-misol. $J[z(x,y)] = \iint_D \left[\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2\right]dxdy$ funksional uchun Eyler - Ostrogradskiy tenglamasini tuzing.

Yechilishi: Funksionalning ifodasidagi funksiya:

$$F(x, y, z, p, q) = p^2 - q^2$$

ko’rinishda bo’lganligidan, Eyler - Ostrogradskiy tenglamasi,

$$-\frac{\partial}{\partial x}(2p) - \frac{\partial}{\partial y}(-2q) = 0$$

yoki

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

ko’rinishni oladi**.**

Mustaqil ishlash uchun misollar

1 – masala. Quyidagi variatsion masala uchun Eyler tenglamalari sistemasini tuzing.

1. $J(y,z)=\int_0^1 \left(2y'z'-y^2+z^2-2ye^x\right)dx;$ $\quad y(0)=0;\ z(0)=1;\ y(1)=1;\ z(1)=0;$

2. $J(y,z)=\int_0^1 \left(2y'z'+y^2+z^2-z\sin x\right)dx;$ $\quad y(0)=0;\ z(0)=1;\ y(1)=1;\ z(1)=0;$

3. $J(y,z)=\int_0^1 \left(2y'z'+y^2-z^2+2z\cos x\right)dx;$ $\quad y(0)=0;\ z(0)=1;\ y(1)=1;\ z(1)=0;$

4. $J(y,z)=\int_0^1 \left(y^2+z^2+2y'z'+ye^x\right)dx;$ $\quad y(0)=0;\ z(0)=1;\ y(1)=1;\ z(1)=0;$

5. $J(y,z)=\int_0^1 \left(y^2+4yz+z^2+y'^2+z'^2+2ze^x\right)dx;$ $\quad y(0)=0;\ z(0)=1;\ y(1)=1;\ z(1)=0;$

6. $J(y,z)=\int_0^1 \left((y+z)^2+y'^2+z'^2+2y\sin x\right)dx;$ $\quad y(0)=0;\ z(0)=1;\ y(1)=1;\ z(1)=0;$

7. $J(y,z)=\int_0^1 \left((y-z)^2+y'^2-z'^2+2z\cos x\right)dx;$ $\quad y(0)=0;\ z(0)=1;\ y(1)=1;\ z(1)=0;$

8. $J(y,z)=\int_{-1}^1 \left(2y'z'-y^2+z^2-2y\cos x\right)dx;$ $\quad y(-1)=2;\ z(-1)=1;\ y(1)=0;\ z(1)=2;$

9. $J(y,z)=\int_{-1}^1 \left(2y'z'+y^2+z^2+2ze^x\right)dx;$ $\quad y(-1)=3;\ z(-1)=0;\ y(1)=1;\ z(1)=2;$

10. $J(y,z)=\int_{-1}^1 \left(2y'z'+y^2-z^2-2z\sin x\right)dx;$ $\quad y(-1)=2;\ z(-1)=0;\ y(1)=0;\ z(1)=2;$

11. $J(y,z)=\int_{-1}^1 \left(y^2+z^2-2y'z'-y\cos x\right)dx;$ $\quad y(-1)=2;\ z(-1)=0;\ y(1)=0;\ z(1)=2;$

12. $J(y,z)=\int_{-1}^1 \left(y^2+4yz+z^2-y'^2-z'^2+2ye^{-x}\right)dx;$ $\quad y(-1)=2;\ z(-1)=0;\ y(1)=0;\ z(1)=2;$

13. $J(y,z)=\int_{-1}^1 \left((y+z)^2-y'^2-z'^2+2xz\right)dx;$ $\quad y(-1)=2;\ z(-1)=0;\ y(1)=0;\ z(1)=2;$

14. $J(y,z)=\int_{-1}^1 \left((y-z)^2+y'^2-z'^2+2xy\right)dx;$ $\quad y(-1)=1;\ z(-1)=0;\ y(1)=0;\ z(1)=2;$

15. $J(y,z)=\int_0^2 \left(2y'z'-y^2+z^2+2y\sin x\right)dx;$ $\quad y(0)=1;\ z(0)=-2;\ y(2)=-1;\ z(2)=1;$

16. $J(y,z)=\int_0^2 \left(2y'z'+y^2+z^2+2z\cos x\right)dx;$ $\quad y(0)=1;\ z(0)=-2;\ y(2)=-1;\ z(2)=1;$

17. $J(y,z)=\int_0^2 \left(2y'z'+y^2-z^2+2ze^x\right)dx;$ $\quad y(0)=1;\ z(0)=-2;\ y(2)=-1;\ z(2)=1;$

18. $J(y,z)=\int_0^2 \left(y^2+z^2+2y'z'+z\sin x\right)dx;$ $\quad y(0)=1;\ z(0)=-2;\ y(2)=-1;\ z(2)=1;$

19. $J(y,z)=\int_0^2 \left(y^2+4yz+z^2-y'^2-z'^2+ze^{3x}\right)dx;$ $\quad y(0)=1;\ z(0)=-2;\ y(2)=-1;\ z(2)=1;$

20. $J(y,z)=\int_0^2 \left((y+z)^2-y'^2-z'^2+2ye^{2x}\right)dx;$ $\quad y(0)=1;\ z(0)=-2;\ y(2)=-1;\ z(2)=1;$

21. $J(y,z)=\int_0^2 \left((y-z)^2+y'^2-z'^2+x^2z\right)dx;$ $\quad y(0)=1;\ z(0)=-2;\ y(2)=-1;\ z(2)=1;$

22. $J(y,z)=\int_{-2}^{2}\left(2y'z'-y^2+z^2+ze^{2x}\right)dx;$ $\quad y(-2)=0; \quad z(-2)=2; \quad y(2)=3; \quad z(2)=1;$

23. $J(y,z)=\int_{-2}^{2}\left(2y'z'+y^2+z^2+2ye^{-x}\right)dx;$ $\quad y(-2)=0; \quad z(-2)=2; \quad y(2)=3; \quad z(2)=1;$

24. $J(y,z)=\int_{-2}^{2}\left(2y'z'+y^2-z^2+2ye^{x}\right)dx;$ $\quad y(-2)=0; \quad z(-2)=2; \quad y(2)=3; \quad z(2)=1;$

25. $J(y,z)=\int_{-2}^{2}\left(y^2+z^2-2y'z'+2ze^{-x}\right)dx;$ $\quad y(-2)=0; \quad z(-2)=2; \quad y(2)=3; \quad z(2)=1;$

26. $J(y,z)=\int_{-1}^{2}\left(2y'z'+y^2-z^2+2z\sin x\right)dx;$ $\quad y(-1)=2; \quad z(-1)=0; \quad y(2)=0; \quad z(2)=2;$

27. $J(y,z)=\int_{-2}^{2}\left(y^2+z^2+2y'z'-y\cos x\right)dx;$ $\quad y(-2)=2; \quad z(-2)=0; \quad y(2)=0; \quad z(2)=2;$

28. $J(y,z)=\int_{-2}^{2}\left(y^2+4yz+z^2-y'^2-z'^2+2ye^{-2x}\right)dx;$ $\quad y(-2)=3; \quad z(-2)=0; \quad y(2)=1; \quad z(2)=2;$

29. $J(y,z)=\int_{-2}^{2}\left((y+z)^2-y'^2-z'^2+4xz\right)dx;$ $\quad y(-2)=2; \quad z(-2)=0; \quad y(2)=0; \quad z(2)=2;$

30. $J(y,z)=\int_{-2}^{2}\left((y-z)^2+y'^2-z'^2+4xy\right)dx;$ $\quad y(-2)=1; \quad z(-2)=0; \quad y(2)=0; \quad z(2)=2;$

<u>2 – masala.</u> Quyidagi variatsion masala uchun Eyler – Puasson tenglamasini tuzing.

1. $J(y)=\int_{0}^{1}\left(y''^2-2y'^2+y^2-2ye^{x}\right)dx;$ $\quad y(0)=2; \quad y(1)=0; \quad y'(0)=1; \quad y'(1)=-1;$

2. $J(y)=\int_{0}^{1}\left(y''^2-y^2+2y\sin x\right)dx;$ $\quad y(0)=2; \quad y(1)=0; \quad y'(0)=1; \quad y'(1)=-1;$

3. $J(y)=\int_{0}^{1}\left(y''^2+4y'y''+y'^2-2ye^{x}\right)dx;$ $\quad y(0)=2; \quad y(1)=0; \quad y'(0)=1; \quad y'(1)=-1;$

4. $J(y)=\int_{0}^{1}\left(y''^2-y'y''+y'^2-2ye^{-x}\right)dx;$ $\quad y(0)=2; \quad y(1)=0; \quad y'(0)=1; \quad y'(1)=-1;$

5. $J(y)=\int_{0}^{1}\left(y''^2-y'^2-4ye^{-x}\right)dx;$ $\quad y(0)=2; \quad y(1)=0; \quad y'(0)=1; \quad y'(1)=-1;$

6. $J(y)=\int_{0}^{1}\left(y''^2-y'^2+yy'-yx\right)dx;$ $\quad y(0)=2; \quad y(1)=0; \quad y'(0)=1; \quad y'(1)=-1;$

7. $J(y)=\int_{0}^{1}\left(y''^2-2y'^2+y^2+2ye^{-x}\right)dx;$ $\quad y(0)=2; \quad y(1)=0; \quad y'(0)=1; \quad y'(1)=-1;$

8. $J(y)=\int_{0}^{1}\left(y''^2-y^2+ye^{x}\right)dx;$ $\quad y(0)=2; \quad y(1)=0; \quad y'(0)=1; \quad y'(1)=-1;$

9. $J(y)=\int_{0}^{1}\left(y''^2+3y'y''+y'^2+2xy\right)dx;$ $\quad y(0)=2; \quad y(1)=0; \quad y'(0)=1; \quad y'(1)=-1;$

10. $J(y)=\int_{0}^{1}\left(y''^2-4y'y''+y'^2+2y\sin x\right)dx;$ $\quad y(0)=2; \quad y(1)=0; \quad y'(0)=1; \quad y'(1)=-1;$

11. $J(y)=\int_{0}^{1}\left(y''^2-y'^2+2ye^{x}\right)dx;$ $\quad y(0)=2; \quad y(1)=0; \quad y'(0)=1; \quad y'(1)=-1;$

12. $J(y)=\int_{0}^{1}\left(y''^2-y'^2+4yy'+2ye^{-x}\right)dx;$ $\quad y(0)=3; \quad y(1)=1; \quad y'(0)=0; \quad y'(1)=1;$

13. $J(y)=\int_{-1}^{1}\left(y''^2-2y'^2+y^2+2y\sin x\right)dx;$ $\quad y(-1)=1; \quad y(1)=2; \quad y'(-1)=-1; \quad y'(1)=1;$

14. $J(y)=\int_{-1}^{1}\left(y''^2-y^2+2y\cos x\right)dx;$ $\quad y(-1)=1;\quad y(1)=2;\quad y'(-1)=-1;\quad y'(1)=1;$

15. $J(y)=\int_{-1}^{1}\left(y''^2+4y'y''+y'^2+2ye^{-x}\right)dx;$ $\quad y(-1)=1;\quad y(1)=2;\quad y'(-1)=-1;\quad y'(1)=1;$

16. $J(y)=\int_{-1}^{1}\left(y''^2-y'y''+y'^2+ye^{x}\right)dx;$ $\quad y(-1)=1;\quad y(1)=2;\quad y'(-1)=-1;\quad y'(1)=1;$

17. $J(y)=\int_{-1}^{1}\left(y''^2-y'^2+4xy\right)dx;$ $\quad y(-1)=1;\quad y(1)=2;\quad y'(-1)=-1;\quad y'(1)=1;$

18. $J(y)=\int_{-1}^{1}\left(y''^2-y'^2+2yy'+ye^{x}\right)dx;$ $\quad y(-1)=1;\quad y(1)=2;\quad y'(-1)=-1;\quad y'(1)=1;$

19. $J(y)=\int_{0}^{2}\left(y''^2-2y'^2+y^2+2ye^{x}\right)dx;$ $\quad y(0)=1;\quad y(2)=4;\quad y'(0)=0;\quad y'(2)=-1;$

20. $J(y)=\int_{0}^{2}\left(y''^2-y^2+2ye^{-x}\right)dx;$ $\quad y(0)=1;\quad y(2)=4;\quad y'(0)=0;\quad y'(2)=-1;$

21. $J(y)=\int_{0}^{2}\left(y''^2+3y'y''+y'^2+4ye^{x}\right)dx;$ $\quad y(0)=1;\quad y(2)=4;\quad y'(0)=0;\quad y'(2)=-1;$

22. $J(y)=\int_{0}^{2}\left(y''^2-4y'y''+y'^2+4ye^{-x}\right)dx;$ $\quad y(0)=1;\quad y(2)=4;\quad y'(0)=0;\quad y'(2)=-1;$

23. $J(y)=\int_{0}^{2}\left(y''^2-y'^2+2ye^{-x}\right)dx;$ $\quad y(0)=1;\quad y(2)=4;\quad y'(0)=0;\quad y'(2)=-1;$

24. $J(y)=\int_{0}^{2}\left(y''^2-y'^2+2yy'+2xy\right)dx;$ $\quad y(0)=1;\quad y(2)=4;\quad y'(0)=0;\quad y'(2)=-1;$

25. $J(y)=\int_{0}^{2}\left(y''^2-2y'^2+y^2+4xy\right)dx;$ $\quad y(0)=1;\quad y(2)=4;\quad y'(0)=0;\quad y'(2)=1;$

26. $J(y)=\int_{0}^{1}\left(y''^2-4y'y''+y'^2+2y\sin x\right)dx;$ $\quad y(0)=2;\quad y(1)=0;\quad y'(0)=1;\quad y'(1)=-1;$

27. $J(y)=\int_{0}^{1}\left(y''^2-y'^2+2ye^{x}\right)dx;$ $\quad y(0)=2;\quad y(1)=0;\quad y'(0)=1;\quad y'(1)=-1;$

28. $J(y)=\int_{0}^{1}\left(y''^2-y'^2+4yy'+2ye^{-x}\right)dx;$ $\quad y(0)=3;\quad y(1)=1;\quad y'(0)=0;\quad y'(1)=1;$

29. $J(y)=\int_{-1}^{1}\left(y''^2-2y'^2+y^2+2y\sin x\right)dx;$ $\quad y(-1)=1;\quad y(1)=2;\quad y'(-1)=-1;\quad y'(1)=1;$

30. $J(y)=\int_{-1}^{1}\left(y''^2-y^2+2y\cos x\right)dx;$ $\quad y(-1)=1;\quad y(1)=2;\quad y'(-1)=-1;\quad y'(1)=1;$

3 – masala. Quyidagi funksional uchun Eyler - Ostrogradskiy tenglamasini tuzing:

1. $J(z)=\iint_D\left(z_x^2+z_y^2+2xyz\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\frac{x^2}{50}+\frac{y^2}{100}.$

2. $J(z)=\iint_D\left(z_x^2+2z_y^2+2z\sin\pi x\sin\frac{\pi y}{2}\right)dS;\quad D:\ \begin{cases}0\le x\le 1;\\ 0\le y\le 2;\end{cases}\quad z|_C=\frac{x^2}{50}-\frac{y}{100}.$

3. $J(z)=\iint_D\left(z_x^2+3z_y^2+2yz\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\frac{x^2}{100}+\frac{y}{50}.$

4. $J(z)=\iint_D\left(z_x^2+4z_y^2+2yz\sin x\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\frac{x^2}{100}+\frac{y}{50}.$

5. $J(z)=\iint_D\left(z_x^2+5z_y^2+2yz\sin x\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\frac{x^2}{200}-\frac{y}{100}.$

6. $J(z)=\iint\limits_D \left(2z_x^2+z_y^2+2yz\cos x\right)dS;\quad D:\ \begin{cases}-1\le x\le 1;\\ 0\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2}{100}+\dfrac{y}{200}.$

7. $J(z)=\iint\limits_D \left(3z_x^2+z_y^2+2xz\cos y\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ -1\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2}{200}-\dfrac{y}{100}.$

8. $J(z)=\iint\limits_D \left(4z_x^2+z_y^2+2xyz\right)dS;\quad D:\ \begin{cases}-1\le x\le 1;\\ 0\le y\le 2;\end{cases}\quad z|_C=\dfrac{x^2+y^2}{100}.$

9. $J(z)=\iint\limits_D \left(5z_x^2+z_y^2+2xyz\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ -1\le y\le 1;\end{cases}\quad z|_C=\dfrac{x+y^2}{200}.$

10. $J(z)=\iint\limits_D \left(2z_x^2+3z_y^2+2x^2yz\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 2;\end{cases}\quad z|_C=\dfrac{x}{100}+\dfrac{y^2}{200}.$

11. $J(z)=\iint\limits_D \left(2z_x^2-z_y^2+2xyz\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2+y^2}{100}.$

12. $J(z)=\iint\limits_D \left(3z_x^2-z_y^2+2z\sin\pi x\sin\dfrac{\pi y}{2}\right)dS;\quad D:\ \begin{cases}0\le x\le 1;\\ 0\le y\le 2;\end{cases}\quad z|_C=\dfrac{x^2-2y}{200}.$

13. $J(z)=\iint\limits_D \left(2z_x^2-z_y^2+yz\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2+y}{100}.$

14. $J(z)=\iint\limits_D \left(5z_x^2-z_y^2+2yz\sin x\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2}{200}+\dfrac{y}{50}.$

15. $J(z)=\iint\limits_D \left(z_x^2-2z_y^2+2xz\sin\pi y\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2}{200}-\dfrac{y}{50}.$

16. $J(z)=\iint\limits_D \left(z_x^2-3z_y^2+2yz\cos x\right)dS;\quad D:\ \begin{cases}-1\le x\le 1;\\ 0\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2}{100}+\dfrac{y}{50}.$

17. $J(z)=\iint\limits_D \left(3z_x^2-4z_y^2+2xz\cos y\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ -1\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2}{200}-\dfrac{y}{100}.$

18. $J(z)=\iint\limits_D \left(5z_x^2-4z_y^2+2xyz\right)dS;\quad D:\ \begin{cases}-1\le x\le 1;\\ 0\le y\le 2;\end{cases}\quad z|_C=\dfrac{x^2}{100}+\dfrac{y^2}{200}.$

19. $J(z)=\iint\limits_D \left(4z_x^2-5z_y^2+2xyz\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ -1\le y\le 1;\end{cases}\quad z|_C=\dfrac{x}{100}+\dfrac{y^2}{50}.$

20. $J(z)=\iint\limits_D \left(2z_x^2-3z_y^2+2x^2yz\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 2;\end{cases}\quad z|_C=\dfrac{x}{50}+\dfrac{y^2}{80}.$

21. $J(z)=\iint\limits_D \left(2z_x^2-z_y^2+z^2+2xyz\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2}{100}+\dfrac{y^2}{160}.$

22. $J(z)=\iint\limits_D \left(3z_x^2-z_y^2+z^2+2z\sin\pi x\sin\dfrac{\pi y}{2}\right)dS;\quad D:\ \begin{cases}0\le x\le 1;\\ 0\le y\le 2;\end{cases}\quad z|_C=\dfrac{x^2}{200}-\dfrac{y}{120}.$

23. $J(z)=\iint\limits_D \left(4z_x^2-z_y^2+z^2+2yz\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2+y}{100}.$

24. $J(z)=\iint\limits_D \left(5z_x^2-z_y^2+z^2+2yz\sin x\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2}{200}+\dfrac{y}{100}.$

25. $J(z)=\iint\limits_D \left(z_x^2-2z_y^2+z^2+2xz\sin\pi y\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ 0\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2}{200}-\dfrac{y}{80}.$

26. $J(z)=\iint\limits_D \left(2z_x^2+z_y^2+z^2+2yz\cos x\right)dS;\quad D:\ \begin{cases}-1\le x\le 1;\\ 0\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2}{100}+\dfrac{y}{60}.$

27. $J(z)=\iint\limits_D \left(z_x^2+2z_y^2+z^2+2xz\cos y\right)dS;\quad D:\ \begin{cases}0\le x\le 2;\\ -1\le y\le 1;\end{cases}\quad z|_C=\dfrac{x^2}{100}-\dfrac{y}{60}.$

28. $J(z)=\iint\limits_D \left(z_x^2+3z_y^2+z^2+2xyz\right)dS;\quad D:\ \begin{cases}-1\le x\le 1;\\ 0\le y\le 2;\end{cases}\quad z|_C=\dfrac{x^2}{200}+\dfrac{y^2}{100}.$

29. $J(z)=\iint\limits_D \left(z_x^2+4z_y^2+z^2+2xyz\right)dS;\quad D: \begin{cases} 0\le x\le 2; \\ -1\le y\le 1; \end{cases} \quad z|_C=\frac{x+y^2}{100}.$

30. $J(z)=\iint\limits_D \left(z_x^2+5z_y^2+z^2+2x^2yz\right)dS;\quad D: \begin{cases} 0\le x\le 2; \\ 0\le y\le 2; \end{cases} \quad z|_C=\frac{x}{100}+\frac{y^2}{160}.$

5-amaliy mashg‘ulot. Chegaralari qo‘zg‘aluvchan variatsion masalalar. Transversallik shartlari

1.Chetlari qo’zg’oluvchan variatsion masalalar.

Endi variatsion hisob asosiy masalasidagi

$$J(y)=\int_{x_0}^{x_1}F(x,y,y')dx \tag{1}$$

funksionalning ekstremumini, chetki nuqtalari $A(x_0,y_0)$ va $B(x_1,y_1)$ lar, mos ravishda,

$$y=\varphi(x),\quad y=\psi(x) \tag{2}$$

silliq egri chiziqlarda yotadigan $y(x)\in C^1[x_0,x_1]$ egri chiziqlarda topish masalasini qaraymiz. (2) shartlardan $y_0=\varphi(x_0)$, $y_1=\psi(x_1)$ munosabatlarni olamiz. Bunda A va B nuqtalarning absissalari x_0 *va* x_1 oldindan tanlanmagan bo’lib, ularni aniqlash talab qilinadi.

Masalaning qo’yilishi va unda hosil bo’ladigan ekstremumning zaruriy shartlari hamda transversallik shartlari ma’ruzada (5-ma’ruza) keltirilgan edi.

Ekstremumning zaruriy shartini qayta keltiramiz: *berilgan ikkita silliq egri chiziqlar*

$$y=\varphi(x),\quad y=\psi(x)$$

ning nuqtalarini tutashtiruvchi barcha $y(x)\in C^1[x_0,x_1]$ *egri chiziqlar ichidan olingan* $y=\tilde{y}(x)$ *egri chiziq (1) funksionalga kuchsiz eksemum berishi uchun:*

1) $y=\tilde{y}(x)$ *(1) funksional uchun tuzilgan Eyler tenglamasning yechimi (ekstremal) bo’lishi ;*
2) $y=\tilde{y}(x)$ *ektremalning* $y=\varphi(x)$, $y=\psi(x)$ *egri chiziqlar bilan kesishish nuqtalari* $A(x_0,y_0)$, $B(x_1,y_1)$ *larda*

$$\left.[F+(\varphi'-y')F_{y'}]\right|_{x=x_0}=0$$
$$\left.[F+(\psi'-y')F_{y'}]\right|_{x=x_1}=0 \tag{3}$$

transversallik shartlarining bajarilishi zarur .

Izohlar .1. (3) transversallik shartlari Eyler tenglamasini yechishda paydo bo’ladigan x_0,x_1 larni aniqlash uchun yetarli emas va ularga ikkita

$$\tilde{y}(y_0)=\varphi(x_0),\ \tilde{y}(y_0)=\psi(x_1), \tag{4}$$

tabiiy shartlatni qo'shish zarur .

2. Agar chetki nuqta (masalan , B nuqta) $x = x_1$ vertikal to'g'ri chiziq bo'ylab o'zgarsa (bunda $\psi' = \infty$), transversallik sharti

$$F_{y'}\Big|_{x=x_1} = 0 \qquad (5)$$

ko'rinishni oladi.

3. Agar chetki nuqta (masalan , A nuqta) $y = y_0(\varphi' = 0)$ gorizantal to'g'ri chiziq bo'ylab o'zgarsa, (3) transversallik sharti

$$F - y'F_{y'}\Big|_{x=x_0} = 0 \qquad (6)$$

ko'rinishni oladi.

4. Agar chetki nuqtalardan biri qo'zg'olmas (masalan, A nuqta) ikkinchi chetki nuqta (B)

$$y = \psi(x)$$

egri chiziq bo'ylab o'zgarsa, (3) transversallik sharti

$$[F + (\psi' - y')F_{y'}]\Big|_{x=x_1} = 0 \qquad (7)$$

va tabiiy chegaraviy shart

$$y_1 = \psi(x_1) \qquad (8)$$

ko'rinishni oladi.

5. Agar chetki nuqtalardan biri (masalan, B nuqta) qo'zg'olmas, ikkinchisi (A nuqta)

$$y = \varphi(x)$$

egri chiziq bo'ylab o'zgarsa, (3) transversallik sharti

$$[F + (\varphi' - y')F_{y'}]\Big|_{x=x_0} = 0 \qquad (9)$$

tabiiy chegaraviy shart

$$y_0 = \varphi(x_0)$$

ko'rinishni oladi.

1-misol . $y=x^2$ parobola va $y=x-5$ to'g'ri chiziq orasidagi masofani toping .

Yechilishi. Masalaning yechimi, $y=y(x)$ egri chiziqning uzunligini ifodalovchi

$$J(y) = \int_{x_0}^{x_1} \sqrt{1 + y'^2}dx \qquad (1)$$

funksionalga minimum beruvchi $\tilde{y}(x)$ egri chiziqdan iborat bo'ladi. Bunda ekstremalning chap cheti $y = \varphi(x) = x^2$ parabola bo'ylab, o'ng cheti esa, $y = \psi(x) = x - 5$ to'g'ri chiziq bo'ylab o'zgaradi.

(1) funksional uchun Eyler tenglamasi

$$y'' = 0 \qquad (2)$$

bo'ladi(Eyler tenglamasining 1- xususiy holi) va uning umumiy yechimi

$$y = C_1x + C_2 \qquad (3)$$

ko'rinishni oladi. Uni

$$\begin{cases} [\sqrt{1+y'^2}+(2x-y')\frac{y'}{\sqrt{1+y'^2}}]|_{x=x_0}=0 \\ [\sqrt{1+y'^2}+(1-y')\frac{y'}{\sqrt{1+y'^2}}]|_{x=x_1}=0 \end{cases} \quad (y'=C_1) \qquad (4)$$

transversallik shartlariga va ekstremalning $y=x^2$ va $y=x-5$ chiziqlar bilan kesishish shartlariga keltirib qo'yish zarur :

$$\begin{cases} C_1x_0+C_2=x_0^2 \\ C_1x_1+C_2=x_1-5 \end{cases} \qquad (5)$$

(4), (5) tenglamalar C_1, C_2, x_0, x_1 parametrlarni aniqlash imkonini beradi:

$$C_1=-1;\ \ C_2=\frac{3}{4};\ \ x_0=\frac{1}{2};\ \ x_1=\frac{28}{8}$$

Shunday qilib, ekstremal

$y=-x+\frac{3}{4}$ to'gri chiziqdan iborat, berilgan parabola va to'g'ri chiziq orasidagi masofa

$$l=\int_{\frac{1}{2}}^{\frac{23}{8}}\sqrt{1+(-1)^2}\,dx=\frac{19\sqrt{2}}{8}$$

ekan .

2-misol . Ushbu

$$J(y)=\int_{x_0}^{x_1}A(x,y)\sqrt{1+y'^2}\,dx \qquad (6)$$

ko'rinishdagi funksional uchun transversallik shartining ko'riinishini toping .

Yechilishi. Bu holda (ekstremalning chap cheti $y=\varphi(x)$ egri chiziqda yotganda) transversallik sharti

$$F+(\varphi'-y')F_{y'}=0$$

quyidagi

$$\frac{A(x,y)(1+\varphi'y')}{\sqrt{1+y'^2}}=0 \qquad (7)$$

ko'rinishda bo'ladi. Bundan $A(x,y)\neq 0$ shartda

$$1+\varphi'y'|_{x=x_0}=0 \qquad (8)$$

ekanligini topamiz, yoki

$$y'\varphi'|_{x=x_0}=-1\ . \qquad (9)$$

Bu esa, berilgan (6) funksional uchun trasversallik sharti

$$y'=\frac{1}{\varphi'} \qquad (10)$$

ortoganallik shartidan iboratligini ko'rsatadi.

Agar ekstremalning o'ng cheti $y=\psi(x)$ egri chiziqda yotishi ma'lum bo'lsa, (9) ga o'xshash shart

$$1 + y'\varphi'|_{x=x_1} = 0$$

ko'rinishida bo'ladi.

1-misolda Eyler tenglamasining umumiy yechimi $y = C_1 x + C_2$ ning C_1, C_2 koeffisientlari hamda $x_0 \; va \; x_1$ nuqtalarni topish uchun bu misolda topilgan shartlardan foydalansak, ushbu

$$\begin{cases} 1 + y'\varphi'|_{x=x_0} = 1 + C_1 * 2x_0 = 0 \\ 1 + y'\varphi'|_{x=x_1} = 1 + C_1 * 1 = 0 \\ C_1 x_0 + C_2 = x_0{}^2 \\ C_1 x_1 + C_2 = x_1 - 5 \end{cases} \quad (11)$$

sistemada ega bo'lamiz .Bu sistemadan, yana avvalgiga qaraganda tezroq

$C_1 = -1; \; x_0 = \frac{1}{2} \; , \; C_2 = \frac{3}{4}; \; x_1 = \frac{23}{8}$

ekanligini olamiz. Shunday qilib, berilgan funksional ko'rinishini hisobga olish C_1, C_2 va $x_0 \, , \, x_1$ larni aniqlash uchun sodda ko'rinishidagi (11) sistemani olish imkonini berdi.

3 – misol. Ushbu

$$J(y) = \int_0^{x_1} \frac{\sqrt{1 + y'^2}}{y} dx$$

funksional $y(0) = 0$ chegaraviy shartda:

a. (x_1, y_1) nuqta $y = x - 5$ to`g`ri chiziq bo`ylab o`zgarganda;

b. (x_1, y_1) nuqta $(x-9)^2 + y^2 = 9$ aylana bo`ylab o`zgarganda;

ekstremumga ega bo`lishi mumkin bo`lgan egri chiziqlar topilsin.

Yechilishi. Bu holda integrant $F = F(y, y') = \dfrac{\sqrt{1 + y'^2}}{y}$ ko`rinishda, Eyler-Lagranj tenglamasi esa,

$$F_y - \frac{d}{dx} F_{y'} = 0$$

dan iboratdir. Tenglamada qatnashayotgan xususiy hosilalarni topamiz:

$$F_y = -\frac{\sqrt{1 + y'^2}}{y^2},$$

$$F_{y'} = \frac{y'}{y\sqrt{1 + y'^2}},$$

$$\frac{d}{dx} F_{y'} = \frac{y'' y \sqrt{1 + y'^2} - y' \left[y' \sqrt{1 + y'^2} + y \dfrac{y' y''}{\sqrt{1 + y'^2}} \right]}{y^2 (1 + y'^2)}$$

va ularning qiymatlarini tenglamaga keltirib qo`yamiz

$$F_y - \frac{d}{dx}F_{y'} = -\frac{\sqrt{1+y'^2}}{y^2} - \frac{y''y(1+y'^2) - y'^2(1+y'^2) - yy'^2 y''}{y^2(1+y'^2)\sqrt{1+y'^2}} = 0,$$

$$-(1+y'^2)^2 - \left[y''y - y'^2 - y'^4\right] = 0,$$

$$-1 - 2y'^2 - y'^4 - y''y + y'^2 + y'^4 = 0,$$

$$yy'' + y'^2 = -1.$$

Oxirgi tenglama

$$(yy')' = -1$$

shaklda yozilishi mumkin. Uni bir marta integrallab,

$$yy' = -x + c_1$$

tenglamaga kelamiz.

Endi oxirgi tenglamani

$$ydy = (-x + c_1)$$

o\`zgaruvchilari ajraladigan tenglama ko\`rinishida yozib, integrallaymiz:

$$\frac{1}{2}y^2 = -\frac{x^2}{2} + c_1 x + c_2,$$

$$y^2 = -x^2 + 2c_1 x + 2c_2.$$

Agar berilgan $y(0) = 0$ chegaraviy shartdan foydalansak, $\Rightarrow c_2 = 0$ bo\`ladi.

a. Bu holda transversallik sharti, $\varphi' = 1$ bo\`lganligidan,

$$\left[F + (1-y')F_{y'}\right]_{x=x_1} = 0,$$

$$\left[\frac{\sqrt{1+y'^2}}{y} + \frac{(1-y')y'}{y\sqrt{1+y'^2}}\right]\Bigg|_{x=x_1} = 0,$$

$$(1 + y'^2 + y' - y'^2)\Big|_{x=x_1} = 0.$$

Buyerdan,

$$\Rightarrow 1 + y'(x_1) = 0 \Rightarrow y'(x_1) = -1.$$

Modomiki, $y^2 = -x^2 + 2c_1 x$ ekan,

$$2yy' = -2x + 2c_1.$$

U holda $x = x_1$ bo\`lganda

$$2y(x_1)y'(x_1) = -2x_1 + 2c_1,$$

$$-2(x_1 - 5) = -2x_1 + 2c_1,$$

$$\Rightarrow c_1 = 5.$$

Demak,

$$y^2 = -x^2 + 10x \qquad \text{yoki} \qquad y = \pm\sqrt{10x - x^2}.$$

b. Bu holda φ' o\`g\`ma

$$(x-9)^2 + y^2 = 9$$

aylanada hisoblansa,

$$2(x-9) + 2yy' = 0,$$

$$yy' = -(x-9)$$

munosabat hosil bo\`ladi. U holda $x = x_1$ bo\`lganda,

$$\varphi'(x_1) = -\frac{x_1 - 9}{y(x_1)}.$$

Endi yechimni esga olsak, $x = x_1$ bo`lganda $y(x_1)$ uchun

$$y(x_1)^2 = -x_1^{\ 2} + 2c_1 x_1$$

ifodani hosil qilamiz. Shuningdek, berilgan aylanadan

$$(x_1 - 9)^2 + y(x_1)^2 = 9$$

ifodani hosil qilamiz. Buyerdan,

$$-x_1^{\ 2} + 2c_1 x_1 = 9 - (x_1 - 9)^2$$

$$\Rightarrow x_1^{\ 2} + 2c_1 x_1 = 9 - x_1^{\ 2} + 18x_1 - 81,$$

$$\Rightarrow c_1 x_1 = 9x_1 - 36.(*)$$

Olingan ifodalarni

$$\left[F + (\varphi' - y')F_{y'}\right]_{x = x_1} = 0$$

transversallik shartiga keltirib qo`ysak,

$$1 + \varphi'(x_1) y'(x_1) = 0,$$

$$1 - \frac{x_1 - 9}{y(x_1)} \frac{-x_1 + c_1}{y(x_1)} = 0,$$

$$1 + \frac{(x_1 - 9)(x_1 - c_1)}{y^2(x_1)} = 0.$$

Buyerdan,

$$y^2(x_1) + (x_1 - 9)(x_1 - c_1) = 0$$

bo`ladi va $y(x_1)^2 = 9 - (x_1 - 9)^2$ ekanligini hisobga olsak,

$$9 - (x_1 - 9)^2 + (x_1 - 9)(x_1 - c_1) = 0,$$

$$9 - (x_1 - 9)\left[x_1 - 9 - x_1 + c_1\right] = 0.$$

Oxirgi tenglamani (*) bilan birgalikda yechsak,

$$c_1 = 9 - \frac{36}{x_1},$$

$$9 - (x_1 - 9)(-\frac{36}{x_1}) = 0,$$

$$9 + 36 = \frac{9 \cdot 36}{x_1},$$

$$x_1 = \frac{9 \cdot 36}{45} \Rightarrow x_1 = \frac{36}{5} \Rightarrow c_1 = 9 - 5 = 4,$$

$$\Rightarrow y^2 = -x^2 + 8x.$$

2.Chetlari erkin variatsion masalalar.

Yuqorida ko‘rib o‘tilgan variatsion hisobning asosiy masalasidagi

$$J(y) = \int_{x_0}^{x_1} F(x, y, y')dx$$

funksionalning ekstremumini $Y = C^1[x_0, x_1]$ fazoda izlash masalasini qaraymiz. Buyerda, chetlari qo‘zg‘olmas variatsion masaladan farqli ravishda, izlanayotgan funksiyaning oraliqning chetidagi qiymatlari haqida hech qanday ma’lumot berilmagan. Shuning uchun, bu masala, *chetlari erkin variatsion masala* deb ataladi.

Bu masalani, biz simvolik ravishda ,

$$\int_{x_0}^{x_1} F(x, y, y')dx \to \min(\max) \tag{12}$$

ko‘rinishda yozamiz.

Chetlari erkin masalaning yechimidan iborat $y_0(x)$ funksiya mavjud bo‘lsin. U holda, ekstremumning zaruriy shartiga asosan,

$$\delta J(y_0, h) \equiv 0 \quad \forall h \in Y,$$

yoki

$$\delta J(y_0, h) = \int_{x_0}^{x_1} [F_y(x, y_0, y_0')h + F_{y'}(x, y_0, y_0')h']dx \equiv 0 \ \ \forall h \in Y = C^1[x_0, x_1]. \tag{13}$$

bo`ladi

Oxirgi tenglikning ikkinchi integralini bo‘laklab integrallasak, ushbu

$$\int_{x_0}^{x_1} [F_y(x, y_0, y_0')h - \frac{d}{dx}F_{y'}(x, y_0, y_0')h']dx +$$

$$[F_y(x, y_0, y_0')h]_{x=x_1} - [F_{y'}(x, y_0, y_0')h']_{x=x_0} = 0 \ \forall h \in Y = C^1[x_0, x_1]$$

Chap tomondagi birinchi qo‘shiluvchi aynan nolga teng , chunki y_0 Eyler tenglamasini qanoatlantiradi. Shuning uchun,

$$F_{y'}(x_1, y_0(x_1), y_0'(x_1))h(x_1) - F_{y'}(x_1, y_0(x_1), y_0'(x_1))h(x_0) = 0 \quad \forall h \in Y = C^1[x_0, x_1]$$

Bu ayniyatga $h(x) = \dfrac{x_1 - x}{x_1 - x_0}$ *va* $h(x) = \dfrac{x - x_0}{x_1 - x_0}$ ifodalarni keltirib qo’ysak,

$$F_{y'}(x_0, y_0, y_0')\Big|_{x=x_0} = 0; \ F_{y'}(x_1, y_0, y_0')\Big|_{x=x_1} = 0 \tag{14}$$

ekanligiga ishonch hosil qilamiz.

Shunday qilib, agar $y_0 \in C^1[x_0, x_1]$ -chetlari erkin variatsion masalaning yechimi bo‘lsa, u

$$F_y(x, y, y') - \frac{d}{dx}F_{y'}(x, y, y') = 0 \tag{15}$$

$$F_{y'}(x, y, y')\Big|_{x=x_0} = 0; \quad F_{y'}(x, y, y')\Big|_{x=x_1} = 0; \tag{16}$$

chegaraviy masalaning yechimidan iborat bo‘lar ekan.

(16) chegaraviy shartlar – *tabiiy chegaraviy shartlar* deb ataladi.

(15), (16) chegaraviy masalaning yechimlari chetlari erkin masalaning *ekstremallari* deb ataladi.

4-misol. Ushbu

$$\int [(y'')^2 + (12x - 6)y]dx \to \min(\max)$$

masalaning ekstremallarini toping.

Yechilishi. Modomiki,

$$F_y = 12x - 6; \quad F_{y'} = 2y'$$

ekan , Eyler tenglamasi va tabiiy chegaraviy shartlar, mos ravishda,

$$y'' = 6x - 3$$

$$y'(0) = 0, \qquad y'(1) = 0$$

ko‘rinishni oladi. Eyler tenglamasining umumiy yechimi

$$y(x) = x^3 - \frac{3}{2}x^2 + C_1 x + C_2$$

bo‘ladi. $y'(x) = 3x^2 - 3x + C_1$ bo‘lganligidan, chegaraviy shartlardan, $C_1 = 0$ ekanligini olamiz.

Shunday qilib, masaaning ekstremallari

$$y(x) = x^3 - \frac{3}{2}x^2 + C$$

egri chiziqlar oilasidan iborat bo‘lar ekan.

5 - misol. Ushbu kvadratik funksional uchun qo’yilgan.

chetlari erkin masala uchun chegaraviy masalani yozing.

Yechilishi. Bu masalada

$$F(x, y, y') = \frac{1}{2}\left[p(x)y'^2 + q(x)y^2\right] - f(x)y$$

$$F_y(x, y, y') = q(x)y - f(x); \;\; F_{y'}(x, y, y') = p(x)y'$$

bo‘gani uchun, Eyler tenglamasi

$$q(x)y - \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) = f(x)$$

ko‘rinishda, tabiiy chegaraviy shartlar

$$p(x_0)y'(x_0) = 0 \;\; p(x_1)y'(x_1) = 0$$

munosabatlardan iborat bo‘ladi.

Mustaqil yechish uchun amaliy topshiriqlar.

1-masala. Agar egri chiziqlarning chap cheti $y(x) = \varphi(x) = ax^2 + 2$ egri chiziqning, o‘ng cheti esa, $y(x) = \psi(x) = a_1 x$ egri chiziqning nuqtalarida yotishi ma’lum bo‘lsa , ular ichida

$$J(y) = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx$$

funksional ekstremumga erishadigan egri chiziqni toping.

Variantlar

var	a	a_1	var	a	a_1	var	a	a_1	var	a	a_1	var	a	a_1
1	2	1	6	½	2	11	2	2	16	1/2	-1	21	1	3
2	3	2	7	2	½	12	-2	2	17	-1	1/2	22	2	3
3	-1	-1	8	1	½	13	2	-2	18	2	-1/2	23	3	2

4	1	-1	9	½	-1	14	-1	2	19	2	-1	24	-2	1
5	-1	1	10	-1	-1/2	15	2	-1	20	3	1	25	1	-2

2 – masala. Quyidagi

$$\begin{cases} J(y) = \int_{-1}^{1} (x^2 + y^2 + y'^2)dx \to \min(\max) \\ y(-1) = 1 \end{cases}$$

chap cheti qo‘zg‘olmas, o‘ng cheti esa, jadvalda berilgan $y = \psi(x)$ funksiya bo‘ylab o‘zgaradigan variatsion masalaning eketremalini toping.

Variantlar

Var	$\psi(x)$	Var	$\psi(x)$	Var	$\psi(x)$
1	$e^x - 2$	8	$e^x - 5$	15	$10 - e^x$
2	$e^{2x} - 9$	9	$-e^x + 5$	16	$50 - e^{\frac{1}{2}x}$
3	$e^{x+\frac{1}{2}} - 3$	10	$2e^{2x} - 2$	17	$10 - 2e^{x-2}$
4	$e^x - 10$	11	$2 - e^{2x}$	18	$50 - e^{x+3}$
5	2-e^x	12	$3 - e^x$	19	$2 + e^{2x}$
6	$e^{x+\frac{1}{2}} - 2$	13	$4 - e^x$	20	$e^{2x-2} - 5.$
7	$-e^{0.5x}$	14	$6 - e^x$	21	$e^{2x-1} - 5.$

3 – masala. Quyidagi

$$\begin{cases} J(y) = \int_{-1}^{1} (x^2 + y^2 + y'^2)dx \to \min(\max) \\ y(1) = 1 \end{cases}$$

o‘ng cheti qo‘zg‘olmas, chap heti esa, jadvalda berilgan $y = \varphi(x)$ funksiya bo‘ylab o‘zgaradigan variatsion masalaning eketremalini toping.

Variantlar

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3	$e^{x+\frac{1}{2}}-3$	10	$2e^{2x}-2$	17	$10-2e^{x-2}$
4	$e^{x}-10$	11	$2-e^{2x}$	18	$50-e^{x+3}$
5	$2-e^{x}$	12	$3-e^{x}$	19	$2+e^{2x}$
6	$e^{x+\frac{1}{2}}-2$	13	$4-e^{x}$	20	$e^{2x-2}-5.$
7	$-e^{0.5x}$	14	$6-e^{x}$	21	$e^{2x-1}-5.$

4- masala. Quyidagi chetlari erkin variatsion masala uchun chegaraviy masalani tuzing va yeching.

Variantlar

Var	$F(x,y,y'),x_0,x_1$	Var	$F(x,y,y'),x_0,x_1$	Var	$F(x,y,y'),x_0,x_1$
1	$(y^2-1)y'^2$;2,3.	8	y'^2; 0,1.	15	$6y'^2-y'^4+yy'$; 0,2.
2	$2y-x^2y'^2$;1,e.	9	$x^2y'^2+12y^2$; 1,2.	16	$x^2y-y+xy^2y'$; 0,2.
3	$y^2y'^2$; 0,1.	10	$y'^2+3yy'+24x^2y$; 0,1.	17	$6x^2y'+y'^2$; 0,1.
4	$y'^2+yy'+12xy$; 0,1.	11	$3y'^2+5yy'+12y$; 0,1.	18	$xy'^4-2yy'^3$; 1,2.
5	xy'^2+yy'; 1,e.	12	$xy'-y'^2$;0,2.	19	$(y')^2+yy'+y^2$; 0,1.
6	$x^2y'^2+12y^2$; 0,1.	13	$4y^2-y'^2+8y$; 0,$\frac{\pi}{4}$.	20	$x(y')^2-yy'+y$;0,1.
7	$xy'-y'^2$;0,1.	14	$\frac{1}{y'}$; 0,2.	21	$(y')^2+4xy'$;1,3.

1′ – masala. (1) masalada integrant $F(x,y,y')$ va x_0,x_1 lar berilgan. Uning uchun (15), (16) chegaraviy masalani yozing va ekstremallarini toping.

Var	$F(x,y,y')$	x_0	x_1	Var	$F(x,y,y')$	x_0	x_1
1	y'^2-y^2	0	1	11	$y-xy'^2$	1	e
2	y'^2-xy	0	1	12	$x^2y'^2$	1	2

3	$y'^2 - x^2y$	0	1	13	$xy'^2 + 2y^2 + 2xy$	1	e
4	xy'^2	1	e	14	$y^2 + y'^2 + 2ye^x$	-1	1
5	$xy'^2 + 2y$	1	e	15	$y'^2 + yy' - 16y^2$	0	1
6	$(1+x)y'^2$	0	1	16	$\frac{y'^2}{x^2}$	1	2
7	$xy'^2 + 2y$	1	e	17	$\frac{1+y^2}{y'^2}$	0	1
8	$y'(1+x^2y')$	1	3	18	$y^2 + y'^2 + 2ye^x$	-1	1
9	$xy' + y'^2$	0	1	19	$y^2 * y'^2$	0	1
10	$\frac{\sqrt{1+y'^2}}{y}$	0	1	20	$2y - x^2y'^2$	1	e

6-amaliy mashg‘ulot. Shartli ekstremumga qo‘yilgan variatsion masalalar. Lagranj ko‘paytuvchilar qoidasi

$$F^*(x,y,y') = F(x,y,y') + \sum_{j=1}^{m} \lambda_j(x)\varphi_j(x,y) \qquad (1)$$

belgilashni kiritsak, bunda $F^*(x,y,y')$ - *Lagranj funksiyasi*, $\lambda_j(x), j=\overline{1,m}$ - *Lagranj ko'paytuvchilari* deb ataladi, oxirgi tenglamani,

$$\int_{x_0}^{x_1} \sum_{i=1}^{n} [F_{y_i}{}^* - \frac{d}{dx} F_{y_i'}{}^*] \delta y_i(x) dx = 0 \qquad (2)$$

Ko`rinishda yozish mumkin.

m ta $\lambda_1(x),...,\lambda_m(x)$ ko`paytuvchilarni shunday tanlaylikki, ular $y^*(x)$ egri chiziq bilan birga m ta

$$F_{y_i}{}^* - \frac{d}{dx} F_{y_i'}{}^* = 0, \quad i = \overline{1,m} \qquad (3)$$

Eyler tenglamalari sistemasini qanoatlantirsin.

1-t e o r e m a $J[y_1*(x), y_2*(x),..., y_n*(x)] = \underset{y(x)\in M}{extr} \int_{x_0}^{x_1} F(x, y_1,..., y_n, y_1', y_2',..., y_n')dx$ masalada ekstremumning zaruriy sharti). Agar $y_i(x_0) = y_{i0}, y_i(x_1) = y_{i1}, i = \overline{1,n}$ chegaraviy shartlarni va $\varphi_j(x, y_1(x),..., y_n(x)) = 0, \ j = \overline{1,m}, \ m < n$ chekli bog’lanishlarni qanoatlantiruvchi $y*(x) = (y_1*(x),..., y_n*(x))^1$ vektor funksiyada, bunda $y_i(x) \in C^1[x_0, x_1], i = \overline{1,n},$ $J[y_1, y_2,..., y_n] = \int_{x_0}^{x_1} F(x, y_1,..., y_n, y_1', y_2',..., y_n')dx$ funksional ekstremumga erishsa, $y_1*(x),..., y_n*(x)$ funksiyalar.

$$J^*[y_1,...,y_n]=\int_{x_0}^{x_1}F^*(x,y_1,...,y_n,y_1',...,y_n')dx=\int_{x_0}^{x_1}[F(x,y_1,...,y_n,y_1',...,y_n')+\sum_{j=1}^{m}\lambda_j(x)\varphi_j(x,y_1,...,y_n)]dx$$

funksional uchun tuzilgan

$$F_{y_i}^{*}-\frac{d}{dx}F_{y_i'}^{*}=0,\quad i=\overline{1,n}$$

Eyler tenglamalari sistemasini qanoatlantiradi.

1. (1) masalada shartli ekstremumning zaruriy shartini qo`llash algoritmi.

1. $F^{**}(x,y,y)=F(x,y,y)+\sum_{j=1}^{m}\lambda_j(x)\varphi_j(x,y)$

Lagranj funksiyasini tuzish, $\lambda_1(x),...,\lambda_m(x)$- Lagranj ko`paytuvchilari.

2. (14) Eyler tenglamalari sistemasi va (2) bog`lanishlar shartlarini yozish:

$$F_{y_i}^{*}-\frac{d}{dx}F_{y_i'}^{*}=0,\quad i=\overline{1,n}$$

$$\varphi_j(x,y_1,...,y_n)=0,\ j=\overline{1,m}.$$

3.Eyler tenglamalari sistemasining

$$y_i(x)=y_i(x,c_1,c_2,...,c_{2n}),\ i=\overline{1,n}\ ,$$

umumiy yechimini va $\lambda_1(x),...,\lambda_m(x)$ Lagranj ko`paytuvchilari uchun ifodalarni topish.

4. $c_1,c_2,...,c_{2n}$ o`zgarmaslarni

$$y_{i0}=y_i(x_0,c_1,c_2,...,c_{2n}),\ i=\overline{1,n}$$

$$y_{i1}=y_i(x_0,c_1,c_2,...,c_{2n}),\ i=\overline{1,n}$$

chegaraviy shartlardan topish va $y^*(x)=(y_1^*(x),...,y_n^*(x))^1$ ekstremal uchun ifoda yozish (ekstremalni yozish).

1-misol. Ushbu

$$J[y_1,y_2]=\int_0^{\pi/2}[y_1^2+y_2^2-y_1'^2-y_2'^2]dx$$

Funksionalning

$$y_1(0)=1.y_2(0)=-1,y_1(\frac{\pi}{2})=1,y_2(\frac{\pi}{2})=1$$

chegaraviy shartlarni va

$$y_1-y_2-2\cos x=0$$

bog`lanishlarni qanoatlantiradigan ekstremalini toping.

Y e ch i l i sh i. 1. Lagranj funksiyasini tuzamiz. Modomiki,

$$F=y_1^2+y_2^2-y_1'^2-y_2'^2,y_1(x,y)=y_1-y_2-2\cos x,\ m=1$$

ekan,

$$F^*=F+\lambda_1(x)y_1(x,y)=y_1^2+y_2^2-y_1'^2-y_2'^2+\lambda_1(x)[y_1-y_2-2\cos x]$$

bo`ladi.

2.Eyler tenglamalari sistemasi va bog`lanishlar tenglamasini yozamiz. Buning uchun,

$$F_{y_1}^{*}=2y_1+\lambda_1(x),\ F_{y_1'}^{*}=-2y_1',\ \frac{d}{dx}F_{y_1'}^{*}=-2y_1'',$$

$$\overline{F_{y_1}^{*}}=2y_1-\lambda_1(x),\ F_{y_2'}^{*}=-2y_1',\ \frac{d}{dx}F_{y_2'}^{*}=-2y_1'',$$

ifodalardan foydalansak, quyidagi

$$F_{y_i}^{*}-\frac{d}{dx}F_{y_i'}^{*}=2y_1+\lambda_1(x)+2y_1''=0$$

$$F_{y_{2i}}^{*}-\frac{d}{dx}F_{y_2'}^{*}=2y_2+\lambda_1(x)+2y_2''=0$$

$$y_1-y_2-2\cos x=0$$

munosabatlarni hosil qilamiz.

3. Sistemaning umumiy yechimini topamiz. Sistemaning birinchi ikkita tenglamasini hadma- had qo`shib,

$$2(y_1''+y_2'')+2(y_1+y_2)=0$$

ekanligini olamiz. Yangi, $z=y_1+y_2$ belgilash kiritib,

$$z''+z=0$$

tenglamani hosil qilamiz. Uning xarakteristik tenglamasi

$$k^2+1=0$$

Bo`lib, u $k_{1,2}=\pm i$ ildizlarga ega ekanligini hisobga olsak,

$$z(x)=c_1\cos x+c_2\sin x=y_1+y_2$$

Bo`ladi.

Ikkinchidan, hosil qilingan sistemaning uchinchi tenglamasidan

$$2\cos x=y_1-y_2$$

Bo`lishi kelib chiqadi. Oxirgi tenglamalarni qo`shib,

$$2y_1=c_1\cos x+c_2\sin x+2\cos x \text{ yoki } y_1(x)=\frac{c_1}{2}\cos x+\frac{c_2}{2}\sin x+\cos x$$

ekanligini olamiz. U holda

$$y_2(x)=y_1(x)-2\cos x,$$

$$\lambda_1(x)=2y_2(x)+2y_2''(x)$$

Bo`lishi kelib chiqadi.

4. c_1 va c_2 ixtiyoriy o`zgarmaslarni chegaraviy shartlardan topamiz:

$$y_1(0)=\frac{c_1}{2}+1=1,$$

$$y_1(\frac{\pi}{2})=\frac{c_2}{2}=1$$

$$y_2^{*}(x)=y_1^{*}(x)-2\cos t=\sin x-\cos x,$$

$$\lambda_1(x)=2\sin x-2\cos x-2\sin x+2\cos x=0$$

bu yerdan $c_1=0,\ c_2=2$ va $y_1^{*}(x)=\sin x+\cos x,$

Shuni e'tirof etish kerakki, masalada chegaraviy shartlar va bog`lanishlar tenglamalari muvofiqlashtirilgan, chunki

$$y_1(0)-y_2(0)-2\cos 0=0,$$

$$y_1(\frac{\pi}{2})-y_2(\frac{\pi}{2})-2\cos\frac{\pi}{2}=0\,.$$

Bu faktni masalani yechishdan oldin tekshirish lozim. Shunday qilib, masalada

$$y_1^{*}(x)=\sin x+\cos x,\ y_2^{*}(x)=\sin x-\cos x$$

ekstremal topildi.

Differensial bog'lanishli variatsion masalalar.

Masalaning qo'yilishi. Quyidagi shartlarni qanoatlantiruvchi joiz $y(x)=(y_1(x),...,y_n(x))'$ vector- funksiyalar to`plami m ni qaraymiz:

a) $y_i(x)$ funksiyalar $[x_0,x_1]$ kesmada aniqlangan va uzluksiz differensiallanuvchi bo`lsin, ya'ni $y_i(x)\in C^1[x_0,x_1],\ i=\overline{1,n},\ x_0,x_1$ lar berilgan;

b) $y_i(x)$ funksiyalar,

$$y_i(x_0)=y_{i0},\ \ y_i(x_1)=y_{i1},\ \ i=\overline{1,n} \qquad (4)$$

chegaraviy shartlarni qanoatlantiradi, bunda $y_{i0},y_{i1},\ i=\overline{1,n}$ sonlar berilgan, ya'ni egri chiziqlarning har biri ikkita mahkamlangan(qo`zg`olmas) chegara nuqtalardan o`tadi;

c) $y_i(x)$ funksiyalar barcha $x\in[x_0,x_1]$ lar uchun

$$\varphi_j(x,y_1,...,y_n,y_1',...,y_n')=0,\ \ j=\overline{1,m};\ m<n \qquad (5)$$

differensial bog`lanishlarni qanoatlantiradi, bunda, $\varphi_j(x,y_1,...,y_n,y_1',...,y_n')$ funksiyalar barcha o`zgaruvchilari bo`yicha uzluksiz differensiallanuvchidir.

Bundan tashqari, (16) tenglamalar o`zaro bog`lanmagan, ya'ni

$$rang\begin{Bmatrix}\frac{\partial\varphi_1}{\partial y_1'}......\frac{\partial\varphi_1}{\partial y_n'}\\ ....\\ \frac{\partial\varphi_m}{\partial y_1'}......\frac{\partial\varphi_m}{\partial y_n'}\end{Bmatrix}=m$$

M to'plamda

$$J[y_1,y_2,...,y_n]=\int_{x_0}^{x_1}F(x,y_1,...,y_n,y_1',y_2',...,y_n')dx \qquad (6)$$

funksional berilgan

(6) funksional ekstremumga erishsin, ya'ni

$$J[y_1*(x),y_2*(x),...,y_n*(x)]=\underset{y(x)\in M}{extr}\int_{x_0}^{x_1}F(x,y_1,...,y_n,y_1',y_2',...,y_n')dx \qquad (7)$$

bo'lsin.

Bu masala - *Lagranj masalasi* deb ataladi.

Masala yechimini izlash tartibi (sxemasi).

funksionalning birinchi variatsiyasi ifodasi

$$\delta J=\int_{x_0}^{x_1}\sum_{i=1}^{n}[F_{y_i}-\frac{d}{dx}F_{y_i'}]\delta y_i(x)dx \qquad (8)$$

ko`rinishda bo`ladi

2-te o r e m a ((7) masalada ekstremumning zaruriy shartlari). Agar (4) chegaraviy shartlar va (5) differensial bog'lanishlarni qanoatlantiruvchi $y*(x)=(y_1*(x),...,y_n*(x))^1$ vector funksiyada (6) funksional ekstremumga erishsa, $y_1*(x),...,y_n*(x)$ funksiyalar,

$$J^*[y_1,...,y_n]=\int_{x_0}^{x_1}F^*(x,y_1,...,y_n,y_1',...,y_n')dx=$$

$$=\int_{x_0}^{x_1}[F(x,y_1,...,y_n,y_1',...,y_n')+\sum_{j=1}^{m}\lambda_j(x)y_j(x,y_1,...,y_n,y_1',...,y_n')]dx$$

funksional uchun tuzilgan,

$$F_{y_i}^{\ *}-\frac{d}{dx}F_{y_i'}^{\ *}=0,\quad i=\overline{1,n}$$

Eyler tenglamalari sistemasini qanoatlantiradi.

3. (7) masalada ekstremumning zaruriy shartini qo'llash algoritmi.

1. $F^*(x,y,y')=F(x,y,y')+\sum_{j=1}^{m}\lambda_j(x)y_j(x,y,y')$

Lagranj funksiyasini tuzish, bunda $\lambda_1(x),...,\lambda_m(x)$ - Lagranj ko'paytuvchilari.

2. Eyler tenglamalari sistemasi va (5) bog'lanishlar tenglamalarini yozish:

$$F_{y_i}^{\ *}-\frac{d}{dx}F_{y_i'}^{\ *}=0,\quad i=\overline{1,n}$$

$$\varphi_j(x,y_1,...,y_n,y_1',...,y_n')=0,\ \ j=\overline{1,m}$$

3. Eyler tenglamalari sistemasining $y_i(x)=y_i(x,c_1,c_2,...,c_{2n}),\ \ i=\overline{1,n}$, umumiy yechimini hamda $\lambda_1(x),...,\lambda_m(x)$ Lagranj ko`paytuvchilari uchun ifodalarni topish.

4. $c_1,c_2,...,c_{2n}$ o`zgarmaslarni

$$y_{i0}=y_i(x_0,c_1,c_2,...,c_{2n}),\ \ i=\overline{1,n},$$

$$y_{i1}=y_i(x_0,c_1,c_2,...,c_{2n}),\ \ i=\overline{1,n}$$

chegaraviy shartlardan topish va $y^*(x)=(y_1^*(x),...,y_n^*(x))^1$ ekstremal uchun ifoda (ekstremalni) yozish.

2-misol . Ushbu

$$J[y_1,y_2]=\int_0^1[y_1^2+y_2^2-y_1'-y_2'^2]dx$$

funksionalning

$$y_1(0)=2,y_2(0)=0,y_1(1)=2ch1,y_2(1)=2sh1$$

chegaraviy shartlarni va

$$y_1'-y_2=0$$

differensial boglanishni qanoatlantruvchi ekstremalini toping.

Eechilishi. 1.Lagranj funksiyasini tuzamiz:
$F(x,y,y')=y_1'^2+y_2'^2,Y_1(x,y,y')=y_1'-y_2,\ m=1$
bo'lganligidan,

$$F^*(x,y,y')=y_1'^2+y_2'^2+\lambda_1(x)\left[y_1'-y_2\right].$$

2.Eyler tenglamalari sestemasini va boglanishlar tenglamasini yozamiz;

$$F_{y_1}^{\ *}=0,,F_{y_1'}^{\ *}=2y_1''+\lambda_1(x),\frac{d}{dx}F_{y_1'}^{\ *}=2y''_1+\lambda_1(x)$$

$$F^*_{y_2}=-\lambda_1(x),F_{y_2'}^{\ *}=2y_2'',\frac{d}{dx}F^*_{y_2'}=2y_2''$$

ekanligini hisobga olsak,

$$\begin{cases} F_{y_1}{}^* - \dfrac{d}{dx} F_{y'_1}{}^* = 0 \Rightarrow \begin{cases} -2y_1''-\lambda_1'(x) = 0 \\ -\lambda_1(x) - 2y_2'' = 0 \end{cases} \\ F_{y_2}{}^* - \dfrac{d}{dx} F_{y_2{}^1}{}^* = 0 \Rightarrow \end{cases}$$

$$y_1' - y_2 = o$$

bo'ladi.

3. Hosil qilingan sistemaning umumiy echimini topamiz. Sistemaning daslabki ikkita tenglamalaridan,

$$\lambda_1(x) = -2y_2'', \lambda_1'(x) = -2y''', 2y_2'' = -\lambda_1'(x) = 2y_2'''$$

ekanligini olamiz, uchinchi tenglamadan esa,

$$y_1' = y_2, y_1'' = y_2'$$

bo'lishi kelib chiqadi. U holda, $2y_1'' = 2y_2' = 2y_2'''$ yoki $y_2''' - y_2' = 0$ bo'ladi. Oxirgi tenglamaning $\lambda^3 - \lambda = 0,, \lambda(x^2 - 1) = 0$ xarakteristik tenglamasi $\lambda_1 = 1, x_2 = -1, \lambda_3 = 0$ ildizlarga ega, shuning uchun,

$$y_2(x) = c_1 e^x + c_2 e^{-x} + c_3,$$

$$y_1(x) = \int y_2(x) dx = c_1 e^x - c_2 e^{-x} + c_3 x + c_4,$$

$$\lambda_1(x) = -2y_2''(x).$$

4. $c_1, c_2, c_3, c_4,$ o'zgarmaslarni chegaraviy shartlardan topamiz;

$$\begin{cases} y_1(0) = c_1 - c_2 + c_4 = 2 \\ y_2(0) = c_1 + c_2 + c_3 = 0, \\ y_1(1) = c_1 e - c_2 e^{-1} + c_3 + c_4 = ch1.ch1 = \dfrac{e + e^{-1}}{2}. \\ y_2(1) = c_1 e + c_2 e^{-2} + c_3 = 2sh1, sh1 = \dfrac{e - e^{-1}}{2} \end{cases}$$

Bu erdan $c_1 = 1, c_2 = -1; c_3 = c_4 = 0$ bo'lishi kelib chiqadi.

Natijada, $y^*(x) = (y^*{}_1(x), y^*{}_2(x))^T$ ekstremal, $y^*{}_1(x) = e^x + e^{-x}; y^*{}_2(x) = e^x - e^{-x}$ ko'rinishda bo'ladi va $\lambda_1(x) = -2y''{}_2(x) = -2e^x + 2e^{-x}$.

Mustaqil ishlash uchun misollar

1-misol. Ushbu

$$J[y_1, y_2] = \int_0^1 [y_1{}^2 + ay_2{}^2 - by_1'^2 + y_2'^2]\, dx$$

funksionalning

$$y_1(0) = 1. y_2(0) = -1, y_1(\frac{\pi}{2}) = 1, y_2(\frac{\pi}{2}) = 1$$

chegaraviy shartlarni va

$$y_1 - y_2 + 2sinx = 0$$

bog`lanishlarni qanoatlantiradigan ekstremalini toping.

a va b lar quyidagi jadvalda keltirilgan

var	a	b	var	a	b	var	a	b	var	a	b	var	a	b
1	2	1	6	1/2	2	11	2	2	16	1/2	-1	21	1	3
2	3	2	7	2	1/2	12	-2	2	17	-1	1/2	22	2	3
3	-1	-1	8	1	1/2	13	2	-2	18	2	-1/2	23	3	2
4	1	-1	9	1/2	-1	14	-1	2	19	2	-1	24	-2	1
5	-1	1	10	-1	-1/2	15	2	-1	20	3	1	25	1	-2

2-misol . Ushbu

$$J[y_1, y_2] = \int_{-1}^{1} [Ay_1{}^2 + y_2{}^2 - y_1{}' + By_2{}'^2]\, dx$$

funksionalning

$$y_1(-1) = 1, y_2(-1) = 0, y_1(1) = 2, y_2(1) = 1$$

chegaraviy shartlarni va

$$y_1' - y_2 = 0$$

differensial boglanishni qanoatlantruvchi ekstremalini toping.

A va B lar quydagi jadvalda berilgan.

var	A	B	var	A	B	var	A	B	var	A	B	var	A	B
1	2	1	6	1/2	2	11	2	2	16	1/2	-1	21	1	3
2	3	2	7	2	1/2	12	-2	2	17	-1	1/2	22	2	3
3	-1	-1	8	1	1/2	13	2	-2	18	2	-1/2	23	3	2
4	1	-1	9	1/2	-1	14	-1	2	19	2	-1	24	-2	1
5	-1	1	10	-1	-1/2	15	2	-1	20	3	1	25	1	-2

7-amaliy mashg‘ulot. Izoperimetrik masalalar

1. Lagranj funksiyasi. Lagranj ko’paytuvchilari qoidasi.

$Q \subset R^3 -$ ochiq to’plam, $S = \{(x, y) : (x, y, z) \in Q\}$, $F_i(x, y, z)$, $i = \overline{0, m}$, funksiyalar Q da uzluksiz, $P_0(x_0, y_0)$, $P_1(x_1, y_1) - S$ to’plamning belgilangan nuqtalari, $x_0 < x_1$ bo’lsin.

Quyidagi:

$$J_0[y] = \int_{x_0}^{x_1} F_0(x, y, y')dx \to \min(\max), \qquad (1)$$

$$J_i[y] = \int_{x_0}^{x_1} F_i(x, y, y')dx = a_i, \; i = \overline{1, m}, \; (a_i = const), \qquad (2)$$

$$y(x_0) = y_0, \; y(x_1) = y_1, \; (x, y(x), y'(x)) \in Q, \; x \in [x_0, x_1], \; y(x) \in C^{(1)}[x_0, x_1]. \qquad (3)$$

ekstremal masalani qaraymiz.

Bu masalaga izoperimetrik masala deyiladi. (2) shartlarga esa, izoperimetrik shartlar (bog`lanishlar) deyiladi. (2), (3) shartlarni qanoatlantiruvchi $y(x) \in C^1[x_0, x_1]$ funksiyalar izoperimetrik masalada joyiz funksiyalar (chiziqlar)dan iborat.

Yechish qoidasi.

1. Lagranj funksiyasini tuzamiz:

$$L = L(t, x, \dot{x}, \lambda) = \sum_{i=0}^{m} \lambda_i f_i(t, x, \dot{x}), \qquad \lambda = (\lambda_0, \lambda_1, ..., \lambda_m).$$

2. L Lagranj uchun ekstremumning zaruriy sharti-Eyler tenglamasini yozish:

$$-\frac{d}{dt}\hat{L}_{\dot{x}}(t) + \hat{L}_x(t) = 0 \Leftrightarrow -\frac{d}{dt}\left(\sum_{i=0}^{m} \lambda_i \hat{f}_{i\dot{x}}(t)\right) + \sum_{i=0}^{m} \lambda_i \hat{f}_{ix}(t) = 0.$$

3. Joiz ekstremalni topish, ya’ni L Lagranj uchun (yozilgan) Eyler tenglamasining, Lagranj ko’paytuvchilari vektori λ nulga teng bo’lmaganda, joiz yechimlarini topish. Bunda $\lambda_0 = 0$ va $\lambda_0 \neq 0$ bo’lgan hollarning har birini alohida qarab chiqish foydali bo’ladi.

4. Topilgan jioz ekstremallar ichida yechimni izlab toppish yoki yechim mavjud emasligini isbotlash.

1-misol.

$$\int_0^1 \dot{x}^2 dt \to \text{extr}; \qquad \int_0^1 x\,dt = 0, \quad x(0)=0, \quad x(1)=1.$$

1. Lagranj funksiyasi:

$L = \lambda_0 \dot{x}^2 + \lambda x.$

2. Zaruriy shart- Eyler tenglamasi:

$$-\frac{d}{dt}L_{\dot{x}}(t) + L_x(t) = 0 \Leftrightarrow -2\lambda_0 \ddot{x} + \lambda = 0\,.$$

3. Agar $\lambda_0 = 0$ bo'lsa $\lambda = 0$ - barcha Lagranj ko'paytuvchilari nollardan iborat bo'ladi. Bu holda joiz ekstremallar yo'q. $\lambda_0 = 1/2$ deb olamiz . U holda , $\ddot{x} = \lambda$ va umumiy yechim $x = C_1 t^2 + C_2 t + C_3$ dan iborat. Umumiy yechimdagi noma'lum C_1, C_2, C_3 o'zgarmaslarni , oraliqning chetlaridagi shartlar hamda izoperimetrik shartlardan topamiz.

$$\left.\begin{aligned} x(0)=0 &\Leftrightarrow C_3 = 0, \\ x(1)=1 &\Leftrightarrow C_1 + C_2 = 1, \\ \int_0^1 x\,dt = 0 \Leftrightarrow \int_0^1 \left(C_1 t^2 + C_2 t\right) dt = 0 &\Leftrightarrow \frac{C_1}{3} + \frac{C_2}{2} = 0 \end{aligned}\right\} \Leftrightarrow \begin{aligned} C_1 &= 3, \\ C_2 &= -2. \end{aligned}$$

Masalada yagona joiz ekstremal $\hat{x} = 3t^2 - 2t$ mavjud.

4. $\hat{x}(\cdot)$ funksiy funksionlga absolyut mimimum berishini ko'rsatamiz. Haqiqatdan , $x(\cdot)$ jioz funksiyani olamiz, u holda, $x(\cdot) - \hat{x}(\cdot) = h(\cdot) \in C_0^1([0,1])$ va $\int_0^1 h\,dt = 0$. So'ngra ,

$$\mathrm{J}(x(\cdot)) - \mathrm{J}(\hat{x}(\cdot)) = \int_0^1 \left(\dot{\hat{x}} + \dot{h}\right)^2 dt - \int_0^1 \dot{\hat{x}}^2 = \int_0^1 2\dot{\hat{x}}\dot{h}\,dt + \int_0^1 \dot{h}^2 dt \ge 2\int_0^1 \dot{\hat{x}}\dot{h}\,dt.$$

Bo'laklab integrallab,

$$\int_0^1 \dot{\hat{x}}\dot{h}\,dt = \int \dot{\hat{x}}\,dh = \dot{\hat{x}}(t)h(t)\Big|_0^1 - \int \ddot{\hat{x}} h\,dt = -6\int_0^1 h\,dt = 0\,.$$

ekanligini olamiz. Shunday qilib, ixtiyoriy joiz $x(\cdot)$ funksiya uchun, $\mathrm{J}(x(\cdot)) \ge \mathrm{J}(\hat{x}(\cdot))$, $S_{\min} = \int_0^1 \dot{\hat{x}}^2\,dt = \int_0^1 (6t-2)^2 dt = 4$ bo'lishini olish qiyin emas.

Javob: $\hat{x} = 3t^2 - 2t$ funksiya masalada absolyut minimum beradi, masalaning qiymati $S_{\min} = 4$ ravshanki $S_{\max} = +\infty$.

2-misol.

$$\int_0^0 \left(y'^2 - y^2\right)dx \to \min, \quad \int_0^1 ydx = 0, \quad y(0) = y(1) = 0.$$

Lagranj ko'paytuvchilari qoidasini qo'llaymiz.

$$F_0 = y'^2 - y^2,\ F_1 = y,\ C^*{}_1 = F_{1y} - \frac{d}{dx}F_{1y} = 1.$$

Demak, $L = \lambda_0 F_0 + \lambda_1 F_1$ Lagranj funksiyasida $\lambda_0 = 1$ deb olish mumkin. Qulaylik uchun $\lambda = \lambda_1$ deb belgilab, $L = F_0 + \lambda F_1 = y'^2 - y^2 + \lambda y$ Lagranj funksiyasi uchun Eyler tenglamasini yozamiz:

$$L_y - \frac{d}{dx}L_{y'} = 0 \Leftrightarrow -2y + \lambda - \frac{d}{dx}(2y') = 0 \Rightarrow y'' + y - \frac{\lambda}{2} = 0.$$

Tuzilgan tenglamaning umumiy yechimi

$$y(x) = c_1 \sin x + c_2 \cos x + \frac{\lambda}{2}.$$

bo'ladi. Chegaraviy va izoperimetrik shartlardan foydalanib, $c_1,\ c_{2,}\ \lambda$ o'zgarmaslarni topamiz. $y(0) = 0 \Rightarrow \frac{\lambda}{2} = -c_2 \Rightarrow y(x) = c_1 \sin x + c_2(\cos x \cdot 1)$.

$$\left.\begin{array}{l} y(1) = 0 \\ \int_{10} ydx = 0 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} c_1 \sin 1 + c_2(\cos 1 - 1) = 0 \\ c_1(1 - \cos 1) + c_2(\sin 1 - 1) = 0 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 0, \\ c_2 = 0. \end{array}$$

Demak, *y(x)**=0* yagona shartli - stasionar funksiyadir.

Mustaqil yechish uchun misollar.

Ushbu

$$J(y) = \int_{-1}^{1} \left(x^2 + y^2 + y'^2\right)dx; \quad \begin{cases} y(-1) = 1; \\ y(1) = 2. \end{cases}$$

funksionalning quyidagi izoperimetrik shartni qanoatlantiruvchi joiz ekstremalini toping.

1. $\int_{x_1}^{x_2} ydx = 1$

2. $\int_{x_1}^{x_2} ydx = 4$

3. $\int_{x_1}^{x_2} y\,dx = 0.5$

4. $\int_{x_1}^{x_2} y\,dx = 2$

5. $\int_{x_1}^{x_2} y\,dx = 1$

6. $\int_{x_1}^{x_2} y\,dx = 1$

7. $\int_{x_1}^{x_2} y\,dx = 3$

8. $\int_{x_1}^{x_2} y\,dx = 3$

9. $\int_{x_1}^{x_2} y\,dx = 2$

10. $\int_{x_1}^{x_2} y\,dx = 2$

11. $\int_{x_1}^{x_2} y\,dx = 2$

12. $\int_{x_1}^{x_2} y\,dx = 6$

13. $\int_{x_1}^{x_2} y\,dx = 4$

14. $\int_{x_1}^{x_2} y\,dx = 4$

15. $\int_{x_1}^{x_2} y\,dx = 1$

16. $\int_{x_1}^{x_2} (x + y)\,dx = 4$

17. $\int_{x_1}^{x_2} y\,dx = 6$

18. $\int_{x_1}^{x_2} (x^2 + y)\,dx = 4$

19. $\int_{x_1}^{x_2} (x^2 + y)\,dx = 5$

20. $\int_{x_1}^{x_2} (\sin x + y)\,dx = 5$

21. $\int_{x_1}^{x_2} (\sin x + y)\,dx = 5$

22. $\int_{x_1}^{x_2} (x^2 + y)\,dx = 8$

23. $\int_{x_1}^{x_2} (y + y')\,dx = 3$

24. $\int_{x_1}^{x_2} (y + y')\,dx = 3$

25. $\int_{x_1}^{x_2}(y+y')dx=16$ 26. $\int_{x_1}^{x_2}(x+y+y')dx=8$

27. $\int_{x_1}^{x_2}(x+y+y')dx=8$ 28. $\int_{x_1}^{x_2}(x^2+y+y')dx=6$

29. $\int_{x_1}^{x_2}(\sin x+y+y')dx=10$ 30. $\int_{x_1}^{x_2}(\sin x+y+y')dx=8$

8-amaliy mashg'ulot. Optimal boshqaruv masalasi. Tezkor masalaga doir soda misol. Optimal boshqaruv masalasining umumiy qo'yilishi, asosiy muammolar

1.Optimal boshqaruv masalasiga sodda misol.

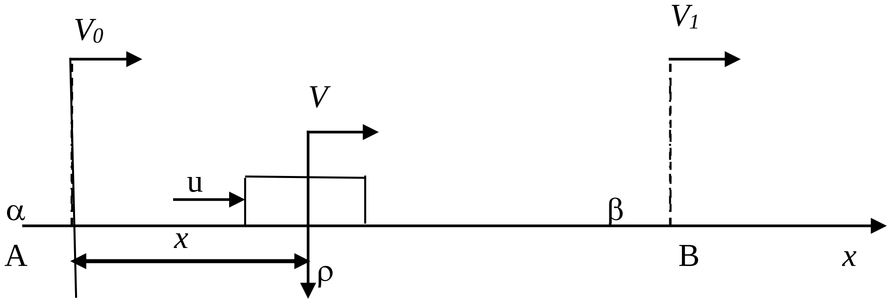
Avvalo optimal boshqaruv amaliy masalalaridan birini keltiramiz:

v_0 boshlang'ich tezlikka ega bo'lgan birlik massali material nuqtani modul bo'yicha birdan oshmaydigan kuch ta'sirida gorizontal to'g'ri chiziq bo'ylab *A* nuqtadan B nuqtaga shunday ko'chirish talab qilinadiki, bunda material nuqta B nuqtaga v_1 tezlik bilan eng qisqa vaqtda yetib kelsin.

Qo'yilgan masala tezkor harakat bo'yicha optimal boshqaruv masalasidan iborat. Uning matematik modelini tuzamiz.

Ox o'qda *A(α)* va B*(β)* nuqtalarni olaylik. Material nuqta $t=t_0$ boshlang'ich vaqtda *A* nuqtada, $t=t_1(t_1>t_0)$ vaqtda esa B nuqtada bo'lsin. (1-rasm)

$T=t_1-t_0$ material nuqtaning ko'chish vaqtidan iborat.



1-rasm.

$x=x(t)$-material nuqtaning *t* vaqtda bosib o'tgan yo'li, $u=u(t)$ material nuqtaga *t* vaqt momentida ta'sir etayotgan kuch miqdori bo'lsin.

U vaqtda $\dot{x} = \dfrac{dx}{dt} = v$ - material nuqtaning tezligi, $\ddot{x} = \dfrac{d^2x}{dt^2} = a$ material nuqtaning tezlanishi bo'ladi.

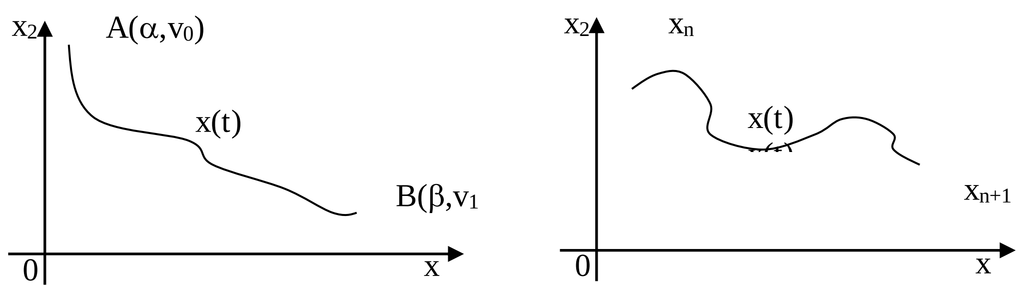
Nyutonning ikkinchi qonuniga ko'ra $m\alpha = u$ tenglik o'rinli, bu yerda m –material nuqtaning massasi $m = 1, a = \ddot{x}$ ekanligini hisobga olsak,

$$\ddot{x} = u \qquad (1)$$

tenglamaga ega bo'lamiz. Masalaning qo'yilishiga ko'ra,

$$\left.\begin{array}{ll} x(t_0) = \alpha, & \dot{x}(t_0) = v_0 \\ x(t_1) = \beta, & \dot{x}(t_1) = v_1 \end{array}\right\} \qquad (2)$$

shartlar kelib chiqadi(2-rasm).



2-rasm.

Bundan tashqari, *u(t)* kuchga

$$|u(t) \le t,\ t \in [t_0, t_1]| \qquad (3)$$

boshqarish funksiyasi (qisqacha, boshqarish) deyiladi. Odatda u, bo'lakli-uzluksiz funksiyalar sinfidan deb qaraladi. Bunday funksiyalar joyiz boshqarishlar sinfini tashkil etadi.

Shunday qilib, qo'yilgan masalaning matematik modeli quyidagicha: shunday $|u^*(t),\ t \in [t_0, t_1^*]|$ joyiz boshqarishni topish talab qilinadiki, (1) tenglamaning unga mos keluvchi $x^*(t)$ yechimi (2) shartlarni qanoatlantirsin va bunda ko'chish vaqti $T = t_1^* - t_0$ minimal bo'lsin.

$x_1 = x,\ x_2 = \dot{x}$ o'zgaruvchilarni kiritib, bu masalani

$$\left.\begin{array}{l} T(u) = t_1 - t_0 \to \min, \\ \dot{x}_1 = x_2, \dot{x}_2 = u, \\ x_1(t_0) = u, x_1(t_1) = \beta, \\ x_2(t_0) = v_0, x_2(t_1) = v_1, \\ |u| \le 1 \end{array}\right\} \qquad (4)$$

ko'rinishda yozish mumkin.

Endi yuqorida ko'rsatilgan differensial bog'lanishlar qatnashmagan optimal boshqaruv masalasini qaraymiz. Bunday masalalar optimal boshqaruvning elementar masalalari deyiladi.

Masalaning qo'yilishi. Optimal boshqaruvning elementar masalasi deb, boshqarish funksiyasiga bog'liq funksionalning minimumini, faqat boshqarishga qo'yilgan bog'lanishlar qatnashganda topish masalasiga aytiladi.

Quyidagi, dinamik bog'lanishlar qatnashmagan, ushbu

$$J(u)=\int_{t_0}^{t_1} h(t,u(t))dt \to \inf \qquad (1)$$

funksionalning minimumini, boshqarishga

$$u(t)\in U(t)\subset R^r, t\in[t_0,t_1]=I, \qquad (2)$$

bog'lanishlar qo'yilganda topish haqidagi ekstremal masalani qaraymiz. Bunda $u(t): I\to R^r$ - I da bo'lakli uzluksiz, $U(t)$ esa, ixtiyoriy $t\in I$ lar uchun R^r fazoning kompakt qism to'plamidan iborat. $h(t,u): I\times R^r\to R$ funksiya t bo'yicha o'lchovli va $u\in U(t)$.bo'yicha uzluksiz funksiyadir.

Teorema(minimumning zaruriy va yetarli sharti). $\bar{u}(t), t\in I$ funksiya (1),(2) masalaning yechimi bo'lishi uchun, u o'zining barcha uzluksizlik nuqtalarida

$$h(t,\bar{u}(t))=\min_{u\in U(t)} h(t,u) \quad t\in I, \qquad (3)$$

minimum shartini qanoatlantirishi zarur va yetarli.

1-misol. Ushbu

$$I(u)=\int_0^T (u^2+tu)dt,$$

funksionalning minimumini, boshqarishga

$$|u(t)|\le 1, t\in[0,T], T>2$$

bog'lanish qatnashganda toping.

Yechilishi. Qaralayotgan misolda $h(t,u)=u^2+tu$ ekanligini hisobga olib, (3) minimum prinsipini yozamiz:

$$\bar{u}^2(t)+t\bar{u}(t)=\min_{|u|\le 1}(u^2+tu).$$

Bu prinsipga binoan, har bir tayinlangan $t\in[0,2]$uchun quyidagi

$$u^2+tu\to\inf, |u|\le 1$$

chekli o'lchovli optimallashtirish masalasini yechish lozim bo'ladi.

Quyidagi

$$\varphi(u)=u^2+tu$$

belgilashni kiritamiz. Bu funksiyaning hosilasini olib va uni nolga tenglashtirib, $\varphi(u)$ funksiyaning minimumini topamiz, unga $\bar{u}(t)=-\frac{t}{2}$ nuqtada erishiladi. $|u(t)|\le 1$ bog'lanishni hisobga olib,

$$u(t)=\begin{cases}-\frac{t}{2}, t\in[0,2]\\ -1, t\in[2,T)\end{cases},$$

optimal boshqarishni hosil qilamiz.

Endi funksionalning minimal qiymatini hisoblaymiz:

$$J(u)=\int_0^2\left(\frac{t^2}{4}-\frac{t^2}{2}\right)dt+\int_2^T(1-t)dt=T-\frac{T^2}{2}-\frac{8}{12}.$$

2-misol. Ushbu

$$I(u)=\int_0^1 (u_1^2+u_2^2)dt,$$

funksionalning minimumini, boshqarishga

$$u_1\ge 1,\ \forall t\in[0,1],\ u_2\in R$$

bog'lanishlar qatnashganda toping.

Yechilishi. Bu masalada $h(u)=u_1^2+u_2^2$. Uning uchun (3) minimum prinsipini yozamiz:

$$\bar{u}_1^2+\bar{u}_2^2=\max_{u_1\ge 1}(u_1^2+u_2^2)=\max_{u_1\ge 1}u_1^2+\max_{u_2\in R}u_2^2.$$

Ravshanki, buyerda optimal boshqarish $[0,1]$ kesmaning barcha nuqtalarida $\bar{u}_1(t)=1,\ \ \bar{u}_2(t)=0$ qiymatlarni qabul qiladi. Funksionalning minimal qiymati esa, $I(\bar{u})=1$.

3-misol. Ushbu

$$I(u)=\int_0^T (4u_1+3u_2)dt$$

funksionalning minimumini

$$u_1^2+u_2^2\le 1,\ t\in[0,T]$$

bog'lanishlar qatnashganda toping.

Yechilishi. Bu masalada $h(u)=4u_1+3u_2$. Uning uchun (3) minimum prinsipini yozamiz:

$$\bar{u}_1(t)+3\bar{u}_2(t)=\min_{u_1^2+u_2^2\le 1}(4u_1+3u_2),t\in[0,T].$$

Boshqarishning joiz qiymatlari to'plami (u_1,u_2) tekislikda radiusi 1 ga teng bo'lgan doiradan iborat. Optimal boshqarish chekli o'lchovli ekstremal masalaning yechimidan iborat. Yechimning geometrik talqinidan foydalanib, u quyidagi

$$\begin{cases} 4u_1+3u_2=c, \\ u_1^2+u_2^2-1=0. \end{cases}$$

algebraik tenglamalar sistemasini qanoatlantirishini olamiz. Bu sistemadan u_2 ni yo'qotib,

$$\frac{25}{9}u_1^2-\frac{8}{9}cu_1+\frac{c^2}{9}-1=0,$$

kvadrat tenglamani hosil qilamiz. Uning diskriminanti

$$D=\frac{16}{81}c^2-\frac{25}{81}c^2+\frac{25}{9}=-\frac{1}{9}c^2+\frac{25}{9}.$$

Kvadrat tenglama yagona yechimga ega bo'lishi uchun, $D=0$ bo'lishi zarur. Bu hol, faqat $c^2=25$ bo'lganda yuz Beradi. Izlanayotgan miqdorning qiymati minimal bo'lishi kerakligidan,

$$c=-5,\quad u_1=-\frac{4}{5},\ \ u_2=-\frac{3}{5}.$$

munosabatlarni hosil qilamiz. Minimallashtirilayotgan funksionalning qiymati $J(u)=-5T$.

Mustaqil yechish uchun misollar.

1-misol. Optimallikning zaruriy va yetarli shartlaridan foydalanib, optimal boshqaruvning quyidagi elementar masalasini yeching.

1. $J(u) = \int_{t_0}^{t_1}(u^2 + t^2u)\,dt \to \inf$,

$B \leq u \leq A$.

2. $J(u) = \int_{t_0}^{t_1}(cosu + \varphi(t)u)dt \to \inf$,

$B \leq u \leq A$.

3. $J(u) = \int_{t_0}^{t_1}(cosu + \varphi(t)u)dt \to \inf$,

$|u| \leq A$.

4. $J(u) = \int_{t_0}^{t_1} u_1u_2dt \to \inf$,

$u_1 + u_2 \leq A$.

5. $J(u) = \int_{t_0}^{t_1} u_1u_2dt \to \inf$,

$u_1 \geq 0,\ \ u_2 \geq 0$.

6. $J(u) = \int_{t_0}^{t_1} u_1u_2dt \to \inf$,

$u_i \geq \alpha_i\ , i = 1,2$.

7. $J(u) = \int_{t_0}^{t_1} \varphi(t)u_1u_2dt \to \inf$,

$u_1^2 + u_2^2 \leq A,\ \ \ \varphi(t) \geq 0$.

8. $J(u) = \int_{-1}^{1}(u^2t - ut^3)dt \to \inf$,

$|u| \leq \frac{1}{2}$.

9. $J(u) = \int_{-10}^{10}(u^3 - 2t^2u)dt \to \inf$,

$|u| \leq 1$.

10. $J(u) = \int_{-1}^{1}(u_1^2 + 2tu_1 - t^2u_2)dt \to \inf$,

$0 \leq u_1 \leq 1,\ \ \ \ \ |u_2| \leq 1$.

11. $J(u) = \int_{0}^{\pi} \cos t\, u_1u_2dt \to \inf$,

$u_i \geq 1\ , i = 1,2;\ \ u_1^2 + u_2^2 \leq 9$.

12. $J(u) = \int_{-10}^{10} tu_1u_2dt \to \inf$,

$\frac{1}{2} \leq u_1 + u_2 \leq 2$.

13. $J(u) = \int_{-1}^{1}(t-1)u_1u_2dt \to \inf$,

$u_1^2 + u_2^2 \leq 1$.

14. $J(u) = \int_0^4 (u^2 - t^3 u)dt \to \inf$,

$0 \leq u \leq 1$,

15. $J(u) = \int_0^{2\pi} (u^2 + \cos t\, u)dt \to \inf$,

$|u| \leq \frac{1}{2}$.

16. $J(u) = \int_0^{\pi} (u^2 + \sin 4t\, u)dt \to \inf$,

$|u| \leq \frac{1}{2}$.

17. $J(u) = \int_0^{2\pi} (u_1^2 - \cos tu_1 + u_2^2 - \cos tu_2)dt \to \inf$,

$|u_i| \leq \frac{1}{2}$, $i = 1,2$.

18. $J(u) = \int_0^{2\pi} (u_1^2 - \cos tu_1 + u_2^2 - \cos tu_2)dt \to \inf$,

$0 \leq u_i \leq \frac{1}{4}$, $i = 1,2$.

19. $J(u) = \int_{-2}^{2} [u_1^3 - tu_1^2 + u_2^3 - tu_2^2]dt \to \inf$,

$|u_i| \leq \frac{1}{2}$, $i = 1,2$.

20. $J(u) = \int_{-1}^{1} [t^2 u^2 - tu_2 + u_1 - |u_1|]dt \to \inf$,

$|u_i| \leq 4$, $i = 1,2$.

21. $J(u) = \int_0^{T} [u_1^2 - tu_1 + u_2^2 - tu_2]dt \to \inf$,

$u_2 \geq 0$, $1 \geq u_1 \geq -2$.

22. $J(u) = \int_0^{T} [u^2 + \ln(t+1)\, u]dt \to \inf$,

$|u| \leq 2$.

23.
7

$|u| \leq 1$.

9-amaliy mashg‘ulot. Terminal boshqarish masalasida Pontryaginning maksimum prinsipi

1.Terminal boshqaruv masalasining qo’yilishi. Maksimum prinsipi.
Boshqarish obyekti

$$\dot{x} = f(x,u,t), \quad t \in [t_0, t_1] \qquad (1)$$

vektorli differensial tenglama bilan berilgan bo’lsin, bu yerda $x = (x_1,...,x_n)$, $u = (u_1,...,u_m)$, $f = (f_1,..., f_n)$, $f_i(x,u,t)$ funksiyalarni $f_{ixj}(x,u,t)$ hususiy hosilalari bilan birga uzluksiz deb hisoblaymiz. Joyiz boshqarishlar $[t_0,t_1]$ oraliqda aniqlangan bo'lakli uzluksiz va $V \subset R^m$ to'plamdan qiymatlar qabul qiluvchi *u=u(t)* *m*– vektor funksiyalardan iborat. (1) tenglamaning har bir *u=u(t)* joyiz boshqarishga mos $x = x(t)$ joyiz trayektoriyasi

$$x(t_0) = x^0 \qquad (2)$$

shartni qanoatlantiradi. Qaralayotgan obyektni boshqarish

$$J(u) = \varphi(x(t_1)) \qquad (3)$$

terminal kriteriy orqali sifat jihatidan baholanadi, bu yerda $\varphi(x) - R^n$ da uzluksiz differensiallanuvchi funksiya. Shunday $u^*(t)$ boshqarishni topish kerakki, $J(u^*) = \inf\limits_{u \in U} J(u)$ bo'lsin, bu yerda *U*– barcha joyiz boshqarishlar to'plami. Shunday qilib, quyidagi

$$\left.\begin{aligned} &J(u) = \varphi(x(t_1)) \to \inf \\ &\dot{x} = f(x,u,t), \ \ t \in [t_0,t_1] \\ &x(t_0) = x^0, \ \ u = u(t) \in V \end{aligned}\right\} \qquad (4)$$

terminal boshqaruv masalasini qaraymiz. Bu masalada trayektoriyalarning chap uchi mahkamlangan ((2) shartga q.), o'ng uchi esa, erkin $(x(t_1) \in R^n)$.

(4) masala avvalgi ma'ruzamizda qaralgan optimal boshqaruv umumiy masalasining xususiy holi bo'lib, Pontryaginning maksimum prinsipi bu masala uchun quyidagicha bo'ladi.

1-teorema. Agar $u^*(t),\ t \in [t_0,t_1]$ – optimal boshqarish, $x^*(t),\ t \in [t_0,t_1]$ optimal trayektoriya bo'lsa,,

$$H(x^*(t),\psi^*(t),u^*(t),t) = \max_{u \in V} H(x^*(t),\psi^*(t),u,t), \quad t \in [t_0,t_1] \qquad (5)$$

maksimum sharti bajariladi, bu yerda

$$H(x,\psi,u,t) = \psi' f(x,u,t) = \sum_{j=1}^{n} \psi_j f_j(x,u,t),$$

$\psi'(t),\ t \in [t_0,t_1]$ funksiya

$$\dot{\psi} = -\frac{\partial H(x^*(t),\psi,u^*(t),t)}{\partial x} \qquad (6)$$

$$\psi(t_1) = -\frac{\partial \varphi(x^*(t_1))}{\partial x} \qquad (7)$$

qo'shma sistemaning yechimidir.

<u>**1-misol.**</u>

$$\left.\begin{aligned} &x_2(1) \to \min, \\ &\dot{x}_1 = u, \ x_2 = -x_1^2, \ \ x_1(0) = x_2(0) = 0, \ \ t \in [0,1], \\ &|u| \le 1. \end{aligned}\right\} \qquad (8)$$

Bu masala (4) ko'rinishdagi terminal boshqarish masalasidir: $f = (f_1, f_2)$, $f_1 = u,\ \ f_2 = -x_1^2,\ \ x^0 = (0,0),\ t_0 = 0,\ t_1 = 1,\ \varphi(x) = x_2,\ \ x = (x_1, x_2)$.

Gamilton-Pontryagin funksiyasini tuzamiz:

$$H(x,\psi,u) = \psi_1 u - \psi_2 x_1^2.$$

Qo’shma sistemani tuzamiz:

$$\left.\begin{aligned}\dot\psi_1 &= -\frac{\partial H}{\partial x_1} = 2\psi_2 x_1,\\ \dot\psi_2 &= -\frac{\partial H}{\partial x_2} = 0\end{aligned}\right\}$$

Bu sistemaning

$$\psi_1(1) = -\frac{\partial\varphi(x(1))}{\partial x_1} = 0, \qquad \psi_2(1) = -\frac{\partial\varphi(x(1))}{\partial x_2} = -1$$

shartlarni qanoatlantiruvchi yechimi,

$$\psi_2(t) = -1,\ \ t\in[0,1],\ \ \psi_1(t) = -2\int_1^t x_2(\tau)d\tau$$

bo’ladi. Qaralayotgan masala uchun (4) maksimum sharti,

$$\psi_1(t)u(t) = \max_{|u|\le 1}\psi_1(t)u$$

ko’rinishga keladi. Demak, har bir

$$u(t) = sign\psi_1(t),\ \ t\in[0,1] \qquad (9)$$

joyiz boshqaruv – ekstremal boshqaruv bo’ladi.

$$u(t)\equiv 1,\ t\in\left[0,\frac{1}{3}\right),\ u(t) = -1,\ \ t\in\left[\frac{1}{3},1\right] \qquad (10)$$

ko’rinishdagi boshqaruv ham ekstremal boshqaruv bo’ladi, chunki unga mos keluvchi trayektoriyaning birinchi komponentasi

$$x_1(t) = t,\ \ t\in\left[0,\frac{1}{3}\right],\ \ x_1(t) = -t+\frac{2}{3},\ \ t\in\left[\frac{1}{3},1\right],$$

bo’lib,

$$\psi_1(t) = 2\int_t^1 x_1(\tau)d\tau = 2\int_t^{1/3} x_1(\tau)d\tau + 2\int_{1/3}^1 x_1(\tau)d\tau = \frac{1}{9} - t^2 \ge 0,\ \ t\in\left[0,\frac{1}{3}\right]$$

$$\psi_1(t) = 2\int_t^1 x_1(\tau)d\tau = 2\int_t^1\left(-\tau+\frac{2}{3}\right)d\tau = t^2 - \frac{4}{3}t + \frac{1}{3} \le 0,\ \ t\in\left[\frac{1}{3},1\right],$$

ya’ni (9) shart bajariladi. (10) boshqaruv ekstremal boshqaruv bo’lsa-da, optimal boshqaruv emas. Haqiqatan ham, $\vec{u}(t)\equiv 1,\ t\in[0,1]$ joyiz boshqaruv uchun $J(\bar u) = -\frac{1}{3}$, (10) uchun esa, $J(u) = -\frac{1}{27}$, ya’ni $J(\bar u) < J(u)$.

2. Chiziqli terminal boshqaruv masalasi.

Chiziqli boshqarish sistemasi uchun chiziqli terminal kriteriyli quyidagi masalani qaraymiz:

$$\left.\begin{aligned}&J(u) = c'x(t_1)\to\min,\\ &\dot x = A(t)x + b(t,u),\ \ t\in[t_0,t_1]\\ &x(t_0) = x^0,\ u = u(t)\in V,\ \ t\in[t_0,t_1]\end{aligned}\right\} \qquad (11)$$

bu yerda $x\in R^n,\ u\in R^m,$ $V\subset R^n,$ $A(t)$ – $n\times n$ –matrisa-funksiya, $b(t,u) = (b_1(t,u),...,b_m(t,u)),\ \ c\in R^n,\ \ x^0\in R^n$.

A(t) matrisaning elementlari $[t_0, t_1]$ da uzluksiz, $b_i(t,u)$, $i=\overline{1,n}$,funksiyalar $[t_0,t_1]\times V$ da uzluksiz deb faraz qilamiz.

(11) masala uchun quyidagi teorema o'rinlidir

2-teorema. $u=u(t)$, $t\in[t_0, t_1]$ bo'lakli-uzluksiz funksiyaning (11) masalada optimal boshqaruv bo'lishi uchun,

$$\max_{v\in V}\psi'(t)b(t,v)=\psi'(t)b(t,u(t)),\ t\in[t_0,t_1] \qquad (12)$$

shartning bajarilishi zarur va yetarlidir, bu yerda $\psi(t), t\in[t_0,t_1]$ funksiya

$$\dot{\psi}=-A'(t)\psi,\ \ \psi(t_1)=-c \qquad (13)$$

qo'shma sistemaning yechimidan iborat.

2-misol.

$$J(u)=x_1(1)+x_2(1)\to \min,$$
$$\dot{x}_1=x_2, \dot{x}_2=x_1+u,$$
$$x_1(0)=x_2(0)=0, |u|\le 1, t\in[0,1].$$

Bu masala (14) ko'rinishdagi masaladir:

$$x=(x_1,x_2), A=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, b(t,u)=(b_1(t,u),b_2(t,u)), b_1(t,u)=0,$$
$$b_2(t,u)=u, c=(c_1,c_2)=(1,1), V=[-1,1].$$

(15) maksimum sharti,

$$\max_{v\in V}\psi'(t)b(t,v)=\max_{v\in V}\psi_2(t)u(t)=\psi_2(t)u(t)$$

ko'rinishda bo'ladi, bu yerda $\psi(t)=(\psi_1(t),\psi_2(t))$

$$\psi_1(t)=-\psi_2,\ \ \psi_2=-\psi_1(1)=-1,\ \psi_2(1)=-1$$

qo'shma sistema yechimidan iborat. Demak, optimal boshqaruv

$$u(t)=sign\psi_2(t),\ \ t\in[0,1]$$

ko'rinishda bo'ladi. Qo'shma sistemaning yechimi

$$\psi^*{}_1(t)=-e^{1-t},\ \ \psi^*{}_2(t)=-e^{1-t}$$

bo'ladi. Shunday qilib, optimal boshqarish

$$u^*(t)=sign\psi^*{}_2=sign\left(-e^{1-t}\right)=-1,\quad t\in[0,1]$$

formula bilan aniqlanadi. Optimal trayektoriya esa,

$$\dot{x}_1=x_2,\ \dot{x}_2+u^*(t)$$

sistemaning $x_1(0)=x_2(0)=0$ shartni qanoatlantiruvchi yechimidan iborat. Bu sistemani yechib,

$$x_1^*(t)=1-\frac{1}{2}e^t-\frac{1}{2}e^{-t},\ \ x_1^*(t)=-\frac{1}{2}e^t-\frac{1}{2}e^{-t}$$

optimal trayektoriyani topamiz. Funksionalning minimal qiymati

$\min_{u\in U} J(u)=J(u^*)=x_1^*(1)+x_2^*(1)=1-e$ bo'ladi.

Mustaqil yechish uchun misollar.

1-masala. Terminal boshqaruv masalasida Gamilton - Pontryagin funksiyasi, qo'shma sistema, optimallik shartini qanoatlantiruvchi boshqarishning ko'rinishini toping.

1. $J(u) = x_1(1) + x_2(1) \to \min,\ \dot{x}_1 = x_2,\ \dot{x}_2 = -x_1^2 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,1]$.

2. $J(u) = x_1^2(1) + x_2(1) \to \min,\ \dot{x}_1 = 2x_1,\ \dot{x}_2 = x_2 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 2,\ t \in T = [0,1]$.

3. $J(u) = 2x_1(1) - x_2^2(1) \to \min,\ \dot{x}_1 = x_1 - x_2,\ \dot{x}_2 = x_2 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 3,\ t \in T = [0,1]$.

4. $J(u) = x_1^2(1) + x_2^2(1) \to \min,\ \dot{x}_1 = x_1^2 - 2u,\ \dot{x}_2 = x_1 + 2x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 4,\ t \in T = [0,1]$.

5. $J(u) = x_1^2(1) + x_1(1)x_2(1) \to \min,\ \dot{x}_1 = x_1 + x_2,\ \dot{x}_2 = x_1 - x_2 - u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 5,\ t \in T = [0,1]$.

6. $J(u) = 3x_1(2) - x_2(2) \to \min,\ \dot{x}_1 = -x_2 - u,\ \dot{x}_2 = x_1 + x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,2]$.

7. $J(u) = 2x_1^2(2) + 3x_2(2) \to \min,\ \dot{x}_1 = x_1^2 + x_2,\ \dot{x}_2 = -x_1 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 2,\ t \in T = [0,2]$.

8. $J(u) = x_1(2) + 2x_2^2(2) \to \min,\ \dot{x}_1 = -x_1 + x_2^2,\ \dot{x}_2 = x_2 + 2u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 3,\ t \in T = [0,2]$.

9. $J(u) = -2x_1^2(2) + x_2^2(2) \to \min,\ \dot{x}_1 = x_2 - 3u,\ \dot{x}_2 = -x_1 + x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 4,\ t \in T = [0,2]$.

10. $J(u) = 3x_1(2)x_2(2) - x_2^2(2) \to \min,\ \dot{x}_1 = 2x_1 - x_2,\ \dot{x}_2 = x_2^2 - 2u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 5,\ t \in T = [0,2]$.

11. $J(u) = x_1(3) + 2x_2(3) \to \min,\ \dot{x}_1 = x_1 + 2u,\ \dot{x}_2 = -x_1 + x_2^2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,3]$.

12. $J(u) = 2x_1^2(3) - x_2(3) \to \min,\ \dot{x}_1 = 2x_1 + x_2,\ \dot{x}_2 = x_1^2 + 2u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 2,\ t \in T = [0,3]$.

13. $J(u) = x_1^2(3) + 4x_2^2(3) \to \min,\ \dot{x}_1 = x_1^2 - 2x_2,\ \dot{x}_2 = x_1 + x_2 - u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 3,\ t \in T = [0,3]$.

14. $J(u) = x_1^2(3) + 3x_2^2(3) \to \min,\ \dot{x}_1 = 2x_1 - x_2^2,\ \dot{x}_2 = 3x_2 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 4,\ t \in T = [0,3]$.

15. $J(u) = 5x_1(3)x_2(3) \to \min,\ \dot{x}_1 = x_1 - x_2 + 3u,\ \dot{x}_2 = -x_1^2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 5,\ t \in T = [0,3]$.

16. $J(u) = 4x_1(4) - x_2(4) \to \min,\ \dot{x}_1 = -x_1^2 + 2u,\ \dot{x}_2 = x_1 + 3x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,4]$.

17. $J(u) = 2x_1^2(4) + 2x_2(4) \to \min,\ \dot{x}_1 = x_2^2 - u,\ \dot{x}_2 = -x_1 + x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 2,\ t \in T = [0,4]$.

18. $J(u) = 3x_1(4) - x_2^2(4) \to \min,\ \dot{x}_1 = x_1 + 2x_2,\ \dot{x}_2 = x_2^2 + 3u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 3,\ t \in T = [0,4]$.

19. $J(u) = x_1^2(4) + 4x_2^2(4) \to \min,\ \dot{x}_1 = x_1^2 - x_2,\ \dot{x}_2 = -x_2^2 + 2u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 4,\ t \in T = [0,4]$.

20. $J(u) = x_1^2(4) + x_1(4)x_2(4) - x_2^2(4) \to \min,\ \dot{x}_1 = x_2 - 2u,\ \dot{x}_2 = x_1^2 + 3x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 5,\ t \in T = [0,4]$.

21. $J(u) = x_1(1) - 5x_2(1) \to \min,\ \dot{x}_1 = x_1 + x_2,\ \dot{x}_2 = -x_1^3 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,1]$

Yechilishi: $H(x,\psi,u) = \psi' f(x,u,t) = \sum_{i=1}^{n} \psi_i f_i(x,u,t)$

$n = 2, \quad f_1(x,u,t) = x_1 + x_2, \quad f_2(x,u,t) = -x_1^3 + u$ bo'lgani uchun, Gamilton – Pontryagin funksiyasi $H(x,\psi,u) = \psi_1(x_1 + x_2) + \psi_2(x_1^3 + u)$ bo'ladi.

U holda qo'shma sistema $\dot{\psi}_1 = -\frac{\partial H}{\partial x_1} = -\psi_1 + 3x_1^2\psi_2, \quad \dot{\psi}_2 = -\frac{\partial H}{\partial x_2} = -\psi_1$ ko'rinishda bo'ladi. $\varphi(x(1)) = x_1(1) - 5x_2(1)$ bo'lgani uchun qo'shma sistema uchun boshlang'ich shartlar

$\psi_1(1) = -\frac{\partial \varphi(x(1))}{\partial x_1} = -1, \quad \psi_2(1) = -\frac{\partial \varphi(x(1))}{\partial x_2} = 5$ ko'rinishga ega. Maksimum shartini yozamiz:

$H(x^0,\psi^0,u^0,t) = \max_{|u|\le 1} H(x^0,\psi^0,u,t)$

$\psi_1^0(x_1^0 + x_2^0) + \psi_2^0(-x_1^{03} + u^0) = \max_{|u|\le 1}\left(\psi_1^0(x_1^0 + x_2^0) + \psi_2^0(-x_1^{03} + u)\right)$

Bu yerdan $\psi_2^0 u^0 = \max_{|u|\le 1} \psi_2^0 u$. Demak, optimal boshqarish

$$u^0 = sign\psi_2^0 = \begin{cases} 1, & агар\, \psi_2^0 > 0 \\ -1, & агар\, \psi_2^0 < 0 \end{cases}$$

ko'rinishga ega.

10-amaliy mashg'ulot. Chiziqli sistemalarni boshqarish. Boshqarish sintezi

1. Chiziqli terminal boshqaruv masalasi.

Chiziqli boshqarish sistemasi uchun chiziqli terminal kriteriyli quyidagi masalani qaraymiz:

$$\left.\begin{aligned} &J(u) = c'x(t_1) \to \min, \\ &\dot{x} = A(t)x + b(t,u), \quad t \in [t_0,t_1] \\ &x(t_0) = x^0, \ u = u(t) \in V, \quad t \in [t_0,t_1] \end{aligned}\right\} \qquad (11)$$

bu yerda $x \in R^n,\ u \in R^m,$ $V \subset R^n,$ $A(t) - n \times n -$matrisa-funksiya, $b(t,u) = (b_1(t,u),...,b_m(t,u)),\ \ c \in R^n,\ \ x^0 \in R^n$.

A(t) matrisaning elementlari $[t_0, t_1]$ da uzluksiz, $b_i(t,u)$, $i=\overline{1,n}$,funksiyalar $[t_0,t_1]\times V$ da uzluksiz deb faraz qilamiz.

(11) masala uchun quyidagi teorema o'rinlidir

2-teorema. $u=u(t)$, $t\in[t_0, t_1]$ bo'lakli-uzluksiz funksiyaning (11) masalada optimal boshqaruv bo'lishi uchun,

$$\max_{v\in V}\psi'(t)b(t,v)=\psi'(t)b(t,u(t)),\ t\in[t_0,t_1] \qquad (12)$$

shartning bajarilishi zarur va yetarlidir, bu yerda $\psi(t), t\in[t_0,t_1]$ funksiya

$$\dot{\psi}=-A'(t)\psi,\ \ \psi(t_1)=-c \qquad (13)$$

qo'shma sistemaning yechimidan iborat.

2-misol.
$$J(u)=x_1(1)+x_2(1)\to\min,$$
$$\dot{x}_1=x_2, \dot{x}_2=x_1+u,$$
$$x_1(0)=x_2(0)=0, |u|\le 1, t\in[0,1].$$

Bu masala (14) ko'rinishdagi masaladir:

$$x=(x_1,x_2), A=\begin{pmatrix}0&1\\0&0\end{pmatrix}, b(t,u)=(b_1(t,u),b_2(t,u)), b_1(t,u)=0,$$

$$b_2(t,u)=u, c=(c_1,c_2)=(1,1), V=[-1,1].$$

(15) maksimum sharti,

$$\max_{v\in V}\psi'(t)b(t,v)=\max_{v\in V}\psi_2(t)u(t)=\psi_2(t)u(t)$$

ko'rinishda bo'ladi, bu yerda $\psi(t)=(\psi_1(t),\psi_2(t))$

$$\psi_1(t)=-\psi_2,\ \ \psi_2=-\psi_1(1)=-1,\ \psi_2(1)=-1$$

qo'shma sistema yechimidan iborat. Demak, optimal boshqaruv

$$u(t)=sign\psi_2(t),\ \ t\in[0,1]$$

ko'rinishda bo'ladi. Qo'shma sistemaning yechimi

$$\psi^*{}_1(t)=-e^{1-t},\ \ \psi^*{}_2(t)=-e^{1-t}$$

bo'ladi. Shunday qilib, optimal boshqarish

$$u^*(t)=sign\psi^*{}_2=sign\left(-e^{1-t}\right)=-1,\quad t\in[0,1]$$

formula bilan aniqlanadi. Optimal trayektoriya esa,

$$\dot{x}_1=x_2,\ \dot{x}_2+u^*(t)$$

sistemaning $x_1(0)=x_2(0)=0$ shartni qanoatlantiruvchi yechimidan iborat. Bu sistemani yechib,

$$x_1^*(t)=1-\frac{1}{2}e^t-\frac{1}{2}e^{-t},\ \ x_1^*(t)=-\frac{1}{2}e^t-\frac{1}{2}e^{-t}$$

optimal trayektoriyani topamiz. Funksionalning minimal qiymati

$\min_{u\in U}J(u)=J(u^*)=x_1^*(1)+x_2^*(1)=1-e$ bo'ladi.

Mustaqil yechish uchun misollar.

1-masala. Terminal boshqaruv masalasida Gamilton - Pontryagin funksiyasi, qo'shma sistema, optimallik shartini qanoatlantiruvchi boshqarishning ko'rinishini toping.

1. $J(u) = x_1(1) + x_2(1) \to \min,\ \dot{x}_1 = x_2,\ \dot{x}_2 = -x_1^2 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,1]$.

2. $J(u) = x_1^2(1) + x_2(1) \to \min,\ \dot{x}_1 = 2x_1,\ \dot{x}_2 = x_2 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 2,\ t \in T = [0,1]$.

3. $J(u) = 2x_1(1) - x_2^2(1) \to \min,\ \dot{x}_1 = x_1 - x_2,\ \dot{x}_2 = x_2 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 3,\ t \in T = [0,1]$.

4. $J(u) = x_1^2(1) + x_2^2(1) \to \min,\ \dot{x}_1 = x_1^2 - 2u,\ \dot{x}_2 = x_1 + 2x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 4,\ t \in T = [0,1]$.

5. $J(u) = x_1^2(1) + x_1(1)x_2(1) \to \min,\ \dot{x}_1 = x_1 + x_2,\ \dot{x}_2 = x_1 - x_2 - u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 5,\ t \in T = [0,1]$.

6. $J(u) = 3x_1(2) - x_2(2) \to \min,\ \dot{x}_1 = -x_2 - u,\ \dot{x}_2 = x_1 + x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,2]$.

7. $J(u) = 2x_1^2(2) + 3x_2(2) \to \min,\ \dot{x}_1 = x_1^2 + x_2,\ \dot{x}_2 = -x_1 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 2,\ t \in T = [0,2]$.

8. $J(u) = x_1(2) + 2x_2^2(2) \to \min,\ \dot{x}_1 = -x_1 + x_2^2,\ \dot{x}_2 = x_2 + 2u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 3,\ t \in T = [0,2]$.

9. $J(u) = -2x_1^2(2) + x_2^2(2) \to \min,\ \dot{x}_1 = x_2 - 3u,\ \dot{x}_2 = -x_1 + x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 4,\ t \in T = [0,2]$.

10. $J(u) = 3x_1(2)x_2(2) - x_2^2(2) \to \min,\ \dot{x}_1 = 2x_1 - x_2,\ \dot{x}_2 = x_2^2 - 2u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 5,\ t \in T = [0,2]$.

11. $J(u) = x_1(3) + 2x_2(3) \to \min,\ \dot{x}_1 = x_1 + 2u,\ \dot{x}_2 = -x_1 + x_2^2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,3]$.

12. $J(u) = 2x_1^2(3) - x_2(3) \to \min,\ \dot{x}_1 = 2x_1 + x_2,\ \dot{x}_2 = x_1^2 + 2u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 2,\ t \in T = [0,3]$.

13. $J(u) = x_1^2(3) + 4x_2^2(3) \to \min,\ \dot{x}_1 = x_1^2 - 2x_2,\ \dot{x}_2 = x_1 + x_2 - u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 3,\ t \in T = [0,3]$.

14. $J(u) = x_1^2(3) + 3x_2^2(3) \to \min,\ \dot{x}_1 = 2x_1 - x_2^2,\ \dot{x}_2 = 3x_2 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 4,\ t \in T = [0,3]$.

15. $J(u) = 5x_1(3)x_2(3) \to \min,\ \dot{x}_1 = x_1 - x_2 + 3u,\ \dot{x}_2 = -x_1^2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 5,\ t \in T = [0,3]$.

16. $J(u) = 4x_1(4) - x_2(4) \to \min,\ \dot{x}_1 = -x_1^2 + 2u,\ \dot{x}_2 = x_1 + 3x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,4]$.

17. $J(u) = 2x_1^2(4) + 2x_2(4) \to \min,\ \dot{x}_1 = x_2^2 - u,\ \dot{x}_2 = -x_1 + x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 2,\ t \in T = [0,4]$.

18. $J(u) = 3x_1(4) - x_2^2(4) \to \min,\ \dot{x}_1 = x_1 + 2x_2,\ \dot{x}_2 = x_2^2 + 3u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 3,\ t \in T = [0,4]$.

19. $J(u) = x_1^2(4) + 4x_2^2(4) \to \min,\ \dot{x}_1 = x_1^2 - x_2,\ \dot{x}_2 = -x_2^2 + 2u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 4,\ t \in T = [0,4]$.

20. $J(u) = x_1^2(4) + x_1(4)x_2(4) - x_2^2(4) \to \min,\ \dot{x}_1 = x_2 - 2u,\ \dot{x}_2 = x_1^2 + 3x_2,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 5,\ t \in T = [0,4]$.

21. $J(u) = x_1(1) - 5x_2(1) \to \min,\ \dot{x}_1 = x_1 + x_2,\ \dot{x}_2 = -x_1^3 + u,$
$x_1(0) = x_2(0) = 0,\ |u(t)| \le 1,\ t \in T = [0,1]$

Yechilishi: $H(x,\psi,u) = \psi' f(x,u,t) = \sum_{i=1}^{n} \psi_i f_i(x,u,t)$

$n = 2,\quad f_1(x,u,t) = x_1 + x_2,\quad f_2(x,u,t) = -x_1^3 + u$ bo'lgani uchun, Gamilton – Pontryagin funksiyasi $H(x,\psi,u) = \psi_1(x_1 + x_2) + \psi_2(x_1^3 + u)$ bo'ladi.

U holda qo'shma sistema $\dot{\psi}_1 = -\frac{\partial H}{\partial x_1} = -\psi_1 + 3x_1^2\psi_2,\quad \dot{\psi}_2 = -\frac{\partial H}{\partial x_2} = -\psi_1$ ko'rinishda bo'ladi. $\varphi(x(1)) = x_1(1) - 5x_2(1)$ bo'lgani uchun qo'shma sistema uchun boshlang'ich shartlar

$\psi_1(1) = -\frac{\partial \varphi(x(1))}{\partial x_1} = -1,\quad \psi_2(1) = -\frac{\partial \varphi(x(1))}{\partial x_2} = 5$ ko'rinishga ega. Maksimum shartini yozamiz:

$$H(x^0,\psi^0,u^0,t) = \max_{|u|\le 1} H(x^0,\psi^0,u,t)$$

$$\psi_1^0(x_1^0 + x_2^0) + \psi_2^0(-x_1^{03} + u^0) = \max_{|u|\le 1}\left(\psi_1^0(x_1^0 + x_2^0) + \psi_2^0(-x_1^{03} + u)\right)$$

Bu yerdan $\psi_2^0 u^0 = \max_{|u|\le 1} \psi_2^0 u$. Demak, optimal boshqarish

$$u^0 = sign\psi_2^0 = \begin{cases} 1, & агар\, \psi_2^0 > 0 \\ -1, & агар\, \psi_2^0 < 0 \end{cases}$$

ko'rinishga ega.

11-amaliy mashg'ulot. Optimal boshqarishning sintezi masalasi. Sodda aperiodic va tebranma sistemalar uchun sintez masalasini yechish

1. Pontryaginning maksimum prinsipi. Maksimum prinsipi –optimal boshqaruv masalalarida optimallikning asosiy zaruriy sharti hisoblanadi. Bu natija XX asrning 50-yillari ikkinchi yarmida akademik L.S.Pontryagin boshchiligidagi sovet matematiklari tomonidan olingan.

Quyidagi:

$$J(u,x) = \int_{t_0}^{t_1} f_0(x(t),u(t)t)dt + g_0(x^0, x(t_1)) \to \inf \qquad (5)$$

$$\left.\begin{array}{l} \dot{x}(t) = f(x(t),u(t),t), t_0 \le t \le t_1 \\ x(t_0) = x^0, g_i(x(t_1)) \le 0,\ i = 1,...,k, \\ g_i(x(t_1)) = 0, i = k+1,...,s \\ u = u(t) \in V, \end{array}\right\} \quad (6)$$

optimal boshqaruv masalasini qaraymiz.

1-teorema. Agar $(x(t),u(t)),\ t_0 \le t \le t_1$ (5)-(6) masalaning yechimi bo'lsa, shunday $a_0, a_1,..., a_n$ sonlar va $\psi(t) = (\psi_1(t),...,\psi_n(t)), t_0 \le t \le t_1$ vektor-funksiya mavjud bo'ladiki, quyidagilar bajariladi:

1) $a = (a_0 a_1,..., a_n) \ne 0,\ a_0 \ge 0,..., a_k \ge 0;$

2) $\psi(t)$ funksiya - (20) qo'shma sistemaning *(x(t),u(t))* ga mos keluvchi yechimidan iborat;

3) *u(t)* optimal boshqarishning barcha $t \in [t_0,t_1]$ uzluksizlik nuqtalarida $H(x(t),u,t,\psi(t),a_0)$ funksiya $u=(u_1,\dots,u_m)$ o'zgaruvchi bo'yicha *V* to'plamda aniq yuqori chegarasiga *u=u(t)* bo'lganda erishadi, ya'ni

$$\sup_{u \in V} H(x(t),u,t,\psi(t),a_0) = H(x(t),u(t),t,\psi(t),a_0),(t_0 \le t \le t_1) \quad (7)$$

4)
$$\psi_1(t_1) = -\sum_{j=0}^{s} a_j g_{jx_i}(x(t_1), i = 1,2,....,n \quad (8)$$

$$a_j g_j(x(t_1)) = 0,\ j = 1,2,...,k \quad (9)$$

(5) shartlarga transversallik shartlari deyiladi.

Endi boshlang'ich yoki oxirgi vaqt momentlari belgilangan quyidagi :

$$J(u,x,t_0,t_1) = \int_{t_0}^{t_1} f_0(x(t),u(t)t)dt + g_0(x(t_1),t_0,t_1) \to \inf \quad (10)$$

$$\left.\begin{array}{l} \dot{x}(t) = f(x(t),u(t),t), t_0 \le t \le t_1 \\ x(t_0) = x^0, g_i(x(t_1),t_0,t_1) \le 0,\ i = 1,...,k, \\ g_i(x(t_1),t_0,t_1) = 0, i = k+1,...,s \\ u = u(t) \in V, t_0 \le t \le t_1 \end{array}\right\} \quad (11)$$

optimal boshqaruv masalasini qaraymiz.

Bu yerda $f = (f_0 f_1,..., f_n), f_0, g_0$ funksiyalarni o'z aniqlanish sohalarida $f_{jx_j}, g_{jx_j}, g_{jt_0}, g_{jt_1}$ xususiy hosilalari bilan birga uzluksiz deb faraz qilamiz.

2-teorema. Agar $(x(t),u(t),t_0,t_1)$ -(10)(11) masalaning yechimi bo'lsa, shunday $a_0 a_1,..., a_s$ sonlar va $\psi(t) = (\psi_1(t),...,\psi_n(t)), t_0 \le t \le t_1$ vektor-funksiya mavjud bo'ladiki, ular 1-teoremaning 1)-3) shartlarini va quyidagi transversallik shartlarini qanoatlantiradi:

$$\psi(t_1) = -\sum_{j=0}^{s} a_j g_{jx_i}(x(t_1),t_0,t_1) \quad (12)$$

$$\max_{u \in V} H(x(t_0),u,t_0,\psi(t_0),a_0) = -\sum_{j=0}^{s} a_j g_{jt_{0i}}(x(t_1),t_0,t_1) \quad (13)$$

(agar t_0 belgilangan bo'lsa, (13) shart qatnashmaydi);

$$\max_{u\in V} H(x(t_1),u,t_1,\psi(t_1),a_0) = -\sum_{j=0}^{s} a_j g_{jt_{1i}}(x(t_1),t_0,t_1) \qquad (14)$$

(agar t_1 belgilangan bo'lsa, (14) shart qatnashmaydi);

$$a_j g_j(x(t_1),t_0,t_1) = 0, \quad j = 1,2,...,k \qquad (15)$$

2.Maksimum prinsipining chegaraviy masalasi. Maksimum prinsipidan amaliyotda qanday foydalanish mumkinligini ko'rib o'tamiz.

$H(x,u,t,\psi,a_0)$ funksiyani $u=(u_0 u_1,...,u_n)$, o'zgaruvchining funksiyasi deb qaraymiz va har bir belgilangan (x,t,ψ,a_0) da

$$H(x,u,t,\psi,a_0) \to \sup, u\in V \qquad (16)$$

maksimallashtirish masalasini yechamiz.

$$u = u(x,t,\psi,a_0)\in V \qquad (17)$$

shu masalaning yechimi bo'lsin, ya'ni

$$H(x,u(x,t,\psi,a_0),t,\psi,a_0) = \sup_{u\in V} H(x,u,t,\psi,a_0) \qquad (18)$$

tenglik bajariilsin. Agar optimal boshqaruv masalasi yechimga ega bo'lsa, maksimum shartiga ko'ra (17) funksiya aniqlangan bo'ladi. Ko'p hollarda (18) funksiyani oshkor ko'rinishda yozish mumkin bo'ladi.

1-masala. Quyidagi optimal boshqaruv masalalarida Gamilton-Pontryagin funksiyasini tuzing va optimal boshqarishni toping.

1. $\int_0^T u^2 dt \to \inf,$

$\dot{x} = u, \quad x(0) = x^0, \quad x(T) = x'.$

2. $\int_0^T u^2 dt + \lambda x^2(T) \to \inf,$

$\dot{x} = u, \quad \dot{x}(0) = x^0.$

3. $\int_0^T u^2 dt \to \inf,$

$\dot{x} = x + u, \quad x(0) = x^0, \quad x(T) = x'.$

4. $\int_0^T (1-u)\sqrt{x}dt \to \inf,$

$\dot{x} = u\sqrt{x} - ux, \quad |u| \le 1, \quad x(0) = x_0, \quad x(T) = x_T.$

5. $\int_0^T \frac{tdt}{1+u^2} \to \inf,$

a) $\dot{x} = u, \quad u(t) \ge 0, \quad x(0) = \xi, \quad x(T) = \xi,$

b) $\dot{x} = u, \quad x(0) = 0, \quad x(T) = \xi.$

6. $I = x_2(1) \to \inf,$

$\dot{x}_1 = u, \quad x_2 = -ux_1, \quad |u(t)| \le 1,$

a) $x_1(0) = 0, \quad x_2(0) = 0,$

b) $x_1(0) = 0, \quad x_2(0) = 0, \quad x_1(1) = 2.$

7. $\frac{1}{2}\int_0^T (x_1u_2 - x_2u_1)dt \to \inf,$

a) $\dot{x}_1 = u_1, \quad \dot{x}_2 = a + u_2 \quad (u_1^{\ 2}) + (u_2)^2 \le u_0^2$

b) $x_1(0) = 0, \quad x_2(0) = 0, \quad x_1(T) = 0, \quad x_2(T) = 0.$

8. $\int_0^T \frac{x^2dt}{2} \to \inf,$

$\dot{x} = 1 - ux, \quad |u| \le A, \quad x(0) = \alpha,$

9. $\int_0^1 xdt \to \sup,$

$\dot{x} = u, \quad |u| \le 1, \quad x(0) = 0, \quad x(1) = 0.$

10. $\int_{S_0}^{S_1} \frac{((x_1)^2 + 1)((u_1)^2 + (u_2)^2)}{(2(u_1)^2 + 2(u_2)^2)^{\frac{1}{2}} - u_1} dt \to \inf,$

$\dot{x}_i = u_i, \quad i = 1,2,$

$x_i(S_j) = x_{ij}, \quad j = 0,1, \quad i = 1,2.$

11. $\int_0^1 t(x^2 + u^2)dt \to \inf,$

$\dot{x} = ax + bu, \quad x(0) = x_0,$

12. $\int_0^1 |u|dt \to \inf,$

$\ddot{x} + ux = 0, \quad |u| \le A,$

$x(0) = 0, \quad x(1) = 0, \quad \dot{x}(0) = 1.$

13. $\int_0^1 t^2x_1dt \to \inf,$

$\dot{x}_1 = x_2 + u^2, \quad \dot{x}_2 = u, \quad x_1(0) = 1.$

14. $\int_0^T (u^2x)dt \to \inf,$

$\dot{x} = u, \quad x(0) = 0, \quad |u| \le 1.$

15. $\int_0^2 xdt \to \inf,$

$\ddot{x} = u, \quad -1 \le u \le 2, \quad \dot{x}(2) = 0, \quad x(2) = 0, \quad \dot{x}(0) = 0.$

16. $\int_0^1 x^2dt \to \max,$

$|\dot{x}| \le 1, \quad x(0) = 0.$

17. $\int_0^1 u^2dt \to \inf,$

$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad |u| \le A,$

$x_1(0) = \xi_0, \quad x_2(0) = \xi_1, \quad x_1(1) = 0, \quad x_2(1) = 0,$

18. $\int_0^1 |u|dt \to \inf,$

$|\ddot{x}| \le A, \quad \ddot{x} = u, \quad x(0) = \xi_0,$
$\dot{x}(0) = \xi_1, \quad x(1) = 0, \quad \dot{x}(1) = 0.$

19. $\int_0^1 x^2 dt \to \sup,$

$|\ddot{x}| \le 1, \quad x(0) = 0, \quad x(1) = 0.$

20. $\int_{t_0}^{t_1} x\sqrt{1+u^2}dt \to \inf,$

$\dot{x} = u, \quad |u| \le A, \quad x(t_i) = x_i, \quad i = 0,1.$

21. $\int_{t_0}^{t_1} (t^2u^2 + \lambda x^2)dt \to \inf,$

$\dot{x} = u, \quad x(t_i) = \zeta_i, \quad i = 0,1$

22. $I = x_1^2(T) \to \max,$
$x_1(0) = u_1 \cos u_2, \quad \dot{x}_2 = u_1 \sin u_2 - 10,$
$\dot{x}_3 = x_1, \quad \dot{x}_2 = x_2,$
$x_1(0) = 0, \quad x_2(0) = 0, \quad x_3(0) = 0, \quad x_4(0) = 0.$

23. $\int_0^{\pi} ((u,u) - (x,x)dt \to \inf,$

$\dot{x} = Bu, \quad x(0) = 0, \quad (x(\pi), x(\pi)) = 1.$

24. $J(u) = \int_{t_0}^{t_1} (u^2(t) - x^2(t))dt \to \inf$

$$\left.\begin{aligned} &\dot{x}(t) = u(t), 0 \le t \le t_1 \\ &x(0) = x(t_1) = 0,\ t_1 > 0 \end{aligned}\right\}.$$

Yechilishi. Gamilton-Pontryagin funksiyasi (gamiltonian)

$$H = -a_0(u^2 - x^2) + \psi u$$

ko`rinishda bo`ladi. Qo`shma sistemani tuzamiz:

$$\dot{\psi}(t) = -H_x = -2a_0 x$$

Agar $a_0=0$ bo`lsa, $H=\psi u$ funksiya $u$ bo`yicha aniq yuqori chegarasiga $V=R^1$ to`plamda faqat $\psi=0$ bo`lganda erishadi. Bu esa, maksimum prinsipiga ziddir. Demak, $a_0>0$ ya’ni $a_0=1$ deb olish mumkin. U vaqtda $H = u^2 - x^2 + \psi u$ funksiyaning $u \in V = R^1$. bo`yicha aniq yuqori chegarasiga $u = \frac{\psi}{2}$ nuqtada erishiladi. U vaqtda maksimum prinsipining chegaraviy masalasi,

$$\dot{x} = \frac{\psi}{2},\ \dot{\psi} = 2x,\ 0 \le t \le t_1, x(0) = x(t_1) = 0$$

ko`rinishda yoziladi. Bu masalaning yagona yechimi $(x(t) \equiv 0, \psi(t) \equiv 0), (0 \le t \le t_1)$ bo’ladi. U vaqtda $(u(t) = 0, \psi(1)/2 \equiv 0), (0 \le t \le t_1)$ -bu bizga ma’lum optimal boshqarishdir.

12-amaliy mashg‘ulot. N intervallar haqidagi teorema. Feldbaum teoremasi. Misollar

Tezkor masala. n ***intervallar haqidagi teorema.*** Obyekt chiziqli bo’lganda tezkor masalani qaraymiz:

$$\dot{x}_i = \sum_{k=1}^{n} a_{ik} x_k + \sum_{j=1}^{r} b_{ij} u_j, \qquad i = 1,2,...,n; \tag{12a}$$

$$\alpha_j \le u_j \le \beta_j, \quad \alpha_j < 0, \quad \beta_j > 0, \quad j = 1,2,...,r; \tag{12b}$$

$$x_i(t_0) = x_i^0, \quad x_i(t_f) = 0, \quad i = 1,2,...,n; \tag{12v}$$

$$J = t_f - t_0 \to \min. \tag{12g}$$

Bu masalaga *chiziqli tezkor masala* deyiladi. Obyektning tenglamalari vektor shaklda

$$\dot{\mathrm{x}} = \mathrm{Ax} + \mathrm{Bu}$$

ko’rinishda yoziladi. Bu tenglamalar, odatda, chetlanishi mavjud tenglamalar- dan iborat, va, shu sababli, obyektni o’tkazish kerak bo’lgan oxirgi holat, koordinatalar boshidan iborat: $(x(t_f) = 0)$.

Pontryagin funksiyasi

$$H = \psi^T (A\mathrm{x} + B\mathrm{u}) = \sum_{i=1}^{n} \psi_i \Big(\sum_{k=1}^{n} a_{ik} x_k + \sum_{j=1}^{r} b_{ij} u_j\Big),$$

ko’rinishda bo’ladi , buyerda $\psi^T = (\psi_1 \psi_2 ... \psi_n)$ ushbu

$$\dot{\psi}^T = -\frac{\partial H}{\partial \mathrm{x}}$$

qo’shma tenglamani qanoatlantiradi.

Maksimum prinsipiga asosan, optimal boshqarish

$$\max_{u \in U} H = \sum_{i=1}^{n} \psi_i \sum_{k=1}^{n} a_{ik} x_k + \max_{u \in U} \sum_{i=1}^{n} \psi_i \sum_{j=1}^{r} b_{ij} u_j,$$

yoki

$$\max_{u \in U} \sum_{j=1}^{r} u_j \sum_{i=1}^{n} b_{ij} \psi_i = \sum_{j=1}^{r} \max_{u \in U} \Big(u_j \sum_{i=1}^{n} b_{ij} \psi_i\Big),$$

shartdan aniqlanadi, buyerda

$$U = \{\mathrm{u} : \alpha_j \le u_j \le \beta_j, \quad j = 1,2,....,r\}.$$

Agar, *normallik sharti* deb ataladigan, shart bajarilsa (uni quyida kelti-ramiz), u holda $\sum_{i=1}^{n} b_{ij} \psi_i$ yig’indi faqat ajralgan nuqtalarda nolga aylanadi. Bu holda oxirgi ayniyatdan $u^*(t)$ optimal boshqarishning u_j^* $(j = 1,...,r)$ koordina-talari bo’lakli o’zgarmas bo’lishi va α_j yoki β_j chetki qiymatlarni qabul qilishi kelib chiqadi:

$$u_j^* = \begin{cases} \alpha_j, & \sum_{i=1}^{n} b_{ij}\psi_i < 0, \\ \beta_j, & \sum_{i=1}^{n} b_{ij}\psi_i > 0, \end{cases} \qquad j = 1,2,...,r.$$

$\beta_j = -\alpha_j$ bo'lgan xususiy holda,

$$u_j^* = \beta_j sign \sum_{i=1}^{n} b_{ij}\psi_i, \quad j = 1,2,....,r$$

bo'lishini olamiz.

Chiziqli tezkor masala uchun maksimum prinsipi, ***normallik sharti*** bajarilganda, optimallikning zaruriy va yetarli shartidan iborat bo'ladi. Bu tushunchani aniqlash uchun, quyidagi

$$N[j] = [B^j (AB)^j ....(A^{n-1}B)^j],$$

$(n \times n)-$ matrisalarni qaraymiz, buyerda $B^j, (AB)^j, ..., (A^{n-1}B)^j -$ mos ravishda, $B, AB, ..., A^{n-1}B$ matrisalarning j-ustunlaridan iborat.

Normallik sharti. *Agar* $N[j]$ *matrisalar maxsusmas ,ya'ni* $\det N[j] \neq 0 \quad (j = 1,2,....,r)$ *bo'lsa,* $\dot{x} = Ax + Bu$ *obyekt uchun **normallik sharti** yoki **holatning umumiylik sharti** bajariladi, deyiladi.*

Ravshanki, boshqarish skalyar skalyar bo'lgan holda, normallik sharti boshqariluvchanlik sharti bilan ustma-ust tushadi. Normallik sharti baja-rilgan obyekt, *normal obyekt* yoki *normal boshqariluvchan sistema* deyiladi.

8- misol. Ushbu obyekt uchun normallik shartining bajarilishini tekshiring:

$$\dot{x}_1 = x_2 + u_1, \quad \dot{x}_2 = u_1 + u_2.$$

Yechilishi. Berilgan misolda

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

$N[j]$ matrisalar

$$N[1] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad N[2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

ko'rinishga ega va ular maxsusmas matrisalardan iborat. Demak, normallik sharti bajariladi.

Optimallikning zaruriy va yetarli sharti. *Agar chiziqli tezkor masalada obyekt normal bo'lsa,* $(u^*(t), x^*(t))$ *joiz juftlik uning yechimi bo'lishi uchun, u maksimum prinsipini qanoatlantirishi zarur va yetarli.*

Chiziqli obyektlar uchun tezkor masalaning optimal boshqarishida $u_j^*(t)$ *funksiyalar, agar obyekt normal bo'lsa,* A *matrisaning ixtiyoriy xos qiymatlarida, faqat chegaraviy qiymatlarni qabul qiladi. Umumiy holda bu funksiyalar ixtiyoriy* *sondagi o'tishlarga -bitta chegaraviy qiymatdan boshqasiga o'tish nuqtalariga ega bo'ladi. Xususiy holda quyidagi teorema o'rinli.*

n ***intervallar haqidagi teorema*** *. Agar chiziqli tezkor masalada obyekt normal bo'lib,*

$$\det(A-\lambda E)=0$$

xarakteristik tenglama faqat haqiqiy ildizlarga ega bo'lsa, u holda optimal boshqarishning $u_j^*(t)$ $(j=1,2,...,r)$ komponentalari bo'lakli o'zgarmas bo'ladi, faqat chetki qiymatlarni qabul qiladi hamda n tadan ortiq bo'lmagan o'zgarmaslik intervallariga ega bo'ladi, yoki $n-1$ tadan ortiq bo'lmagan o'tishlarga ega bo'ladi.

Masalalar

19. Quyidagi optimal boshqaruv masalalarida $u^*(t)$ programm optimal boshqarish va $x^*(t)$ optimal trayektoriyani toping:

1) $\dot{x}_1=x_2,\ \dot{x}_2=u,\ |u|\le 1,\ x(0)=0,\ x_1(t_f)=10,\ x_2(t_f)=0,\ J=t_f\to\min.$

2) $\dot{x}_1=x_2,\ \dot{x}_2=u,\ |u|\le 1,\ x(0)=0,\ x_1(t_f)=10,\ J=t_f\to\min.$

3) $\dot{x}_1=x_2,\ \dot{x}_2=u,\ |u|\le 1,\ x(0)=0,\ x_1(t_f)=10,\ x_2(t_f)=5,\ J=t_f\to\min.$

4) $\dot{x}_1=x_2,\ \dot{x}_2=u,\ |u|\le 1,\ x_1(0)=0,\ x_2(0)=5,\ x_1(t_f)=10,\ x_2(t_f)=0,\ J=t_f\to\min.$

5) $\dot{x}_1=x_2,\ \dot{x}_2=u,\ |u|\le 1,\ x_1(0)=5,\ x_2(0)=0,\ x_1(t_f)=10,\ x_2(t_f)=0,\ J=t_f\to\min.$

6) $\dot{x}_1=x_2,\ \dot{x}_2=u-1,\ |u|\le 2,\ x(0)=0,\ x_1(t_f)=10,\ x_2(t_f)=0,\ J=t_f\to\min.$

7) $\dot{x}_1=x_2,\ \dot{x}_2=u-1,\ |u|\le 2,\ x(0)=0,\ x_1(t_f)=10,\ J=t_f\to\min.$

8) $\dot{x}_1=x_2,\ \dot{x}_2=u-1,\ |u|\le 2,\ x(0)=0,\ x_2(0)=0,\ x_1(t_f)=10,\ x_2(t_f)=5,\ J=t_f\to\min.$

9) $\dot{x}_1=x_2,\ \dot{x}_2=u-1,\ |u|\le 2,\ x_1(0)=0,\ x_2(0)=5,\ x_1(t_f)=10,\ x_2(t_f)=0,\ J=t_f\to\min.$

10) $\dot{x}_1=x_2,\ \dot{x}_2=u-1,\ |u|\le 2,\ x_1(0)=5,\ x_2(0)=0,\ x_1(t_f)=10,\ x_2(t_f)=0,\ J=t_f\to\min.$

13-amaliy mashg'ulot. Dinamik programmalashtirish usulining optimal boshqaruv masalalarida qo'llanilishi

1.Dinamik programmalashtirish usuli (resurslarni taqsimlash masalasi).
Dinamik programmalashtirish deb, matematik modellari ko'p bosqichli va dinamik jarayonli xarakterga ega bo'lgan chiziqli bo`lmagan programmalash-tirishning maxsus masalalari va optimal boshqaruv masalalarini yechishning hisoblash usuliga aytiladi. Bu usul jarayonlarning ketma-ket tahliliga asoslangan bo'lib, ekstremal masalalarni yechishda amerikalik olim R. Bellman tomonidan XX asrning 50-

yillaridan boshlab dastlab sistematik va prinsipial keng qo`llanila boshlandi. Mazkur bobda bu usulning asosiy qoidalari chiziqli bo`lmagan programmalashtirishning qator maxsus masalalari uchun bayon qilinadi. Dinamik programmalashtirishning optiml boshqaruv masalalariga ba’zi tatbiqlari haqida ham so`z yuritamiz.

Resurslarni taqsimlash masalasi. Aytaylik,*c* hajmli xomashyo va *n* ta texnologik jarayon mavjud bo`lsin. Agar xomashyoning *x* miqdorini *i* — texnologik jarayonda sarflansa, $f_i(x_i)$ foyda olinadi. Maksimal foyda olish uchun xom ashyoni jarayonlar o’rtasida qanday taqsimlash kerak?

Faraz qilaylik, x_i – i-jarayon uchun ajratilgan xomashyo miqdori bo`lsin. U holda qo`yilgan *resurslarni taqsimlash masalasining* matematik modeli

$$\sum_{i=1}^{n} f_i(x_i) \to \max, \quad \sum_{i=1}^{n} x_i = c, \quad x_i \geq 0, \quad i = \overline{1,n} \qquad (1)$$

ko`rinishni oladi.

(1) chiziqli bo`lmagan programmalashtirish masalasining o’ziga xosligi shundan iboratki, uning maqsad funksiyasi $f(x)$ va asosiy bog`lanish funksiyasi $g(x)$ *separabeldir,* ya’ni ular bir o’zgaruvchili funksiyalar yig`indisi shaklida ifodalangan.

Ekstremal masalani dinamik programmalashtirish usuli bilan yechishning birinchi bosqichi — berilgan masalani unga o`xshash masalalar oilasiga *invariant turkumlashdan* iboratdir. Bu bosqich ma’lum ma’noda san’at bo`lib, har bir muayyan holda tadqiqotchining tajribasi, sezgisi va mahoratiga bog`liqdir. U (1) masala uchun ixtiyoriy $k,\ 1 \leq k \leq n$ sondagi texnologik jarayonlarga va $y,\quad 0 \leq y \leq c$ xomashyo g`amlamasiga ega bo`lgan resurslarni taqsimlashning ushbu

$$\sum_{i=1}^{n} f_i(x_i) \to \max, \quad \sum_{i=1}^{k} x_i = y, \quad x_i \geq 0, \quad i = \overline{1,k} \qquad (2)$$

masalalarini qarashdan iboratdir. $k = n,\quad y = c$ bo`lganda (2) masalalar oilasidan boshlang`ich (1) masala olinadi.

(2) masalalar oilasidan olingan ixtiyoriy masala maqsad funksiyasining optimal qiymati $B_k(y)$ *Bellman funksiyasi* deyiladi:

$$B_k(y) = \max \sum_{i=1}^{k} f_i(x_i), \quad \sum_{i=1}^{k} x_i = y, \quad x_i \geq 0, \quad i = \overline{1,k}\ . \qquad (3)$$

Masalani dinamik programmalashtirish usuli bilan yechishning ikkinchi bosqichi — Bellman funksiyasi uchun tenglamani olishdan iboratdir. Bu bosqichda *Bellmanning optimallik prinsipi* umumiy holda qo’llaniladi. . 1, 2, . . . $,k-1$ nomerli jarayonlar uchun esa, $y - z$ miqdordagi xomashyo qoladi. Aytaylik, bu xomashyo qolgan jarayonlarga optimal taqsimlangan bo`lsin. (3) ning aniqlanishiga ko`ra, $k-1$ ta jarayondan keladigan foydaning maksimal miqdori $B_{k-1}(y-z)$ ga teng bo`ladi. Shunday qilib, $k$ jarayonga $z$ miqdorda xomashyo ajratilganda barcha $k$ jarayonlar va *y* xomashyo g`amlamasidan

$$f_k(z)+B_{k-1}(y-z) \quad (4)$$
foyda olamiz.

z miqdorni $0\le z\le y$ chegarasida o'zgartirib, (4) umumiy foyda maksimal bo`ladigan $x_k^0(y)$ (k- jarayon uchun xomashyoning optimal miqdori) qiymatni topamiz:

$$f_k\left(x_k^0(y)\right)+B_{k-1}\left(y-x_k^0(y)\right)=\max_{0\le z\le y}[f_k(z)+B_{k-1}(y-z)] \quad (5)$$

Ikkinchi tomondan, (3) ga asosan, xomashyo miqdori y bo'lganda k ta jarayondan olinadigan maksimal foyda $B_k(y)$ ga tengdir. Bu qiymatni (5) ifodaning o`ng tomoniga tenglashtirib, $B_k(y)$ funksiya uchun

$$B_k(y)=\max_{0\le z\le y}[f_k(z)+B_{k-1}(y-z)],\quad k=\overline{1,n},\quad 0\le y\le c \quad (6)$$

tenglamani olamiz. Bu *Bellman tenglamasi* deb ataladi. (6) tenglama $B_k(y)$ funksiyaning k argumentiga nisbatan rekurrent bo`lganligidan uni yechish uchun boshlang`ich shart berilishi kerak. Uni (3) dan $k=1$ bo`lganda topish mumkin:

$$B_1(y)=\max f_1(x_1),\quad x_1=y,\ x_1\ge 0.$$

Shunday qilib, (6) Bellman tenglamasi uchun boshlang`ich shart

$$B_1(y)=f_1(y)$$

ko`rinishga ega bo`ladi.

1-misol. Resurslar taqsimoti haqidagi

$$\sum_{i=1}^{n} f_i(x_i)\to\max,\qquad \sum_{i=1}^{n} x_i=b,\qquad x_i\ge 0, i=\overline{1,n}. \qquad (2)$$

masalada

$n=3, b=4, f_1(x)=x, f_2(x)=x^2+x, f_3(x)=x^2-x^3+24x$ bo'lsin. Shu masalani yechamiz.

Demak, quyidagi masala berilgan:

$$\sum_{i=1}^{3} f_i(x_i)=x_1+x_2^2+x_2+x_3^2+24x_3\to\max,\quad x_1+x_2+x_3=4,\quad x_i\ge 0, i=1,2,...$$

Unga o'xshash masalalar oilasi quyidagicha bo'ladi.

$$\sum_{i=1}^{k} f_i(x_i)\to\max,\quad \sum_{i=1}^{k} x_i=y,\quad x_i\ge 0, i=\overline{1,k},\quad k=1,2,3,\qquad 0\le y\le 4.$$

Bellman funksiyasi

$$B_k(y)=\max\sum_{i=1}^{k} f_i(x_i),\quad \sum_{i=1}^{k} x_i=y,\quad x_i\ge 0, i=\overline{1,k},\quad k=1,2,3,\qquad 0\le y\le 4.$$

uchun Bellman tenglamasini yozamiz.

$$B_k(y)=\max_{0\le z\le y}\left[f_k(z)+B_{k-1}(y-z)\right],\quad k=2,3,\quad 0\le y\le 4.$$

Bu tenglama uchun boshlang'ich shart $B_1(y) = f_1(y) = y$ bo'ladi. Bellman tenglamasidan ketma-ket $B_2(y)$ va $B_3(y)$ funksiyalarni topamiz.

$$B_2(y) = \max_{0\le z\le y}\left[f_2(z) + B_1(y-z)\right] = \max_{0\le z\le y}\left[z^2 + z + y - z\right] = y^2 + y$$

$$B_3(y) = \max_{0\le z\le y}\left[f_3(z) + B_2(y-z)\right] = \max_{0\le z\le y}\left[z^2 - z^3 + 24z + (y-z)^2 + y - z\right]$$

Endi optimal taqsimotni aniqlaymiz.

$$B_3(4) = \max_{0\le z\le 4}\left[z^2 - z^3 + 24z + (4-z)^2 + 4 - z\right] = \max_{0\le z\le y}\left[2z^2 - z^3 + 15z + 20\right] = 56$$

bu yerda maksimumga **z = 3** da erishiladi. Demak, $b_1 = b - x_3^0 = 1;\ \ B_2(1) = \max_{o\le z\le 1}[z^2 + 1] = 2$,

bu yerda maksimumga **z = 1** da erishiladi. Demak, **x^o_2= 1** . U vaqtda $x_1^0 = 4 - x_3^0 - x_2^0 = 0$.

Shunday qilib, optimal taqsimot quyidagicha bo'ladi. **x^o_1=0, x^o_2= 1 , x^o_3= 3 :** maksimal foyda $B_3(4) = 56$ ga teng.

Mustaqil yechish uchun misollar.

Dinamik programmalashtirish usulidan foydalanib, quyidagi masalalarni yeching.

1) $x_1^2 + x_2^2 + \frac{1}{2}x_2 + 4x_3 \to \max, \quad 2x_1 + x_2 + x_3 = 11, \quad x_i \ge 0, i = 1,2,3;$

2) $\frac{1}{2}x_1^2 + 3x_2 + x_3^3 - x_3^2 \to \max, \quad x_1 + 2x_2 + x_3 = 9 \quad x_i \ge 0, i = 1,2,3;$

3) $x_1^3 + 2x_2^2 + 2x_3^3 - x_3 \to \max, \quad x_1 + 2x_2 + 3x_3 = 10, \quad x_i \ge 0, i = 1,2,3;$

4) $2x_1^2 + \frac{1}{3}x_2^3 + 10x_3 \to \max, \quad 3x_1 + x_2 + x_3 = 12, \quad x_i \ge 0, i = 1,2,3;$

5) $\frac{1}{4}x_1^2 + \frac{1}{6}x_2^2 + \frac{1}{12}x_3^2 \to \max, \quad 3x_1 + 2x_2 + 4x_3 = 16, \quad x_i \ge 0, i = 1,2,3;$

6) $e^{x_1} + x_2^4 + 5x_3^2 \to \max, \quad x_1 + 3x_2 + 2x_2 = 14, \quad x_i \ge 0, i = 1,2,3;$

7) $2^{x_1}x_2^2 + x_3^2 - x_3 \to \max, \quad 3x_1 + x_2 + 3x_3 = 16, \quad x_i \ge 0, i = 1,2,3;$

8) $\frac{1}{3}x_1^3 + e^{x_2} - x_2 + 8x_3 \to \max, \quad x_1 + 4x_2 + x_3 = 15, \quad x_i \ge 0, i = 1,2,3;$

1) $2^{x_1} + \frac{1}{3}x_2^3 + 4x_3^2 + \frac{1}{2}x_4^3 \to \max, \quad x_1 + 2x_2 + x_3 + 3x_4 \le 10, \quad x_i \ge 0, i = \overline{1,4};$

2) $e^{x_1} + 2x_2^2 + 3x_3^2 - x_3 + x_4^2 \to \max, \quad 2x_1 + x_2 + x_3 + 2x_4 \le 10, \quad x_i \ge 0, i = \overline{1,4};$

3) $5x_1^2+3x_2^4-3x_2^2+x_3^2+\frac{1}{2}x_4^2\to\max,\qquad x_1+3x_2+2x_3+x_4\le 9,\qquad x_i\ge 0, i=\overline{1,4};$

4) $\frac{1}{3}x_1^3-x_1+3^{x_2}+2x_3^2+9x_4\to\max,\qquad x_1+x_2+3x_3+x_4\le 10,\qquad x_i\ge 0, i=\overline{1,4};$

5) $3x_1^2+2x_2^2+e^{x_3}-6x_3+x_4^2\to\max,\qquad x_1+2x_2+3x_3+4x_4\le 8,\qquad x_i\ge 0, i=\overline{1,4};$

6) $4x_1^2+5x_2^2+x_3^2+\frac{1}{3}x_4^4\to\max,\qquad 2x_1+3x_2+x_3+x_4\le 10,\qquad x_i\ge 0, i=\overline{1,4};$

7) $6x_1^2+12x_2+e^{x_3}+x_4^2\to\max,\qquad x_1+x_2+2x_3+x_4\le 9,\qquad x_i\ge 0, i=\overline{1,4};$

8) $\frac{1}{2}x_1^3+x_2^2+x_3^3+x_3+7x_4\to\max,\qquad x_1+2x_2+3x_3+4x_4\le 8,\qquad x_i\ge 0, i=\overline{1,4};$

Terminal boshqaruv masalasini dinamik programmalshtirish usuli bilan yechish.

Terminal boshqaruvning eng sodda masalasi quyidagi ko`rinishda qo`yilgan edi:

$$\dot{x}=f(x,u,t), x(t_0)=x_0, u(t)\in U, t\in T=[t_0,t_1], \qquad (7)$$

$$I(u)=\varphi(x(t_1))\to\min .$$

Bu masalani skalar τ va *n* vector x parametrga bog`liq bo`lgan,

$$\dot{x}=f(x,u,t), x(\tau)=x, u(t)\in U, t\in T_\tau=[\tau,t_1],$$

$$I(u)=\varphi(x(t_1))\to\min , \qquad (8)$$

masalalar oilasiga turkumlaymiz.

$$-\frac{\partial B(x,\tau)}{\partial\tau}=\min_{v\in U}\frac{\partial B'(x,\tau)}{\partial x}f(x,v,\tau). \qquad (9)$$

Bellman tenglamasi.

Optimallikning yetarlilik sharti. Oldingi bandda (9) tenglamadan Pontryaginning maksimum prinsipi ancha kuchli talablarda olindi. Hozirgi ishlarda (9) tenglamadan, odatda, optimallikning zaruriyliki shartlarini emas, balki yetarlilik shartlarini ifodalash uchun foydalaniladi.

Teorema. Aytaylik, $B(x,t)$ -(9) Bellman tenglamasining

$$B(x,t_1)=\varphi(x)+\lambda' g(x)(\lambda\ge 0) \qquad (10)$$

chegaraviy shartli silliq yechimi $u(x,t)$ quyidagi

$$\frac{\partial B'(x,t)}{\partial x}f(x,y(x,t)t)=\min_{u\in U}\frac{\partial B'(x,t)}{\partial x}f(x,u,t) \qquad (11)$$

shartni qanoatlantiruvchi boshqarish qonuni bo`lsin. Agar

$$\dot{x}=f(x,u(x,t),t), x(t_0)=x_0$$

tenglama shunday $x(t), t\in T$ yechimga ega bo`lsaki, u yechim bo'ylab $u(t)=u(x(t),t)$ bo`lakli-uzluksiz va

$$g(x(t_1))\le 0, \lambda' g(x(t_1))=0 \qquad (12)$$

bo`lsa, $u(t), t\in T$ boshqarish

$$I(u)=\varphi(x(t_1))\to\min, \dot{x}=f(x,u,t), \qquad (13)$$

$$x(t_0)=x_0, u(t)\in U, t\in T, g(x(t_1))\leq 0$$

masalada optimal bo`ladi.

1- **misol.** Bog'lanishlar qatnashmagan bir o'lchovli masala. Optimal boshqaruvning quyidagi masalasini qaraymiz:

$$\frac{dx}{dt}=u(t), \qquad t\in[0,T],$$

$$x(0)=x_0,$$

$$J(x,u)=\int_0^T u^2(t)dt+px^2(T)\to min,$$

Bunda $x(t)\in E^1, u(t)$ -bo'lakli uzluksiz skalyar funksiya, x_0, p, T berilgan sonlar va $p\geq 0$. Shu masala uchun Bellman usulini qo'llaymiz.

Faraz qilaylik, $\psi(y,\tau)-$ qaralayotgan masala uchun Bellman funksiyasi bo'lsin. Bu funksiya uchun Koshi-Bellman masalasini yozamiz:

$$\min_u\left\{\frac{\partial\psi(y,\tau)}{\partial y}u+\frac{\partial\psi(y,\tau)}{\partial\tau}+u^2\right\}=0$$

$$\psi(y,T)=py^2.$$

Quyidagi belgilashni kiritamiz:

$$R(y,u,\tau)=\frac{\partial\psi(y,\tau)}{\partial y}u+\frac{\partial\psi(y,\tau)}{\partial\tau}+u^2.$$

Bu funksiya belgilangan y,τ lar uchun skalyar u o'zgaruvchining funksiyasidan iborat. Modomiki, boshqarishga bog'lanioshlar qo'yilmagan ekan, $R(y,u,\tau)$ Funksiya minimumga erishadigan nuqtani topish uchun, $\frac{\partial R(y,u,\tau)}{\partial u}$, hosilani nolga tenglashtirish kerak bo'ladi, ya'ni

$$\frac{\partial\psi(y,\tau)}{\partial y}+2u=0,$$

Bu yerdan

$$u=-\frac{1}{2}\frac{\partial\psi(y,\tau)}{\partial y} \qquad (1)$$

Ekanligini olamiz. Haqiqatan bu nuqta minimum nuqtasi bo'ladi, chunki

$$\frac{\partial^2R(y,u,\tau)}{\partial u^2}=2>0\ .$$

$R(y,u,\tau)$ funksiyaning minimumini hisoblaymiz:

$$\min_u R(y,u,\tau)=-\frac{1}{2}\left(\frac{\partial\psi(y,\tau)}{\partial y}\right)^2+\frac{\partial\psi(y,\tau)}{\partial\tau}+\frac{1}{4}\left(\frac{\partial\psi(y,\tau)}{\partial y}\right)^2$$

$$=-\frac{1}{4}\left(\frac{\partial\psi(y,\tau)}{\partial y}\right)^2+\frac{\partial\psi(y,\tau)}{\partial\tau}.$$

Shunday qilib, $\psi(\zeta,\tau)$ Bellman funksiyasi uchun quyidagi masalani hosil qilamiz:

$$-\frac{1}{4}\left(\frac{\partial\psi(y,\tau)}{\partial y}\right)^2+\frac{\partial\psi(y,\tau)}{\partial\tau}=0 \qquad (2)$$

$$\psi(y,T)=py^2 \qquad (3)$$

$\psi(y,\tau)$ funksiyani *y ga* nisbatan ko'phad ko'rinishida izlaymiz, ya'ni

$$\psi(y,\tau)=\varphi_0(\tau)+\varphi_1(\tau)y+\varphi_2(\tau)y^2, \qquad (4)$$

Bo'lsin deb faraz qilamiz, bu yerda $\varphi_0,\varphi_1,\varphi_2$ lar aniqlanishi kerak bo'lgan noma'lum funksiyalar. Bu funksiya ifodasini (2), (3) larga keltirib qo'yib, ko'phadlar uchun quyidagi tengliklarni olamiz:

$$-\frac{1}{4}(\varphi_1(\tau)+2\varphi_2(\tau)y)^2+\frac{d\varphi_0}{d\tau}+\frac{d\varphi_1}{d\tau}y+\frac{d\varphi_2}{d\tau}y^2=0$$

$$\varphi_0(T)+\varphi_1(T)y+\varphi_2(T)y^2=py^2.$$

y ning bir xil darajalari oldidagi koeffisiyentlarini tenglashtirib, quyidagi 3ta differensial tenglamalardan iborat sistema va boshlang'ich shartlarni olamiz:

$$\frac{d\varphi_2}{d\tau}-\frac{1}{4}\varphi_2^2(\tau)=p, \qquad (5)$$

$$\frac{d\varphi_1}{d\tau}-\varphi_1(\tau),\varphi_2(\tau)=0, \qquad (6)$$

$$\frac{d\varphi_0}{d\tau}-\frac{1}{4}\varphi_1^2(\tau)=0, \qquad (7)$$

$$\varphi_2(T)=p, \qquad (8)$$

$$\varphi_1(T)=0, \qquad (9)$$

$$\varphi_0(T)=0. \qquad (10)$$

(1) tenglamani integrallaymiz va quyidagiga ega bo'lamiz:

$$\frac{d\varphi_2}{d\tau}=\varphi_2^2(\tau),$$

yoki

$$\frac{d\varphi_2}{\varphi_2^2(\tau)}=d\tau, \qquad -\frac{1}{\varphi_2}=\tau+C.$$

Bu yerdagi C ixtiyoriy o'zgarmasni (7) shartdan topamiz:

$$\varphi_2(T)=-\frac{1}{T+C}=p,$$

Bu yerdan

$$C=-\frac{1+Tp}{p}.$$

Demak,

$$\varphi_2(T)=-\frac{1}{\tau+C}=-\frac{p}{p(\tau-T)-1}=\frac{p}{p(T-\tau)+1}.$$

$\varphi_1(\tau)$ funksiyani topish uchun (6), (9) Koshi masalasiga ega bo'lamiz. Bu tenglama nolga teng bo'lgan bosslang'ich shartli bir jinsli diferensial tenglamadan iborat, shu sababli $\varphi_1(\tau)=0$ shunga o'xshash (7), (10) lardan $\varphi_0(\tau)=0$.

ekanligi kelib chiqadi. Topilgan $\varphi_0, \varphi_1, \varphi_2$ larni (4) tenglikka keltirib qo'yib, qaralayotgan masala uchun Bellman funksiyasini olamiz:

$$\psi(y,\tau) = \frac{py^2}{p(T-\tau)+1}.$$

$y = x_0$ bo'lganda bu formula sifat kriteriysining minimal qiymatini beradi. (1) dan foydalanib, optimal boshqarishni topamiz.

$$u(\tau) = -\frac{1}{2}\frac{\partial\psi(x_0,\tau)}{\partial y} = -\frac{px_0}{p(T-\tau)+1}.$$

Mustaqil yechish uchun misollar.

Quyidagi terminal boshqaruv masalalarida Bellman funksiyasini aniqlab, Bellman tenglamasini yozing.

1.
$$\begin{aligned}
&I(\omega) = x_1^2(T) \to \max,\\
&\dot{x}_1 = u_1 \cos u_2, \quad \dot{x}_2 = u_1 \sin u_2 - 10,\\
&\dot{x}_3 = x_1, \quad \dot{x}_4 = x_2,\\
&x(0) = 0, \ \ x_2(0) = 0, \ \ x_3(0) = 0, \ \ x_4(0) = 0.
\end{aligned}$$

2.
$$\begin{aligned}
&I(\omega) = x_5(T) \to \inf,\\
&\dot{x}_1 = -u_1, \quad \dot{x}_2 = -u_1 + 2, \quad \dot{x}_3 = -u_2,\\
&\dot{x}_4 = -u_2 + 2, \quad \dot{x}_5 = -u_1 x_1 x_2 - u_2 x_3 x_4 + u_2,\\
&x_i(0) = 0, \quad i = 1,\ldots,,3,\\
&|u| \le 1, \quad i = 1,2.
\end{aligned}$$

3.
$$\begin{aligned}
&I(\omega) = x_1^2(2) + 4x_2^2(2) \to \inf,\\
&\dot{x}_1 = 1 + x_2 u, \quad \dot{x}_2 = -t x_1 u,\\
&|u| \le 2, \quad t \in [1,2],\\
&x_1(1) = 0, \quad x_2(1) = 1.
\end{aligned}$$

4.
$$\begin{aligned}
&I(\omega) = x_5(T) \to \inf,\\
&t \in [0,T],\\
&\dot{x}_1 = -u_1, \quad \dot{x}_2 = -u_1 + 2, \quad \dot{x}_3 = -u_2,\\
&\dot{x}_4 = -u_2 + 2, \quad \dot{x}_5 = -u_1 x_1 x_2 - u_2 x_1 x_3 + u_2,\\
&|u| \le 10, \quad i = 1,2, \ \ x_i(0) = 0, \quad i = 1,\ldots,5.
\end{aligned}$$

5.
$$I(\omega) = x_2(1) \to \max,$$
$$\dot{x}_1 = -u_1 x_1 + u_1^2 x_2, \quad \dot{x}_2 = u_2 x_1 - 3u_1^2 x_2,$$
$$|u_i| \le 1, \quad i = 1, 2, \quad x_1(0) = 1, \quad x_2(0) = 0.$$

6.
$$I(\omega) = x_1(T) \to \max,$$
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{u - x_2}{x_3}, \quad \dot{x}_3 = u,$$
$$t \in [0, T], \quad 0 \le u \le A,$$
$$x_1(0) = \vartheta, \quad x_2(0) = \vartheta, \quad x_3(0) = m_0, \quad x_3(T) = m_k,$$
$$\vartheta, m_0, m_k, A - const.$$

7.
$$I(\omega) = x_1(T) \to \max,$$
$$\dot{x}_1 = x_2, \quad \dot{x}_2 = \frac{u - x_2^2}{x_3}, \quad \dot{x}_3 = u, \quad u \ge 0,$$
$$x_1(0) = 0, \quad x_2(0) = \vartheta, \quad x_3(0) = 1, \quad x_3(T) = m,$$
$$m, \vartheta - const.$$

8.
$$I(\omega) = x_1^2(2) + x_2^2(2) \to \inf,$$
$$\dot{x}_1 = 1 + x_2 u, \quad \dot{x}_2 = -t x_1 u,$$
$$|u| \le 1, \quad t \in [1, 2],$$
$$x_1(1) = 1, \quad x_2(1) = 5.$$

9.
$$I(\omega) = x_2(1) \to \inf,$$
$$\dot{x}_1 = u, \quad \dot{x}_2 = -u x_1,$$
$$a)\, x_1(0) = 0, \quad x_2(0) = 0,$$
$$b)\, x_1(0) = 0, \quad x_2(0) = 0, \quad x_1(1) = 2.$$

10.
$$I(\omega) = \sum_{i=1}^{N} x_i^2(0) \to \inf,$$
$$\dot{x} = t x_i u_i^2, \quad |u_i| \le 1, \quad i = 1, 2.$$

11. $I = x_2(1) \to \inf,$

$\dot{x}_1 = u, \quad x_2 = -u x_1, \quad |u(t)| \le 1,$

a) $x_1(0) = 0, \quad x_2(0) = 0,$

b) $x_1(0) = 0, \quad x_2(0) = 0, \quad x_1(1) = 2.$

12. $$T \to \inf, \quad |\ddot{x}| \le 2, \quad x(-1) = 1, \quad x(T) = -1, \quad \dot{x}(-1) = 0, \quad \dot{x}(T) = 0.$$

13. $$T \to \inf, \quad -1 \le \ddot{x} \le 3, \quad x(0) = 1, \quad x(T) = -1, \quad \dot{x}(0) = 0, \quad \dot{x}(T) = 0.$$

14. $$T \to \inf, \quad |\ddot{x}| \le 1, \quad x(0) = \xi_1, \quad \dot{x}(0) = \xi_2, \quad \dot{x}(T) = 0.$$

14-amaliy mashg‘ulot. Differensial o‘yinlar haqida asosiy tushunchalar

1. Matrisaviy o’yin. O’yining quyi va yuqori baholari

Ikkita A va B o’yinchilar qatnashgan antogonistik o’yinni qaraymiz. O’yinchilar qarama-qarshi maqsadni ko’zlaydi. Biri qandaydir yutuqqa ega bo’lsa, ikkinchisi shu miqdorda yutqazadi. Demak A o’yinchining yutug’i B o’yinchi yutug’ining qarama-qarshi ishora bilan olinganiga teng bo’lgani uchun, bu o’yinda A o’yinchining yutugini tahlil qilish yetarli.

A o’yinchi (biz uni I o’yinchi deymiz) m ta $A_1, A_2, \ldots, A_m$ strategiyalariga, B o’yinchi (biz uni II o’yinchi deymiz) n ta $B_1, B_2, \ldots, B_m$ strategiyalarga ega bo’lsin. Bunday o’yinga $m \times n$ o’lchamli o’yin (ba’zan qisqacha $m \times n$-o’yin) deyiladi.

I o’yinchi o’zining mumkin bo’lgan strategiyalaridan biri A_i ni, $i = 1,2,\ldots,m,$ II o’yinchi esa, I o’yinchining tanlash natijasidan bexabar holda, B_j strategiyani $(j = 1,2,\ldots,n)$ tanlangan bo’lsin. Strategiyalarni tanlash natijasida I o’yinchining yutug’i $W_1(A_i, B_j)$ va II o’yinchining yutug’i $W_2(A_i, B_j)$ bo’lsa, ular $W_1(A_i, B_j) + W_2(A_i, B_j) = 0$ munosabatni qanoatlantiradi. Agar $W(A_i, B_j) = W_1(A_i, B_j)$ deb olsak, $W_2(A_i, B_j) = -W(A_i, B_j)$ bo’ladi. $W(A_i, B_j) = a_{ij}$ deb belgilaylik. Bu qiymatlarni satrlari I o’yinchining strategiyalariga, ustunlari esa II o’yinchining strategiyalariga mos keladigan jadval (1-jadval) ko’rinishida yozamiz. Bunday jadval to’lovlar matrisasi deb ataladi.

1-jadval

B / A	B_1	B_2	...	B_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
...	…	…	…	…

A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

To'lovlar matrisasining har bir musbat a_{ij} elementi mos strategiyalar qo'llanilganda I o'yinchining yutug'i (yoki II o'yinchining yutqazig'i) miqdorini bildiradi. Matrisaning har bir manfiy a_{ij} elementi esa mos strategiyalar qo'llanilganda I o'yinchining yutqazig'i (yoki II o'yinchining yutug'i) miqdorini bildiradi. Ikkala o'yinchining ham maqsadi – o'z yutug'ini maksimallashtirishdan (yoki o'z yutqazig'ini minimallashtirishdan) iborat.
1-misol. O'yinda I va II o'yinchilar ishtirok qiladilar. O'yinchilardan har biri boshqasidan bexabar holda 1, 2 yoki 3 ta barmog'ini ko'rsatishi mumkin. Agar I va II o'yinchilar ko'rsatgan barmoqlar soni o'rtasidagi ayirma musbat bo'lsa, I o'yinchi shu sonlar ayirmasi qadar ochko yutadi va aksincha, agar ayirma manfiy bo'lsa, II o'yinchi shuncha yutadi. Agar sonlar o'rtasidagi ayirma nol bo'lsa, o'yin durang bilan tugaydi.

O'yinda har bir o'yinchining uchtadan shaxsiy yurishi bor. I o'yinchi strategiyalari: A_1 – 1 ta barmoqni ko'rsatish, A_2 – 2 ta barmoqni ko'rsatish, A_3 – 3 ta barmoqni ko'rsatish. II o'yinchining (ya'ni, I o'yinchi raqibining) strategiyalari esa, B_1 – 1 ta, B_2 – 2 ta, B_3 – 3 ta barmoqni ko'rsatishdan iborat. O'yinchilarning ular tegishli strategiyalarni qo'llaganlardagi yutuqlarini to'lov matrisasi (2-jadval) ko'rinishida yozib qo'yamiz.
2-jadval

II I	B_1	B_2	B_3
A_1	0	-1	-2
A_2	1	0	-1
A_3	2	1	0

Bu matrisa elementlari qanday hosil qilinganligini ko'ramiz. Agar I o'yinchi A_1 strategiyasini, II o'yinchi B_3 strategiyasini qo'llasa, u vaqtda I o'yinchi ikki ochko yutqazadi. Bu yutqazish to'lov matrisasida birinchi satr va uchinchi ustunlar kesishishidagi (1;3) katakka yozilgan ($a_{13} = 2$). Agar I o'yinchi A_2 strategiyasini, raqib esa B_1 strategiyani qo'llasa, u holda I o'yinchi bir ochko yutadi. To'lov matrisasida bu yutuq (2;1) katakka musbat ishora bilan yozib qo'yilgan ($a_{21} = 1$). Jadvalning boshqa elementlari ham shu tariqa hosil qilingan.

To'lov matrisasi 1-jadvalda keltirilgan $m \times n$ – o'yinni qaraymiz. Masala $A_1, A_2, \ldots, A_m$ strategiyalar orasidan I o'yinchining eng yaxshi strategiyasini, $B_1, B_2, \ldots, B_m$ strategiyalardan esa II o'yinchining eng yaxshi strategiyasini topishdan iborat. Bu masalani yechishda o'yinda ishtirok etuvchi raqiblar bir xil aql-idrokka ega va ulardan har biri o'z maqsadiga erishish uchun hamma chora-tadbirlarni ko'radi deb hisoblaymiz. Bu tamoyildan foydalanib I o'yinchining eng yaxshi strategiyasini topamiz. Buning uchun uning hamma strategiyalarini ketma-ket tahlil qilamiz. I o'yinchi o'zining A_i strategiyasini tanlaganda biz unga II o'yinchi o'zining I o'yinchi yutug'ini minimallashtiruvchi B_j strategiyasi bilan javob beradi deb hisoblashimiz kerak. Shunga ko'ra to'lov matrisasining har bir satridagi a_{ij} sonlardan minimalini topamiz va uni $\alpha_i, i = \overline{1, m}$, bilan belgilab, to'lov matrisasining yonida qo'shimcha ustunga yozib qo'yamiz:

$$\alpha_i = \min_{j=\overline{1,n}} a_{ij}, i = 1,2,\ldots,m \quad (1)$$

α_i sonlarni bilgan I o'yinchi o'zining strategiyalaridan shundayini tanlaydiki, bu unga eng katta yutuq bersin. Bu maksimal yutuqni α deb belgilaymiz, ya'ni $\alpha = \max_{i=\overline{1,m}} \alpha_i$. Shunday qilib, (1) ni hisobga olsak, $\alpha = \max_{i=\overline{1,m}} \min_{j=\overline{1,n}} a_{ij}$ hosil bo'ladi.

α soni I o'yinchining kafolatli yutug'i bo'lib, o'yinning quyi bahosi (maksimini) deb ataladi. O'yinning quyi bahosi α ni ta'minlovchi strategiya maksimin strategiya deb ataladi. Agar I o'yinchi o'zining maksimin strategiyasiga amal qilsa, II o'yinchi qanday yo'l tutishidan qa'tiy nazar, unga α dan kam bo'lmagan yutuq ta'minlanadi.

II o'yinchi o'z yutqazig'ini kamaytirishga, ya'ni I o'yinchi yutug'ini minimumga aylantirishga harakat qiladi. Shu sababli o'zining eng yaxshi strategiyasini tanlab olish uchun u to'lov matrisasi ustunlarining har biridagi maksimal sonni topishi va bu qiymatlar orasidan eng kichigini tanlab olishi kerak.

Har bir ustundagi maksimal elementni β_j deb belgilaymiz va bu elementlarni 3-jadvalning qo'shimcha satriga yozib qo'yamiz. β_j lar orasidan eng kichik qiymatlisini β deb belgilaymiz. β - o'yinning yuqori bahosi (minimaksi) bo'lib, u $\beta = \min_{j=\overline{1,n}} \max_{i=\overline{1,m}} a_{ij}$ formula bo'yicha topiladi.

II o'yinchiga β "yutuqni" ta'minlaydigan strategiya uning minimaks strategiyasi deb ataladi. Agar II o'yinchi o'zining minimaks strategiyasiga amal qilsa, har qanday holda ham uning yutqazig'i β dan oshmaydi.

3-jadval

B A	B_1	B_2	...	B_n	α_i
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}	α_1
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}	α_2
...	…	…	…	…	...
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}	α_m
β_j	β_1	β_2	...	β_n	α β

O'yinning quyi va yuqori baholari uchun $\max_{1\le i\le m} \min_{1\le j\le n} a_{ij} \le \min_{1\le j\le n} \max_{1\le i\le m} a_{ij}$, tengsizlikning hamisha o'rinli, ya'ni $\alpha \le \beta$ ekanligini ko'rsatish mumkin.

2. O'yinning egar nuqtasi. Sof strategiyalar

Quyi va yuqori baholari o'zaro teng, ya'ni $\alpha = \beta$ bo'lgan o'yinlar mavjud. Bunday o'yinlar egar nuqtali o'yinlar deb ataladi. Egar nuktali o'yilarda yuqori va quyi baholarining umumiy qiymati o'yining sof bahosi, bu qiymatga erishishni ta'minlovchi A_{i^*} va B_{j^*} strategiyalar esa optimal strategiyalar deyiladi.

Optimal strategiyalarning (A_{i^*}, B_{j^*}) jufti matrisaviy o'yinning egar nuqtasi (muvozanat vaziyati) deb ataladi. To'lovlar matrisasida shu egar nuqtaga mos $a_{i^*j^*} = \gamma$ element bir vaqtning o'zida i - satrda minimal, j - ustunda maksimaldir.

2-misol. To'lov matrisasi 2-jadvalda keltirilgan o'yinning yechimini topamiz. Shu jadvalga mos α_i va β_j larning kiymatlarini topib, ular yordamida 4-jadvalni hosil qilamiz.

4- jadval

II I	B_1	B_2	B_3	α_i
A_1	0	-1	-2	-2
A_2	1	0	-1	-1
A_3	2	1	0	0
β_j	2	1	0	

O'yinning quyi bahosi

$$\alpha = \max_i \alpha_i = max(-2;-1;0) = 0, \quad \alpha = \alpha_3 .$$

O'yinning yuqori bahosi

$$\beta = \min_j \beta_j = min(2;1;0) = 0, \quad \beta = \beta_3 .$$

$\alpha = \beta$ bo'lgani uchun o'yin egar nuqtaga ega. O'yinning sof bahosi $\gamma = 0$. Optimal strategiyalar: I o'yinchining A_3 strategiyasi va II o'yinchining B_3 strategiyasi. Egar nuqta esa (A_3, B_3) bo'ladi.

3-misol. To'lov matrisasi 5-jadvalda keltirilgan o'yinning yechimi topilsin.

α_i va β_j larning qiymatlarini topamiz va ularni 5-jadvalga kiritamiz.

5- jadval

II I	B_1	B_2	B_3	B_4	α_i
A_1	6	5	8	5	5
A_2	7	3	2	3	2
A_3	6	5	7	5	5
β_j	7	5	8	5	

O'yinning quyi va yuqori baholarini topamiz:

$$\alpha = \max_i \alpha_i = max(5,2,5) = 5, \quad \alpha = \alpha_1 = \alpha_3 = 5$$

$$\beta = \min_j \beta_j = min(7;5;8) = 5, \quad \beta = \beta_2 = \beta_4 = 5 .$$

α va β ning qiymatlaridan ko'rinib turibdiki, o'yinda optimal strategiyalarning A_1B_2, A_1B_4, A_3B_2, A_3B_4 juftlariga mos to'rtta egar nuqta mavjud. O'yinning sof bahosi $\gamma = 5$.

Muammoli masala va topshiriqlar

O’yin vaziyatini sifat jihatidan aks ettiruvchi masalani qo’ying va uning matematik modelini tuzing.
1-topshiriqda qo’yilgan o’yin masalasida tomonlarni, ularning sonini, o’yin xarakterini va o’yinchilar strategiyalarini aniqlang.
O’yinda I va II o’yinchilar ishtirok qiladilar. O’yinchilardan har biri boshqasidan bexabar holda 1, 2 yoki 3 ta barmog’ini ko’rsatishi mumkin. Agar I va II o’yinchilar ko’rsatgan barmoqlar soni yig’indisi juft bo’lsa, I o’yinchi shu yig’indiga teng ochko yutadi va aksincha, agar yig’indi toq bo’lsa, II o’yinchi shuncha ochko yutadi.Shu o’yinda:
a) tomonlarning strategiyalari va ularga mos yutuqlarini aniqlang;
b) o’yinchilarning minimaks va maksimin strategiyalarini toping.
To’lovlar matrisasi H quyidagicha bo’lgan o’yinda quyi va yuqori baholarni toping va egar nuqta (muvozanat vaziyati) mavjudligini tekshiring:

$$H = \begin{pmatrix} 9 & -5 & 2 & 6 & -7 \\ -1 & 5 & 8 & -2 & 4 \\ 5 & 7 & -5 & 0 & 5 \\ 6 & 1 & -2 & 3 & 8 \end{pmatrix}.$$

Mustaqil ishlash uchun savollar

O’yinlar nazariyasi nima bilan shug’ullanadi. O’yinlarning turlari.
Nol yig’indili o’yin. Strategiya, optimal strategiya tushunchalari.
Matrisaviy o’yin, to’lovlar matrisasi, minimaks va maksimin strategiyalar.
O’yinning quyi va yuqori baholari, sof baho, sof optimal strategiyalar.
Quyidagi to’lovlar matritsasi bilan berilgan o’yinda muvozanat vaziyati (egar nuqta) bor yoki yo’qligini tekshiring, bor bo’lsa egar nuqtani va maksimin, minimaks strategiyalarni toping.

1) $H = \begin{pmatrix} 4 & 3 & 5 & 6 \\ 2 & 1 & 7 & 8 \\ 6 & 1 & 7 & 9 \end{pmatrix}$ 2) $H = \begin{pmatrix} 3 & 2 & 5 & 6 \\ 1 & 0 & 6 & 7 \\ 5 & 1 & 5 & 4 \end{pmatrix}$

3) $H = \begin{pmatrix} 4 & 1 & 5 & 8 \\ 0 & 1 & 6 & 5 \\ 3 & 2 & 4 & 3 \end{pmatrix}$ 4) $H = \begin{pmatrix} 1 & 2 & 0 & 5 \\ 1 & 4 & 6 & 7 \\ 2 & 5 & 7 & 9 \end{pmatrix}$

5) $H = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 4 & 2 & 5 & 3 \\ 6 & 3 & 7 & 8 \end{pmatrix}$ 6) $H = \begin{pmatrix} 2 & 5 & 6 & 7 \\ 3 & 6 & 7 & 8 \\ 4 & 4 & 5 & 6 \end{pmatrix}$

7) $H = \begin{pmatrix} 5 & 1 & 4 & 8 \\ 4 & 2 & 5 & 3 \\ 5 & 4 & 3 & 7 \end{pmatrix}$ 8) $H = \begin{pmatrix} 5 & 0 & 3 & 7 \\ 3 & 1 & 4 & 2 \\ 3 & 2 & 4 & 5 \end{pmatrix}$

9) $H = \begin{pmatrix} 4 & 2 & 5 & 3 \\ 3 & 1 & 1 & 4 \\ 5 & 0 & 4 & 3 \end{pmatrix}$ 10) $H = \begin{pmatrix} 3 & 1 & 0 & 5 \\ 4 & 1 & 5 & 0 \\ 6 & 2 & 2 & 3 \end{pmatrix}$

11) $H = \begin{pmatrix} 10 & 1 & 2 & 3 \\ 8 & 3 & 4 & 6 \\ 9 & 4 & 8 & 7 \end{pmatrix}$ 12) $H = \begin{pmatrix} 7 & 1 & 3 & 4 \\ 6 & 2 & 6 & 5 \\ 8 & 4 & 7 & 8 \end{pmatrix}$

13) $H = \begin{pmatrix} 6 & 1 & 3 & 2 \\ 8 & 2 & 5 & 4 \\ 7 & 3 & 6 & 7 \end{pmatrix}$ 14) $H = \begin{pmatrix} 5 & 0 & 2 & 3 \\ 6 & 1 & 5 & 6 \\ 7 & 3 & 7 & 9 \end{pmatrix}$

15) $H = \begin{pmatrix} 0 & 1 & 2 & 4 \\ 1 & 2 & 0 & 6 \\ 1 & 3 & 2 & 1 \end{pmatrix}$ 16) $H = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 2 & 6 & 7 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix}$

17) $H = \begin{pmatrix} 1 & 2 & 4 & 9 \\ 0 & 4 & 3 & 6 \\ 2 & 7 & 9 & 6 \end{pmatrix}$ 18) $H = \begin{pmatrix} 1 & 6 & 8 & 5 \\ 2 & 7 & 5 & 9 \\ 6 & 9 & 9 & 10 \end{pmatrix}$

19) $H = \begin{pmatrix} 1 & 3 & 6 \\ 2 & 5 & 6 \\ 4 & 3 & 2 \\ 5 & 7 & 9 \end{pmatrix}$ 20) $H = \begin{pmatrix} 3 & 1 & 5 \\ 6 & 2 & 4 \\ 3 & 0 & 1 \\ 4 & 2 & 5 \end{pmatrix}$

21) $H = \begin{pmatrix} 4 & 2 & 5 \\ 1 & 4 & 7 \\ 4 & 5 & 0 \\ 6 & 5 & 8 \end{pmatrix}$ 22) $H = \begin{pmatrix} 5 & 4 & 2 \\ 8 & 1 & 3 \\ 6 & 4 & 2 \\ 4 & 5 & 3 \end{pmatrix}$

23) $H = \begin{pmatrix} 4 & 2 & 5 & 6 \\ 1 & 4 & 7 & 8 \\ 0 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{pmatrix}$ 24) $H = \begin{pmatrix} 5 & 4 & 2 & 3 \\ 3 & 1 & 8 & 6 \\ 0 & 5 & 4 & 2 \\ 10 & 9 & 8 & 6 \end{pmatrix}$

25) $H = \begin{pmatrix} 3 & 6 & 3 & 4 \\ 1 & 2 & 0 & 2 \\ 5 & 4 & 1 & 5 \end{pmatrix}$ 26) $H = \begin{pmatrix} 1 & 5 & 4 & 5 \\ 3 & 2 & 3 & 7 \\ 6 & 6 & 2 & 9 \end{pmatrix}$

27) $H = \begin{pmatrix} 1 & 2 & 3 & 5 \\ 3 & 5 & 3 & 7 \\ 6 & 6 & 2 & 9 \end{pmatrix}$ 28) $H = \begin{pmatrix} 1 & 1 & 4 & 6 \\ 2 & 4 & 5 & 5 \\ 2 & 7 & 9 & 8 \end{pmatrix}$

29) $H = \begin{pmatrix} 1 & 1 & 4 & 6 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 7 & 9 \\ 5 & 6 & 8 & 7 \end{pmatrix}$ 30) $H = \begin{pmatrix} 5 & 2 & 3 & 1 \\ 6 & 5 & 6 & 3 \\ 7 & 8 & 9 & 2 \\ 6 & 4 & 5 & 4 \end{pmatrix}$

31) $H = \begin{pmatrix} 1 & 2 & 4 & 3 \\ 3 & 2 & 5 & 6 \\ 4 & 8 & 7 & 5 \\ 4 & 4 & 8 & 6 \end{pmatrix}$ 32) $H = \begin{pmatrix} 1 & 1 & 4 & 6 \\ 3 & 6 & 3 & 4 \\ 5 & 6 & 8 & 9 \\ 4 & 2 & 6 & 7 \end{pmatrix}$

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